σ-POROUS SETS OF GENERALISED NONEXPANSIVE MAPPINGS

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Abstract. In the paper we show that many generic results obtained in the last 40 years in the field of fixed point theory, are consequences of one general theorem. As an application we get some extensions of them and we study the generic aspect of the existence of attractors of nonexpansive IFSs. We also give a detailed discussion on the relationship between two results obtained by De Blasi and Myjak, and Reich and Zaslavski.
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1. Introduction

A well known Banach fixed point theorem states that every Banach contraction on a complete metric space has a unique fixed point, and every sequence of iterates of the mapping converges to this fixed point. On the other hand, there are nonexpansive selfmappings of complete metric spaces, which do not have a fixed point. Hence the following question arises:
How big (in the space of all nonexpansive mappings) is the set of all mappings which satisfy the thesis of the Banach fixed point theorem?
In fact, we can formulate many similar questions since we can replace nonexpansiveness by other nonexpansive–type conditions, and the thesis of the Banach fixed point theorem, with some other fixed point properties.
The answers to these questions have been established for the last 40 years, mainly by Benavides, Alvárez, De Blasi, Myjak, Reich and Zaslavski ([3], [4], [10], [11], [13], [5]). It turns out that in general, the set of all mappings with some fixed point property is big in the space of all nonexpansive mappings in the sense that its complement is of the first category or even σ-lower porous. On the other hand, the set of all Banach contractions can be σ-lower porous in the space of all nonexpansive mappings.

This paper can be considered as a discussion on part of the results, and is organized as follows:
In the next section be present some notions of porosity, recall the definitions of the Hausdorff metric and the Kuratowski measure of noncompactness, and define some spaces of nonexpansive–type mappings and theirs particular subsets.
In Section 3 we recall some known results. Note that we formulate them in a bit different way than originally, in order to point out the similarities between them.
In Section 4 we give some further definitions and construct a space of generalized nonexpansive–type mappings and its particular subset. This section can be consider as a background for the next one.
In Section 5 we prove a general theorem (Theorem 5.3), and, as an application, we give some extensions of results mentioned in Section 3.
In Section 6 we show that the result of Reich and Zaslavskii [11], which is considered as an extension of the earlier result of De Blasi and Myjak [4], is in fact a consequence of it.
In Section 7 we study the size of the set of all nonexpansive IFS's, which generate a Hutchinson–Barnsley fractal. The main result is another application of Theorem 5.3

2. Notation and basic facts

At first we will present some notions of porosity. Let X be a metric space. In the following, $B(x,R)$ will stand for an open ball centered in x with a radius $R>0$.
We say that $M\subset X$ is *lower porous*, if

$$\forall_{x\in M}\ \exists_{\alpha>0}\ \exists_{R_0>0}\ \forall_{R\in(0,R_0)}\ \exists_{z\in X}\ B(z,\alpha R)\subset B(x,R)\backslash M, \tag{1}$$

If M is a countable union of lower porous sets, then we say that M is *σ-lower porous.*

Remark 2.1. In mentioned papers of Benavides, Alvárez, De Blasi, Myjak, Reich and Zaslavski, there was defined another notion of porosity. Namely, we say that $M\subset X$ is *porous*, if

$$\exists_{\alpha>0}\ \exists_{R_0>0}\ \forall_{x\in X}\ \forall_{R\in(0,R_0)}\ \exists_{z\in X}\ B(z,\alpha R)\subset B(x,R)\backslash M.$$

Additionally, we define *σ-porosity* in an obvious way.
This notion seems to be stronger than the lower porosity. However, by [16, Proposition 2.2], the following conditions are equivalent:

(i) M is σ-lower porous;
(ii) M is σ-porous.

Clearly, the σ-lower porosity implies meagerness, but the converse need not be true – in all "reasonable" complete metric spaces there are sets which are meager and are not σ-lower porous. Hence if we know that a particular set is not only meager but also σ-lower porous, then we know that it is even smaller. In fact, there are many notions of porosity – for more information we refer the reader to survey papers [15] and [16].

If X is a metric space, then $\mathbf{B}(X)$, $\mathbf{CB}(X)$ and $\mathbf{K}(X)$ will stand, respectively, for spaces of all bounded, nonempty bounded and closed, and nonempty and compact subsets of X. If X is a normed space, then we additionally consider the space $\mathbf{CK}(X)$ of all nonempty compact and convex subsets of X. We consider $\mathbf{CB}(X)$, $\mathbf{K}(X)$ and $\mathbf{CK}(X)$ as metric spaces endowed with the Hausdorff metric H:

$$H(A,B) := \max\left\{\sup_{x\in A}\operatorname{dist}(x,B), \sup_{y\in B}\operatorname{dist}(y,A)\right\}, \quad A,B\subset X;$$

It is well known that if X is complete, then $\mathbf{CB}(X)$, $\mathbf{K}(X)$ and $\mathbf{CK}(X)$ are also complete. Assume now that (X,d) and (Y,ρ) are metric spaces. Then we denote by $\mathbf{C}^b(X,Y)$ the set of all continuous bounded (i.e., of bounded images) functions from X to Y. We consider $\mathbf{C}^b(X,Y)$ as a metric space with the standard supremum metric h:

$$\forall_{f,g\in\mathbf{C}^b(X,Y)}\ \ h(f,g) := \sup\{\rho(f(x),g(x)) : x\in X\}.$$

It is well known that if Y is a complete metric space, then $(\mathbf{C}^b(X,Y),h)$ is also complete. Moreover, if Y is a normed space, then the metric h is induced by the standard supremum norm.

We define the space of all nonexpansive bounded mappings from X to Y by:

$$\mathbf{\Omega}(X,Y) := \{f\in\mathbf{C}^b(X,Y) : \forall_{x,y\in X}\ \rho(f(x),f(y))\le d(x,y)\}.$$

We consider $\mathbf{\Omega}(X,Y)$ as a metric subspace of $\mathbf{C}^b(X,Y)$. We also define the subsets of $\mathbf{\Omega}(X,Y)$:

$$\mathbf{kB}(X,Y) := \{f\in\mathbf{\Omega}(X,Y) : \exists_{k<1}\ \forall_{x,y\in X}\ \rho(f(x),f(y))\le kd(x,y)\}$$

and

$$\mathbf{Ps}(X,Y) := \bigcap_{c>0}\ \bigcup_{f\in\mathbf{kB}(X,Y)} B(f,(1-k_f)c),$$

where $k_f := \inf\{k>0 : \forall_{x,y\in X}\ \rho(f(x),f(y))\le kd(x,y)\}$.

Remark 2.2. Clearly, $\mathbf{\Omega}(X,X)$ is the space of all bounded nonexpansive mappings, and $\mathbf{kB}(X,X)$ is the set of all bounded Banach contractions.

If A is a subset of a metric space, then we define the *Kuratowski measure of noncompactness* of A by:

$$\psi(A) := \inf\{\epsilon > 0 : A \text{ can be covered by finitely many sets of diameter } \leq \epsilon\}.$$

We define the following space of continuous, bounded ψ-nonexpansive mappings by

$$\mathbf{\Omega}_{\psi}(X,Y) := \{f \in \mathbf{C}^b(X,Y) : \forall_{C \in \mathbf{B}(X)}\ \psi(f(C)) \leq \psi(C)\}. \tag{2}$$

We consider $\mathbf{\Omega}_{\psi}(X,Y)$ as a metric subspace of $\mathbf{C}^b(X,Y)$. We also define the subsets of $\mathbf{\Omega}_{\psi}(X,Y)$:

$$\mathbf{kB}_{\psi}(X,Y) := \{f \in \mathbf{\Omega}_{\psi}(X,Y) : \exists_{k<1}\ \forall_{C \in \mathbf{B}(X)}\ \psi(f(C)) \leq k\psi(C)\} \tag{3}$$

and

$$\mathbf{Ps}_{\psi}(X,Y) := \bigcap_{c>0} \bigcup_{f \in \mathbf{kB}_{\psi}(X,Y)} B(f,(1-k_f)c), \tag{4}$$

where $k_f := \inf\{k > 0 : \forall_{C \in \mathbf{B}(X)}\ \psi(f(C)) \leq k\psi(C)\}$.

Remark 2.3. It is easy to see that

$$\mathbf{Ps}(X,Y) = \bigcap_{n \in \mathbb{N}} \bigcup_{f \in \mathbf{kB}(X,Y)} B(f,(1-k_f)\epsilon_n)$$

and

$$\mathbf{Ps}_{\psi}(X,Y) = \bigcap_{n \in \mathbb{N}} \bigcup_{f \in \mathbf{kB}_{\psi}(X,Y)} B(f,(1-k_f)\epsilon_n),$$

where (ϵ_n) is any sequence of positive reals which converges to 0.

3. Known results

We will recall the results of Benavides, Alvárez, De Blasi, Myjak, Reich and Zaslavski.
We say that for a selfmapping f of a metric space (X,d), the fixed point problem is *well posed*, if f has a unique fixed point, and every sequence of approximated fixed points (i.e., a sequence (x_n) for which $\lim_{n\to\infty} d(f(x_n),x_n) = 0$) converges to this fixed point.
Let K be a nonempty bounded convex and closed subset of a Banach space. De Blasi and Myjak proved that the set of all mappings from $\mathbf{\Omega}(K,K)$ which satisfy the thesis of the Banach fixed point theorem, forms the complement of a σ-lower porous set [4, Theorem 4], and so does the set of all mappings from $\mathbf{\Omega}(K,K)$, for which the fixed point problem is well posed [4, Theorem 8]. In fact, if we take a closer look at their proofs, we will see that they proved something more:

Theorem 3.1. *If K is nonempty bounded convex and closed subset of a Banach space, then the set $\mathbf{\Omega}(K,K) \setminus \mathbf{Ps}(K,K)$ is σ-lower porous in $\mathbf{\Omega}(K,K)$, and for every $f \in \mathbf{Ps}(K,K)$, the thesis of the Banach fixed point theorem is satisfied and the fixed point problem is well posed.*

We say that for a selfmapping f of a metric space X, the fixed point problem is *weakly well posed*, if the set of fixed points of f is nonempty and compact, and every sequence of approximated fixed points is precompact.
Now let K be as above. De Blasi and Myjak [4, Theorem 13] proved that the set of mappings from $\mathbf{\Omega}_\psi(K,K)$, for which the fixed point problem is weakly well posed, forms the complement of σ-lower porous subset of $\mathbf{\Omega}_\psi(K,K)$. In fact, they proved something more:

Theorem 3.2. *If K is a nonempty bounded convex and closed subset of a Banach space, then the set $\mathbf{\Omega}_\psi(K,K) \setminus \mathbf{Ps}_\psi(K,K)$ is σ-lower porous in $\mathbf{\Omega}_\psi(K,K)$, and for every $f \in \mathbf{Ps}_\psi(K,K)$, the fixed point problem is weakly well posed.*

If X, Y are metric spaces, then we say that $f : X \to Y$ is *ψ-condensing*, if for every nonprecompact set $C \in \mathbf{B}(X)$, we have $\psi(f(C)) < \psi(C)$. Now let X be a Banach space and $K \subset X$ be nonempty closed bounded and convex, and $A \subset X$ be nonempty closed and convex. Benavides and Alvárez [3, Theorem 1] showed that the set of all mappings from $\mathbf{\Omega}_\psi(K,A)$ which are not condensing, is a σ-lower porous subset of $\mathbf{\Omega}_\psi(K,A)$. In fact, they proved something more:

Theorem 3.3. *Let K be a nonempty closed bounded and convex subset of a Banach space X, and A be a nonempty closed and convex subset of X. The set $\mathbf{\Omega}_\psi(K,A) \setminus \mathbf{Ps}_\psi(K,A)$ is σ-lower porous in $\mathbf{\Omega}_\psi(K,A)$, and every $f \in \mathbf{Ps}_\psi(K,A)$ is condensing.*

Now let us switch the attention to set valued mappings. Recall that we denote by $\mathbf{P}(X)$ the family of all subsets of a set X. If $f : X \to \mathbf{P}(X)$, then we say that $x \in X$ is a *fixed point* of f, if $x \in f(x)$. Now let S be a closed and bounded subset of a Banach space, which is additionally *star shaped*, i.e., $st(S) := \{x \in S : \forall_{y \in S}\ [x,y] \subset S\} \neq \emptyset$. De Blasi, Myjak, Reich and Zaslavski [5, Theorems 2.1 and 2.2] proved that the set of all mappings from $\mathbf{\Omega}(S,\mathbf{K}(S))$ [from $\mathbf{\Omega}(S,\mathbf{CK}(S))$] which have nonempty and compact subset of fixed points is a generic subset of $\mathbf{\Omega}(S,\mathbf{K}(S))$ [of $\mathbf{\Omega}(S,\mathbf{CK}(S))$]. In fact, they proved something more:

Theorem 3.4. *Assume that S is a closed bounded and star shaped subset of a Banach space. Sets $\mathbf{Ps}(S,\mathbf{K}(S))$ and $\mathbf{Ps}(S,\mathbf{CK}(S))$ are generic subsets of $\mathbf{\Omega}(S,\mathbf{K}(S))$ and $\mathbf{\Omega}(S,\mathbf{CK}(S))$, respectively. Moreover, if $f \in \mathbf{Ps}(S,\mathbf{K}(S))$ (and, in particular, $f \in \mathbf{Ps}(S,\mathbf{CK}(S))$), then f has a nonempty and compact set of fixed points.*

As we can see, on one hand, all presented results are different, because they deal with different spaces, but on the other hand, they share some similarities. In the following, we will formulate and prove a general result which implies a slight strengthening of them.

4. Further notation

At first, we will present the definition of a hyperbolic space, and introduce the definition of a semi-hyperbolic space. Put $\mathbb{R}_+ := [0, \infty)$ and let (X, d) be a metric space. The functions $c : \mathbb{R} \to X$ and $c_+ : \mathbb{R}_+ \to X$ are *metric embeddings*, if

$$d(c(t), c(s)) = |t - s| \text{ for every } t, s \in \mathbb{R}$$

and

$$d(c_+(t), c_+(s)) = |t - s| \text{ for every } t, s \in \mathbb{R}_+.$$

The image $c(\mathbb{R})$ is called a *metric line*, and the image $c_+(\mathbb{R}_+)$ is called a *metric half-line.*

The images of a closed interval $[a, b]$ under c or c_+ is called a *metric segment* (and denoted by $[c(a), c(b)]$ or $[c_+(a), c_+(b)]$).

Now let (X, d) be a metric space and let L be a family of metric lines or half lines such that for each $x, y \in X$, $x \neq y$, there is at most one $l \in L$ such that $x, y \in l$. From now on we will consider only metric segments which are subsets of elements of L.

If $[x, y]$ is a metric segment and $t \in [0, 1]$, then there exists a unique $z \in [x, y]$, such that $d(x, z) = td(x, y)$ (and, consequently, $d(z, y) = (1 - t)d(x, y)$). Then we write $z = (1 - t)x \oplus ty$.

If $A \subset X$, $x \in X$ and $t > 0$ are such that for each $y \in A$, there is a metric segment joining x and y, then we put $(1 - t)x \oplus tA := \{(1 - t)x \oplus ty : y \in A\}$.

We first recall the definition of a hyperbolic space [12] (in fact, we give here an equivalent definition – cf. [12, p. 558]):

Definition 4.1. Assume that (X, d) is a metric space which contains a family L of metric lines such that for every $x, y \in X$, $x \neq y$, there exists a unique metric line in L which passes through x and y. If, moreover, for every x, z, w and every $t \in [0, 1]$,

$$d((1 - t)x \oplus tz, (1 - t)x \oplus tw) \leq td(z, w),$$

then we say that X is a *hyperbolic space.* If $M \subset X$ is such that for any $x, y \in M$, $[x, y] \subset M$, then we say that M is *h-convex.*

Now we will introduce the notion of a semi-hyperbolic space:

Definition 4.2. Assume that (X, d) is a metric space which contains a family L^+ of metric half-lines in X such that for each $x, y \in X$, $x \neq y$, there is at most one metric half line in L^+ which passes through x and y. Assume that

$$St_{sh}(X) := \left\{ x \in X : \forall_{y \in X} \ \exists_{l_+ \in L^+} \ x, y \in l_+ \right\} \neq \emptyset.$$

If for every $x \in St_{sh}(X)$, $z, w \in X$ and every $t \in [0, 1]$,

$$d((1-t)x \oplus tz, (1-t)x \oplus tw) \leq td(z, w),$$

then we say that X is *semi-hyperbolic.* If $M \subset X$ is such that

$$st_{sh}(M) := \{x \in M \cap St_{sh}(X) : \forall_{y \in M} \ [x, y] \subset M\} \neq \emptyset,$$

then we say that M is *sh-star shaped.* We call any $x_0 \in st_{sh}(M)$ a *star center of* M.

Remark 4.3. Since the definitions of a hyperbolic space and a semi-hyperbolic space involve the families L and L^+, respectively, we should define them as a triples (X, d, L) and (X, d, L^+), respectively. However, our simplification will not lead to any confusions.

Proposition 4.4. *The following statements holds:*

(i) *every hyperbolic space is semi hyperbolic;*
(ii) *every nonempty h-convex subset of a hyperbolic space is sh-star shaped.*

Proof. We will only sketch the proof of (i). Let L be the family of metric lines which appears in the definition of hyperbolic space, and let $x_0 \in X$. Let L_{x_0} be the family of all lines in L, which passes through x_0. It is obvious that we can assume that each $l \in L_{x_0}$ is an image of metric embedding $c_l : \mathbb{R} \to X$ such that $c_l(0) = x_0$. Define the family

$$L^+_{x_0} := \{c_l([0, \infty)) : l \in L_{x_0}\} \cup \{c_l((-\infty, 0]) : l \in L_{x_0}\}.$$

It can be easily seen that the family $L^+_{x_0}$ has all needed properties. Hence we have (i). The proof of (ii), in view of the above construction, is obvious. □

Remark 4.5. By the above result, every hyperbolic space is semi-hyperbolic. In particular, normed spaces are semi-hyperbolic, and star shaped subsets of normed spaces are *sh*-star shaped. The reason for defining semi-hyperbolic spaces and its *sh*-star shaped subsets comes from the fact that if X is a normed space, then $\mathbf{CB}(X)$ and $\mathbf{K}(X)$ are semi-hyperbolic and need not be hyperbolic, as will be shown in the sequel. Moreover, as will be recalled in the sequel, there are generic results concerning hyperbolic spaces. Since we wanted to generalize also them, we needed a definition which "gathers" hyperbolic spaces and $\mathbf{CB}(X)$ spaces.

Proposition 4.6. *Let X be a normed linear space and S be a star shaped subset of X. The following statements hold:*

(i) *the spaces $\mathbf{CB}(X)$, $\mathbf{K}(X)$ and $\mathbf{CK}(X)$ (considered as metric spaces with Hausdorff metric H) are semi-hyperbolic.*

(ii) *the sets $\mathbf{CK}(S)$ and $\mathbf{CK}(S)$ are sh-star shaped subsets of $\mathbf{CB}(X)$.*

Proof. Directly from the definition of a Hausdorff metric we get that for every $x \in X$, $A \in \mathbf{CB}(X)$,

$$H(A, \{x\}) = \sup\{\| y - x \|: y \in A\},$$

so

$$\forall_{x\in X}\ \forall_{A\in\mathbf{CB}(X)}\ \forall_{t\geq 0}\ H((1-t)x + tA, \{x\}) = tH(A, \{x\}). \tag{5}$$

Now observe that

(a) $\forall_{x\in X}\ \forall_{A\in\mathbf{CB}(X)}\ \forall_{t_1,t_2\geq 0}\ H((1-t_1)x + t_1A, (1-t_2)x + t_2A) = |t_1 - t_2| H(A, \{x\});$

(b) $\forall_{x\in X}\ \forall_{A,B\in\mathbf{CB}(X)}\ \forall_{t\in[0,1]}\ H((1-t)x + tA, (1-t)x + tB) = tH(A, B);$

(c) $\forall_{x\in X}\ \forall_{A\subset X}\ \forall_{t\in\mathbb{R}}\ (A \in \mathbf{CB}(X)$ iff $(1-t)x + tA \in \mathbf{CB}(X))$ and $(A \in \mathbf{K}(X)$ iff $(1-t)x+tA \in \mathbf{K}(X))$ and $(A \in \mathbf{CK}(X)$ iff $(1-t)x+tA \in \mathbf{CK}(X))$.

We first show (a). Take any $x \in X$, $A \in \mathbf{CB}(X)$ and $t_1, t_2 \geq 0$. Clearly, we can assume that $x = 0$. For every $y \in A$, we have:

$$\inf\{\| t_1y - t_2z \|: z \in A\} \leq |t_1 - t_2|\ \| y \|,$$

and

$$\inf\{\| t_2y - t_1z \|: z \in A\} \leq |t_1 - t_2|\ \| y \|,$$

so

$$H(t_1A, t_2A) \leq |t_1 - t_2| H(A, \{0\}). \tag{6}$$

Moreover, by (5), we get

$$\begin{aligned} t_1H(A, \{0\}) &\overset{(5)}{=} H(t_1A, \{0\}) \leq H(t_1A, t_2A) + H(t_2A, \{0\}) \\ &\overset{(5)}{=} H(t_1A, t_2A) + t_2H(A, \{0\}). \end{aligned}$$

Hence

$$H(t_1A, t_2A) \geq (t_1 - t_2)H(A, \{0\}).$$

In the same way we get

$$H(t_1A, t_2A) \geq (t_2 - t_1)H(A, \{0\}),$$

so finally,

$$H(t_1A, t_2A) \geq |t_2 - t_1| H(A, \{0\}). \tag{7}$$

Now (6) and (7) imply

$$H(t_1A, t_2A) = |t_2 - t_1|H(A, \{0\}),$$

which proves (a).
Now we show (b). Let $x \in X$, $A, B \in \mathbf{CB}(X)$ and $t \in [0, 1]$. Again, assume that $x = 0$. For every $y \in A$, we get

$$\inf\{\| ty - tz \|: z \in B\} = t \inf\{\| y - z \|: z \in B\},$$

and for every $y \in B$,

$$\inf\{\| ty - tz \|: z \in A\} = t \inf\{\| y - z \|: z \in A\},$$

which easily implies (b).
The fact (c) is trivial.
Now we show that $\mathbf{CB}(X)$ is a semi-hyperbolic space. Set $x_0 \in X$ and for each $A \in \mathbf{CB}(X)$, put $l^A_{x_0} := \{(1 - t)x_0 + tA : t \geq 0\}$. Define the family

$$L^+_{x_0} := \left\{l^A_{x_0} : A \in \mathbf{CB}(X),\ H(A, \{x_0\}) = 1\right\}.$$

By (a) and (c), each element of $L^+_{x_0}$ is a metric half line in $\mathbf{CB}(X)$. Now if $C \in \mathbf{CB}(X)$ and $C \in l^A_{x_0} \cap l^B_{x_0}$, where $H(A, \{x_0\}) = H(B, \{x_0\}) = 1$, then for some $t, s \geq 0$, we have $C = (1 - t)x_0 + tA$ and $C = (1 - s)x_0 + sB$. Hence and by (a), $H(C, \{x_0\}) = t$ and $H(C, \{x_0\}) = s$. Thus $t = s$, so if $C \neq \{x_0\}$, then $A = B$. This shows that the metric half lines in $L^+_{x_0}$ are equal or their intersection contains only $\{x_0\}$. In particular, there is at least one metric half line from $L^+_{x_0}$, which passes through any two fixed sets.
Now let $B \in \mathbf{CB}(X) \setminus \{\{x_0\}\}$ and set $t := \frac{1}{H(B,\{x_0\})}$. Then $B \in l^A_{x_0}$, where $A := (1 - t)x_0 + tB$. Since $H(A, \{x_0\}) = 1$, we get $\{x_0\} \in St(\mathbf{CB}(X))$.
Now assume that $A \in St(\mathbf{CB}(X))$ and $A \neq \{x_0\}$. Then, in particular, for every $B \in \mathbf{CB}(X)$ with $H(B, \{x_0\}) = 1$, there is $C \in \mathbf{CB}(X)$ with $H(C, \{x_0\}) = 1$ and $A, B \in l^C_{x_0}$. Since $H(B, \{x_0\}) = 1$, we get that $C = B$, so $A \in l^B_{x_0}$. Hence each metric half line in $L^+_{x_0}$ contains two established points (A and $\{x_0\}$). This is a contradiction. Hence $\{\{x_0\}\} = St(\mathbf{CB}(X))$. Finally, by (b), we have that $\mathbf{CB}(X)$ is semi hyperbolic. The proofs of the facts that $\mathbf{K}(X)$ and $\mathbf{CK}(X)$ are semi hyperbolic are similar. Now if S is a star shaped subset of X, then for every $A \subset S$, $x \in st(S)$ and $t \in [0, 1]$, we get that $(1 - t)x + tA \subset S$. Hence and by the above construction, we have that $\mathbf{CB}(S)$, $\mathbf{K}(S)$ and $\mathbf{KC}(S)$ are sh-star shaped subsets of $\mathbf{CB}(X)$. □

Now we show that spaces $\mathbf{CB}(X)$ and $\mathbf{K}(X)$ need not be hyperbolic.

Example 4.7. Let $X = \mathbb{R}^2$ and $\| \cdot \|$ be the Euclidean norm. Define $A = \{0\} \times [-1, 1]$, $F = \{2\} \times [-1, 1]$ and $C = \{(0, 1), (2, 1)\}$. Then $H(A, F) = H(A, C) =$

$H(F, D) = 2$. It is easy to see that if for some closed set D, $H(A, D) = H(D, F) = \frac{1}{2}H(A, F) = 1$, then $D = \{1\} \times [-1, 1]$, and if E is such that $H(A, E) = H(E, C) = \frac{1}{2}H(A, C) = 1$, then E must be contained in the set $E' = \overline{B}((0, 1), 1) \cup \overline{B}((2, 1), 1)$. Since $\operatorname{dist}((1, -1), E') > 1$, we have that $H(D, E) > 1 = \frac{1}{2}H(F, C)$. This shows that $\mathbf{CB}(X)$, $\mathbf{K}(X)$ are not hyperbolic.

Recall that by $\mathbf{B}(X)$ we denote the family of all bounded subsets of X.

Definition 4.8. Let X be a metric space and $\alpha_X : \mathbf{B}(X) \to [0, \infty)$.
We say that α_X is *monotonic*, if

$$\forall_{C,D\in\mathbf{B}(X)} \; (C \subset D \Rightarrow \alpha_X(C) \leq \alpha_X(D)).$$

We say that α_X is *Lipschitzian*, if

$$\exists_{L>0} \; \forall_{C\in\mathbf{B}(X)} \; \forall_{r>0} \; \alpha_X\left(\bigcup_{y\in C} B(y, r)\right) \leq \alpha_X(C) + Lr.$$

If, additionally, X is a *sh*-star shaped subset of a semi-hyperbolic space, then we say that α_X is *star-controlled*, if for any star center x_0 of X,

$$\forall_{C\in\mathbf{B}(X)} \; \forall_{t\in(0,1)} \; \alpha_X((1-t)x_0 \oplus tC) \leq t\alpha_X(C).$$

Remark 4.9. It is easy to see that if C is a bounded subset of a *sh*-star shaped subset of a semi-hyperbolic space, then for every $t \in [0, 1]$ and a star center x_0, the set $(1-t)x_0 \oplus tC$ is also bounded. This omits a possible problem with the definition of a star-controlled functional.

Remark 4.10. Clearly, the diameter, defined on bounded subsets of a *sh*-star shaped set, is monotonic, Lipschitzian and star-controlled.

Now we present the definition of an *abstract* measure of noncompactness [1, p. 160].

Definition 4.11. Let X be a normed space. We say that a functional $\phi : \mathbf{B}(X) \to [0, \infty)$ is a *measure of noncompactness*, if for every $M, N \in \mathbf{B}(X)$ and $\lambda \in \mathbb{R}$, the following condition are satisfied:

(1.1) $\phi(M \cup N) = \max\{\phi(M), \phi(N)\}$;
(1.2) $\phi(M + N) \leq \phi(M) + \phi(N)$;
(1.3) $\phi(\lambda M) = |\lambda|\phi(M)$;
(1.4) $\phi(M) \leq \phi(N)$ for $M \subset N$;
(1.5) $\phi([0, 1] \cdot M) = \phi(M)$;
(1.6) $\phi(\overline{\operatorname{conv}} M) = \phi(M)$;

(1.7) $\phi(M) = 0$ iff M is precompact.

Remark 4.12. Note that all "reasonable" measures of noncompactness (the Kuratowski measure of noncompactness, the Hausdorff measure of noncompactness, the lattice measure of noncompactness – [1, p. 161]) satisfy (1.1)–(1.7).

Remark 4.13. Any measure of noncompactness defined on a star shaped subset of a Banach space, is monotonic (by (1.4)), Lipschitzian (by (1.2), (1.3) and (1.4)), and star-controlled (by (1.2), (1.3) and (1.7)).

5. Result

Let X and Y be metric spaces, $\alpha_X : \mathbf{B}(X) \to [0, \infty)$ and $\alpha_Y : \mathbf{B}(Y) \to [0, \infty)$. We will define the space of continuous bounded functions which are, let us say, *(α_X-α_Y)*-nonexpansive:

$$\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X, Y) := \left\{ f \in \mathbf{C}^b(X, Y) : \forall_{C \in \mathbf{B}(X)}\ \alpha_Y(f(C)) \leq \alpha_X(C) \right\}.$$

As usually, we consider $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X, Y)$ as a metric subspace of $\mathbf{C}^b(X, Y)$.

Remark 5.1. It is easy to see that if α_Y is monotonic and Lipschitzian, then $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X, Y)$ is a closed subset of $\mathbf{C}^b(X, Y)$. Hence if Y is a complete *sh*-star shaped subset, then the space $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X, Y)$ is complete.

Now let us define the set of all generalized strict contractions:

$$\mathbf{kB}_{\alpha_X}^{\alpha_Y}(X, Y) := \{ f \in \mathbf{\Omega} : \exists_{k<1}\ \forall_{C \in \mathbf{B}(X)}\ \alpha_Y(f(C)) \leq k\alpha_X(C) \}.$$

If $f \in \mathbf{kB}_{\alpha_X}^{\alpha_Y}(X, Y)$, then by k_f we define

$$k_f := \inf\{ k : \forall_{C \in \mathbf{B}(X)}\ \alpha_Y(f(C)) \leq k\alpha_X(C) \}.$$

Clearly, for every $C \in \mathbf{B}(X)$, $\alpha_Y(f(C)) \leq k_f \alpha_X(C)$.
Finally, we define the following subset of $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X, Y)$:

$$\mathbf{Ps}_{\alpha_X}^{\alpha_Y}(X, Y) := \bigcap_{c>0} \bigcup_{f \in \mathbf{kB}_{\alpha_X}^{\alpha_Y}(X,Y)} B(f, (1 - k_f)c).$$

Remark 5.2. Assume that X and Y are metric spaces. If α_X and α_Y are the diameters. It is easy to observe that in this case (recall the notation from Section 2)

$$\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X, Y) = \mathbf{\Omega}(X, Y),$$
$$\mathbf{kB}_{\alpha_X}^{\alpha_Y}(X, Y) = \mathbf{kB}(X, Y),$$
$$\mathbf{Ps}_{\alpha_X}^{\alpha_Y}(X, Y) = \mathbf{Ps}(X, Y).$$

Assume that X and Y are subsets of the same normed space and ϕ is a measure of nonompactness. Define $\mathbf{\Omega}_\phi(X, Y)$, $\mathbf{kB}_\phi(X, Y)$ and $\mathbf{Ps}_\phi(X, Y)$ in a similar way as we defined them for Kuratowski measure of nonompactness (cf. (2), (3), (4)). If $\alpha_X := \phi_{|\mathbf{B}(X)}$ and $\alpha_Y := \phi_{|\mathbf{B}(Y)}$, then

$$\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X,Y) = \mathbf{\Omega}_{\phi}(X,Y),$$
$$\mathbf{kB}_{\alpha_X}^{\alpha_Y}(X,Y) = \mathbf{kB}_{\phi}(X,Y),$$
$$\mathbf{Ps}_{\alpha_X}^{\alpha_Y}(X,Y) = \mathbf{Ps}_{\phi}(X,Y).$$

The main result of this section is the following:

Theorem 5.3. *Assume that (X,d) is a metric space, (Y,ρ) is a sh-star shaped subspace of a semi-hyperbolic space, $\alpha_X : \mathbf{B}(X) \to [0,\infty)$ is monotonic and $\alpha_Y : \mathbf{B}(Y) \to [0,\infty)$ is monotonic and star-controlled. Then the set $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X,Y) \setminus \mathbf{Ps}_{\alpha_X}^{\alpha_Y}(X,Y)$ is a σ-lower porous subset of $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X,Y)$.*

The proof is a refinement of the proofs from the mentioned papers.

Proof. For simplicity, we will write $\mathbf{\Omega}$, $\mathbf{kB}$ and $\mathbf{Ps}$ instead of $\mathbf{\Omega}_{\alpha_X}^{\alpha_Y}(X,Y)$, $\mathbf{kB}_{\alpha_X}^{\alpha_Y}(X,Y)$ and $\mathbf{Ps}_{\alpha_X}^{\alpha_Y}(X,Y)$, respectively.

It is easy to see that $\mathbf{Ps} = \bigcap_{n\in\mathbb{N}} \bigcup_{g\in\mathbf{kB}} B\left(g,(1-k_g)\frac{1}{n}\right)$. Hence we only have to show that $\mathbf{\Omega} \setminus \bigcup_{g\in\mathbf{kB}} B\left(g,(1-k_g)\frac{1}{n}\right)$ is lower porous in $\mathbf{\Omega}$ for every $n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$ and $f \in \mathbf{\Omega}$. Since f is bounded, there exists $M < \infty$ such that

$$\sup_{x\in X} \rho(f(x), z_0) < M, \tag{8}$$

where z_0 is a star center of Y. Now let $\delta > 0$ be such that

$$n\delta < \frac{1-\delta}{M}, \tag{9}$$

and let $r_0 := \frac{M}{1-\delta}$. Take any $r \in (0, r_0)$. By (9), we can take t so that

$$nr\delta < 1-t < r\frac{1-\delta}{M}. \tag{10}$$

Since $r < r_0$, we also have that $1-t < 1$. Thus $t \in (0,1)$. Now let f^t be defined in the following way:

$$f^t(x) := (1-t)z_0 \oplus tf(x) \text{ for } x \in X.$$

By Remark 4.9 and a fact that $f^t(X) = (1-t)z_0 \oplus tf(X)$, we get that $f^t(X)$ is bounded. Now let us note that f^t is continuous. Indeed, let $x \in X$ and (x_n) be a sequence with $x_n \to x$. Then

$$\rho\left(f^t(x_n), f^t(x)\right) \le t\rho\left(f(x_n), f(x)\right).$$

Hence and by the continuity of f, we get that $f^t(x_n) \to f^t(x)$, so f^t is continuous. Now by (8) and (10), for every $x \in X$, we have

$$\rho(f^t(x), f(x)) = (1-t)\rho(f(x), z_0) \overset{(8)}{\le} (1-t)M \overset{(10)}{\le} (1-\delta)r,$$

so $B(f^t, \delta r) \subset B(f, r)$. Thus we only have to show that

$$B(f^t, \delta r) \subset \bigcup_{g \in kB} B\left(g, (1-k_g)\frac{1}{n}\right). \tag{11}$$

Note that $f^t \in \mathbf{kB}$ and $k_{f^t} \leq t$. Indeed, if $C \in \mathbf{B}(X)$, then by the fact that α_Y is star-controlled and $f^t(C) = (1-t)z_0 \oplus tf(C)$, we get

$$\alpha_Y\left(f^t(C)\right) \leq t\alpha_Y(f(C)) \leq t\alpha_X(C).$$

Hence and by (10), $\delta r \overset{(10)}{<} \frac{1}{n}(1-t) \leq (1-k_{f^t})\frac{1}{n}$. The above facts show (11) and the proof of the theorem is finished. □

Now we will show that Theorem 5.3 implies slight strengthenings of mentioned results. We will start by recalling the Sadovskiĭ's theorem in a very general form (in fact, there are also successful attempts to further generalizations – cf. [8]) . It seems to be a mathematical folklore, but since we did not find such formulation in the literature, we will sketch its proof. If X and Y are subsets of normed spaces and ϕ is a measure of noncompactness, then we say that $f : X \to Y$ is *ϕ-condensing*, if for every nonprecompact, bounded set $C \subset X$, $\phi(f(C)) < \phi(C)$.

Proposition 5.4. *Assume that ϕ is a measure of noncompactness. Then for every ϕ-condensing selfmapping f of a nonempty closed convex bounded subset of a Banach space, the fixed point problem is weakly well posed.*

Proof. By the standard proof of Sadovskiĭ [14], we get that the set of all fixed points of f, namely $Fix(f)$, is nonempty. Since f is continuous, $Fix(f)$ is closed. Now since f is ϕ-condensing and $f(Fix(f)) = Fix(f)$, we have that $Fix(f)$ is compact. Now let (x_n) be the sequence of approximated fixed points of f, i.e., $\lim_{n\to\infty} \| f(x_n) - x_n \| = 0$. Then by (1.1), (1.2), (1.3) and (1.4) (from Definition 4.11), we get $\phi(f(\{x_n : n \in \mathbb{N}\})) = \phi(\{x_n : n \in \mathbb{N}\})$. Hence $\{x_n : n \in \mathbb{N}\}$ is precompact. □

Theorem 5.5. *Let X be a Banach space and*

- *K, A be nonempty closed and convex subsets (not necessarily bounded) of X;*
- *S be a closed, starshaped subset (not necessarily bounded) of X;*
- *ϕ be any measure of noncompactness.*

Then the following statements hold:

(i) *the set $\mathbf{\Omega}(K, K) \setminus \mathbf{Ps}(K, K)$ is σ-lower porous in $\mathbf{\Omega}(K, K)$, and for every $f \in \mathbf{Ps}(K, K)$, the thesis of the Banach fixed point theorem is satisfied and the fixed point problem is well posed;*

(ii) *the set $\mathbf{\Omega}_\phi(K, K) \setminus \mathbf{Ps}_\phi(K, K)$ is σ-lower porous in $\mathbf{\Omega}_\phi(K, K)$, and for every $f \in \mathbf{Ps}_\phi(K, K)$ the fixed point problem is weakly well posed;*

(iii) *the set* $\mathbf{\Omega}_\phi(K,A)\setminus\mathbf{Ps}_\phi(K,A)$ *is* σ*-lower porous in* $\mathbf{\Omega}_\phi(K,A)$*, and every* $f\in\mathbf{Ps}_\phi(K,A)$ *is* ϕ*-condensing;*

(iv) *the set* $\mathbf{\Omega}(S,\mathbf{K}(S))\setminus\mathbf{Ps}(S,\mathbf{K}(S))$ *is* σ*-lower porous in* $\mathbf{\Omega}(S,\mathbf{K}(S))$*, and every* $f\in\mathbf{Ps}(S,\mathbf{K}(S))$ *has a nonempty and compact set of fixed points;*

(v) *the set* $\mathbf{\Omega}(S,\mathbf{CK}(S))\setminus\mathbf{Ps}(S,\mathbf{CK}(S))$ *is* σ*-lower porous in* $\mathbf{\Omega}(S,\mathbf{CK}(S))$*, and every* $f\in\mathbf{Ps}(S,\mathbf{CK}(S))$ *has a nonempty and compact set of fixed points.*

Proof. By Proposition 4.6, Remarks 4.10, 4.13, 5.2 and Theorem 5.3, the sets considered in (i)–(v) are σ-lower porous. Hence we only have to show the other part of thesis.

Now we show (i). Let $f\in\mathbf{Ps}(K,K)$. Then the set $K':=\overline{conv}(f(K)+B(0,1))\cap K$ is a nonempty convex closed and bounded subset of K. Set $f':=f_{|K'}$. Then $f'\in\mathbf{\Omega}(K',K')$. Let $c>0$. Clearly, we may assume that $c<1$. Since $f\in\mathbf{Ps}(K,K)$, there exists $g\in\mathbf{kB}(K,K)$ such that $f\in B(g,(1-k_g)c)$. In particular, for every $x\in K$, $\| f(x)-g(x)\|<(1-k_g)c<1$, so

$$g(K)\subset(f(K)+B(0,1))\cap K\subset K'. \tag{12}$$

Now let $g':=g_{|K'}$. By (12), $g':K'\to K'$. Hence $g'\in\mathbf{kB}(K',K')$ and $k_{g'}\le k_g$. Moreover,

$$\sup\{\| f(x)-g(x)\|: x\in K'\}\le\sup\{\| f(x)-g(x)\|: x\in K\}<(1-k_g)c\le(1-k_{g'})c.$$

Hence $f'\in\mathbf{Ps}(K',K')$. By Theorem 3.1, f' satisfies the thesis of Banach fixed point theorem, and so does f. Now let (x_n) be a sequence for which $\lim_{n\to\infty}\| f(x_n)-x_n\|=0$. Then there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$, $\| x_n-f(x_n)\|<1$. Hence

$$\{x_n : n\ge n_0\}\subset(f(K)+B(0,1))\cap K\subset K',$$

so again by Theorem 3.1, we get that (x_n) is convergent to the unique fixed point of f', so it is convergent to the fixed point of f. The proof of (i) is finished.

Now we prove part (iii). Let $f\in\mathbf{Ps}_\phi(K,A)$ and $C\in\mathbf{B}(K)$ be nonprecompact. Pick $c>0$ with $c\phi(B(0,1))<\phi(C)$. Then there exists $g\in\mathbf{kB}_\phi(K,X)$ such that $f\in B(g,(1-k_g)c)$. Hence

$$k_g+\frac{(1-k_g)c\phi(B(0,1))}{\phi(C)}<1 \tag{13}$$

and

$$f(C)\subset g(C)+B(0,(1-k_g)c). \tag{14}$$

By (13), (14) and the definition of measure of noncompactness, we get

$$\phi(f(C))\overset{(14)}{\le}\phi(g(C))+(1-k_g)c\phi(B(0,1))\le k_g\phi(C)+(1-k_g)c\phi(B(0,1))\overset{(13)}{<}\phi(C).$$

Hence f is condensing.
Part (ii) follows from (iii) (used for $A = K$) and a very similar argument to that from part (i). The difference is that we have to make use of Proposition 5.4 instead of Theorem 3.1.
Now we prove part (iv). Let $f \in \mathbf{Ps}(S, \mathbf{K}(S))$ and $x_0 \in st(S)$. Put

$$S' := \overline{\text{conv}}\left(\left(\bigcup f(S) + B(0,1)\right) \cup \{x_0\}\right) \cap S.$$

Now we show that S' is closed bounded and star shaped. Since $f(S)$ is bounded in $\mathbf{K}(S)$, the set $\bigcup f(S)$ is bounded. Hence

$$\overline{\text{conv}}\left(\left(\bigcup f(S) + B(0,1)\right) \cup \{x_0\}\right).$$

is closed bounded and convex. Now since x_0 is an element of the above set, and $x_0 \in st(S)$, we get that S' is star shaped. It is also easy to see that if $h(f,g) < 1$, then $g(S') \subset \mathbf{K}(S')$. Then we proceed as in the proof of (i), and by Theorem 3.4, we get that the set of fixed points of $f_{|S'}$ is nonempty and compact. Since the set of fixed points of $f_{|S'}$ is the same as the set of fixed points of f, the proof of (iv) is finished.
The proof of (v) is very similar to the proof of (iv). □

6. Final remarks and other applications

Define the following family of functions:

$$\mathbf{Ra} := \{\eta : [0,\infty) \to [0,1] : \eta \text{ is nonincreasing and } \eta(t) < 1 \text{ for } t > 0\}.$$

Now if (X,d) and (Y,ρ) are metric spaces, then we define the set:

$$\mathbf{kR}(X,Y) := \{f \in \mathbf{C}^b(X,Y) : \exists_{\eta \in \mathbf{Ra}} \, \forall_{x,y \in X} \ \rho(f(x), f(y)) \le \eta(d(x,y))d(x,y)\}.$$

Note that if $X = Y$, then elements of $\mathbf{kR}(X,Y)$ are called *Rakotch contractions.* The following fixed point theorem is known. The proof can be found in [7].

Proposition 6.1. *Let X be a complete metric space. For every $f \in \mathbf{kR}(X,X)$, the thesis of the Banach fixed point theorem is satisfied, and the fixed point problem is well posed.*

Reich and Zaslavski proved the following (part (i) is stated in [11, Theorem 2.2] and (ii) is stated in [12, Theorem 4.3]).

Theorem 6.2. *The following conditions hold:*

(i) *If K is a nonempty closed bounded and convex subset of a Banach space, then the set $\mathbf{\Omega}(K,K) \setminus \mathbf{kR}(K,K)$ is σ-lower porous subset of $\mathbf{\Omega}(K,K)$;*

(ii) *If C is a nonempty closed bounded and h-convex subset of a complete hyperbolic space, then the set $\mathbf{\Omega}(C,C) \setminus \mathbf{kR}(C,C)$ is σ-lower porous subset of $\mathbf{\Omega}(C,C)$.*

In view of Proposition 6.1, this result (in particular the part (i)) seems to be an extension of the mentioned results of De Blasi and Myjak (Theorem 3.1). However, as will be seen, $\mathbf{Ps}(K,K) \subset \mathbf{kR}(K,K)$, so the result of Reich and Zaslavski is not an extension of the result of De Blasi and Myjak. In fact, the inclusion $\mathbf{Ps}(K,K) \subset \mathbf{kR}(K,K)$ is a special case of a more general statement.
Assume that X, Y are metric spaces, $\alpha_X : \mathbf{B}(X) \to [0,\infty)$, $\alpha_Y : \mathbf{B}(Y) \to [0,\infty)$. Define

$$\mathbf{kR}^{\alpha_Y}_{\alpha_X}(X,Y) := \left\{f \in \mathbf{\Omega}^{\alpha_Y}_{\alpha_X}(X,Y) : \exists_{\eta\in\mathbf{Ra}}\ \forall_{C\in\mathbf{B}(X)}\ \alpha_Y(f(C)) \le \eta(\alpha_X(C))\alpha_X(C)\right\}.$$

Remark 6.3. Assume that α_X and α_Y are diameters. Obviously, $\mathbf{kR}^{\alpha_Y}_{\alpha_X}(X,Y) \subset \mathbf{kR}(X,Y)$. An easy argument shows that the inverse inclusion also holds. Since it is not important for our considerations, we skip the proof.

Proposition 6.4. *Assume that X and Y are metric spaces, $\alpha_X : \mathbf{B}(X) \to [0,\infty)$ is monotonic and $\alpha_Y : \mathbf{B}(Y) \to [0,\infty)$ is monotonic and Lipschitzian. Then $\mathbf{Ps}^{\alpha_Y}_{\alpha_X}(X,Y) \subset \mathbf{kR}^{\alpha_Y}_{\alpha_X}(X,Y)$.*

Proof. For simplicity, we write $\mathbf{Ps}$, $\mathbf{kB}$ and $\mathbf{kR}$ instead of $\mathbf{Ps}^{\alpha_Y}_{\alpha_X}(X,Y)$, $\mathbf{kB}^{\alpha_Y}_{\alpha_X}(X,Y)$ and $\mathbf{kR}^{\alpha_Y}_{\alpha_X}(X,Y)$, respectively.
It is easy to see that $\mathbf{kR} = \bigcap_{n\in\mathbb{N}} M_n$, where for every $n \in \mathbb{N}$,

$$M_n := \left\{f \in \mathbf{\Omega} : \exists_{\lambda\in(0,1)}\ \forall_{C\in\mathbf{B}(X)}\ \left(\alpha_X(C) \ge \frac{1}{n}\ \Rightarrow\ \alpha_Y(f(C)) \le \lambda\alpha_X(C)\right)\right\}.$$

Hence it is enough to show that for any $n \in \mathbb{N}$, $\mathbf{Ps} \subset M_n$. So fix $n \in \mathbb{N}$ and take any $f \in \mathbf{Ps}$. Since α_Y is Lipschitzian, there exists $L > 0$ such that

$$\forall_{C\in\mathbf{B}(Y)}\ \forall_{r>0}\ \alpha_Y\left(\bigcup_{y\in C} B(y,r)\right) \le \alpha_Y(C) + Lr.$$

Now take $c > 0$ so that $nLc < 1$. Since $f \in \mathbf{Ps}$, there exists $g \in \mathbf{kB}$ such that $d_s(f,g) < c(1-k_g)$. Set $\lambda := k_g + (1-k_g)nLc$. Clearly, $\lambda < 1$. Now take $C \in \mathbf{B}(X)$ so that $\alpha_X(C) \ge \frac{1}{n}$. Since $h(f,g) < (1-k_g)c$, we have that $f(C) \subset \bigcup_{y\in g(C)} B(y,(1-k_g)c)$. Thus and by our assumptions, we have

$$\alpha_Y(f(C)) \le \alpha_Y(g(C)) + Lc(1-k_g) \le k_g\alpha_X(C) + Lc(1-k_g) = \left(k_g + \frac{Lc(1-k_g)}{\alpha_X(C)}\right)\alpha_X(C) \le \lambda\alpha_X(C),$$

so $g \in M_n$ and the result follows. □

Problem 6.5. The question arises, whether the inverse inclusion holds (we were able to show this only for the case $X = Y = [0, 1]$). We state it as an open problem.

The above result together with Remark 6.3, gives us the following corollary:

Corollary 6.6. *Let X and Y be metric spaces. Then $\mathbf{Ps}(X, Y) \subset \mathbf{kR}(X, Y)$.*

Theorem 5.3, Proposition 6.1 and Corollary 6.6, give us the following extension of Theorem 5.5 (i) (and extension of Theorem 6.2):

Theorem 6.7. *Let D be a closed sh-star shaped subset of a complete semi-hyperbolic space. The set $\mathbf{\Omega}(D, D) \setminus \mathbf{Ps}(D, D)$ is σ-lower porous in $\mathbf{\Omega}(D, D)$, and for every $f \in \mathbf{Ps}(D, D)$, the thesis of the Banach fixed point theorem is satisfied, and the fixed point problem is well posed.*

Now let K be a nonempty bounded closed and convex subset of a Banach space. Recall that $\mathbf{CB}(K)$ denotes the space of all nonempty closed and bounded subsets of K. By Proposition 4.6, $\mathbf{B}(K)$ is a *sh*-star shaped subset of a semi-hyperbolic space. Reich and Zaslavski [13] proved that $\mathbf{kR}(K, \mathbf{B}(K))$ is a generic subset of $\mathbf{\Omega}(K, \mathbf{B}(K))$, and that elements of $\mathbf{kR}(K, \mathbf{B}(K))$ have some interesting properties. Hence Theorem 5.3 and Corollary 6.6 imply the following slight extension of this result:

Corollary 6.8. *Let K be a nonempty bounded closed and convex subset of a Banach space. The set $\mathbf{\Omega}(K, \mathbf{B}(K)) \setminus \mathbf{Ps}(K, \mathbf{B}(K))$ is σ-lower porous.*

7. Generic existence of attractors of IFSs

If (X, d) is a metric space and $f_1, ..., f_n : X \to X$ are mappings which are "in some sense" contractive, then we call $S := (f_1, ..., f_n)$ an *iterated function system* (IFS in short). IFS generates a natural mapping $F_S : \mathbf{K}(X) \to \mathbf{K}(X)$ in the following way:

$$F_S(D) := f_1(D) \cup ... \cup f_n(D).$$

It turns out that the function F_S takes over many contractive conditions from the functions $f_1, ..., f_n$ (for example, if all $f_1, ..., f_n$ are Banach contractions, so is F_S; cf., also, Theorem 7.1). Hence in many cases, the function F_S has a fixed point. Such fixed points are called *attractors* or *Hutchinson–Barnsley fractals* (Hutchinson introduced the notion of IFS in [6] and Barnsley popularized it in [2]).

In this section we apply Theorem 5.3 and Corollary 6.6 to show that the set of all nonexpansive bounded IFSs S, for which F_S satisfy the thesis of the Banach fixed point theorem, is a complement of a σ-lower porous set.

We omit a standard proof of the following:

Theorem 7.1. *Assume that X is a complete metric space and $f_1, ..., f_n : X \to X$ are Rakotch contractions. If $S = (f_1, ..., f_n)$, then F_S is also a Rakotch contraction.*

Define the space of all nonexpansive IFSs:

$$\mathbf{\Omega}_n(X) := \{(f_1, ..., f_n) : f_i \in \mathbf{\Omega}(X, X)\ i = 1, ..., n\}.$$

Consider $\mathbf{\Omega}_n(X)$ as a metric space with a supremum metric. Then we define:

$$\mathbf{kR}_n(X) := \{(f_1, ..., f_n) \in \mathbf{\Omega}_n(X) : f_i \in \mathbf{kR}(X, X)\ i = 1, ..., n\}$$

Since each n-tuple of function $(f_1, ..., f_n)$ can be considered as a function from X to X^n, and each function $f : X \to X^n$ can be considered as an n-tuple $f = (f_1, ..., f_n)$, we can easily see that $\mathbf{\Omega}_n(X) = \mathbf{\Omega}(X, X^n)$ and $\mathbf{kR}_n(X) = \mathbf{kR}(X, X^n)$ (**in the Cartesian product** X^n **we consider the maximum metric**). Therefore, applying Theorem 5.3, Proposition 6.1, Corollary 6.6 and Theorem 7.1, we get:

Theorem 7.2. *Assume that X is a sh-star shaped subset of a semi hyperbolic space and $n \in \mathbb{N}$. Then the set $\mathbf{\Omega}_n(X) \setminus \mathbf{kR}_n(X)$ is a σ-lower porous subset of $\mathbf{\Omega}_n(X)$. In particular, the set of all n-tuples $(f_1, ..., f_n)$ from $\mathbf{\Omega}_n(X)$, such that the function $F_{\mathcal{S}}$ (for $\mathcal{S} := (f_i)_{i=1}^n$) satisfies the thesis of the Banach fixed point theorem, is a complement of a σ-lower subset of $\mathbf{\Omega}_n(X)$.*

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References

[1] J. Appell, *Measures of noncompactness, condensing operators and fixed points: an application-oriented survey*, Fixed Point Theory **6** (2005), no. 2, 157-229.

[2] M. F. Barnsley, *Fractals everywhere*. Academic Press Professional, Boston, MA, 1993.

[3] T. Domínguez Benavides, J. Carmona Alvárez, *Porosity and K-set contractions*, Boll. Un. Mat. Ital. A (7) **6** (1992), no. 2, 227-232.

[4] F. De Blasi, J. Myjak, *Sur la porosité de l'ensemble des contractions sans point fixe*, C. R. Acad. Sci. Paris, **308** (1989), 51-54.

[5] F. S. De Blasi, J. Myjak, S. Reich, A. Zaslavski, *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set Valued Anal. **17** (2009), no. 1, 97-112.

[6] J. Hutchinson, *Fractals and self-similarity*. Indiana Univ. Math. J. **30** (1981), no 5, 713-747.

[7] J. Jachymski, I. Jóźwik, *Nonlinear contractive conditions: a comparison and related problems*, Banach Center Publ., 77, Polish Acad. Sci., **77** (2007), 123-146.

[8] M. A. Khamsi, *Sadovskiĭ's fixed point theorem without convexity*, Nonlinear Anal. **53** (2003), 829-837.

[9] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459-465.

[10] S. Reich, A. Zaslavski, *Almost all nonexpansive mappings are contractive,* C. R. Math. Acad. Sci. Soc. R. Can. **22** (2000), no. 3, 118-124.

[11] S. Reich, A. Zaslavski, *The set of noncontractive mappings is σ-porous in the space of all nonexpansive mappings,* C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 6, 539-544.

[12] S. Reich, A. Zaslavski, *Generic aspects of metric fixed point theory*, Handbook of Metric Fixed Point Theory, 557-575, Kluwer Acad. Publ., Dordrecht, 2001.

[13] S. Reich, A. Zaslavski, *Convergence of iterates of nonexpansive set-valued mappings*, Set Valued Mappings with Applications in Nonlinear Analysis, Ser. Math. Anal. Appl. **4**, Taylor & Francis, London, 2002, 411-420.

[14] B. N. Sadovskiĭ, *On a fixed point principle*, (Russian) Funkcional. Anal. i Priložen. 1 (1967), no. **2**, 74-76.

[15] L. Zajíček, *Porosity and σ-porosity*, Real Anal. Exchange **13** (1987/88), no. 2, 314-350.

[16] L. Zajíček, *On σ-porous sets in abstract spaces*, Abstr. Appl. Anal. 2005, no. **5**, 509-534.

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