IV Conference

GEOMETRY AND TOPOLOGY OF MANIFOLDS

Krynica, 29 April – 4 May 2002

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FOREWORD

Thank you for coming to Krynica!

This is the conference of the cycle initiated in 1998 with a meeting in Konopnica (`http://im0.p.lodz.pl/konferencje`) and is organized as in 2001 in Krynica (Poland) from 29.04.2002 to 4.05.2002.

The main purpose of the conference is to present an overview of principal directions of research conducted in differential geometry, topology and analysis on manifolds and their applications, mainly (but not only) to Lie algebroids and related topics.

We would like to attract attention to:

- Riemannian, symplectic and Poisson manifolds,
- Lie groups, Lie groupoids, Lie algebroids and Lie-Rinehart algebras, Poisson algebras,
- foliations,
- characteristic classes.

The organizers of the conference are grateful to the following sponsors:

- Rector of the Technical University of Łódź,
- Rector of the Jagiellonian University,
- Rector of the Stanisław Staszic University of Mining and Metallurgy,
- State Committee for Scientific Research,
- Committee on Mathematics of the Polish Academy of Sciences.

We hope that all of you will enjoy your stay in Krynica. We wish you success with your debates. Have a good time!

Organizers

ORGANIZERS

IV International Conference
GEOMETRY AND TOPOLOGY OF MANIFOLDS

is organized by

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LIST OF PARTICIPANTS

1. **ALEKSEEVSKY, Dimitri**, Department of Mathematics, University of Hull, Cottingham Road, HU6 7RX, Hull, UNITED KINGDOM, e-mail: D.V.Alekseevsky@Hull.ac.uk

2. **BAJGUZ, Wiesław**, Institute of Computer Science, University of Białystok, ul. Sosnowa 64, PL–15–887 Białystok, POLAND, e-mail: wbajguz@ii.uwb.edu.pl

3. **BALCERZAK, Bogdan**, Institute of Mathematics, Technical University of Łódź, aleja Politechniki 11, 90–924 Łódź, POLAND, e-mail: bogdan@ck-sg.p.lodz.pl

4. **BELKO, Ivan**, Belarusian State Economics University, High Mathematics Department, 26 Partizansky avenue, 220070 Minsk, BELARUS, e-mail: niipulm@users.med.by

5. **BENAMEUR, Moulay**, Institut G. Desargues, Université Lyon 1, 43, Bd du 11-nov-1918, Bat. 101, 69622 Villeurbanne cedex, FRANCE, e-mail: benameur@desargues.univ–lyon1.fr, `http://www.desargues.univ-lyon1.fr/home/benameur/`

6. **BROWN, Ronald**, School of Informatics, Mathematics Division, University of Wales, Bangor, Dean St., Bangor, Gwynedd LL57 1UT, UNITED KINGDOM, e-mail: r.brown@bangor.ac.uk, `http://www.bangor.ac.uk/~mas010/`

7. **BRZYCHCZY, Stanisław**, Faculty of Applied Mathematics of the University of Mining and Metallurgy (AGH), Cracow, al. A. Mickiewicza 30, 30-059 Kraków, POLAND, e-mail: brzych@uci.agh.edu.pl

8. **BURSZTYN, Henrique**, MSRI, Berkeley / ULB, Brussels, USA, e-mail: henrique@math.berkeley.edu

9. **CA, Augusto**, Guinea–Bissau Institute of Mathematics, 26 Rua Vitorino costa, Bissau, GUINEA–BISSAU, e-mail: gomes_foundation@hotmail.com

10. **CARIÑENA, José F.**, Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, SPAIN, e-mail: jfc@posta.unizar.es

11. **CHIKE PRECIOUS AMAH,** College of Continuing Education, Benin Study Centre, Upper Lawani Quarters, P. O. Box 14517, Benin City Edo State, NIGERIA

12. **COLLETTI NEGREIROS, Caio José**, University of Campinas (Unicamp), BRASIL, e-mail: caione@ime.unicamp.br

13. **CORREIA, Patrico**, Guinea–Bissau Institute of Mathematics, 26 Rua Vitorino costa, Bissau, GUINEA–BISSAU, e-mail: gomes_foundation@hotmail.com

14. **DESZCZ, Ryszard**, Department of Mathematics, Agricultural University of Wrocław, ul. Grunwaldzka 53, 50–357 Wrocław, POLAND, e-mail: rysz@ozi.ar.wroc.pl

15. **DOLIWA, Adam**, Institute of Theoretical Physics, Warsaw University, 00–681 Warszawa, ul. Hoża 69, POLAND, e-mail: doliwa@fuw.edu.pl, `http://www.fuw.edu.pl/~doliwa`

16. **DOMITRZ, Wojciech**, Faculty of Mathematics and Information Science, Warsaw University of Techology, plac Politechniki 1, 00–661 Warszawa, POLAND, e-mail: wdomitrz@ise.pw.edu.pl, domitrz@mini.pw.edu.pl

17. **EHIGATOR TINO PRIASE,** College of Continuing Education, Benin Study Centre, Upper Lawani Quarters, P. O. Box 14517, Benin City Edo State, NIGERIA

18. **EWERT–KRZEMIENIEWSKI, Stanisław**, Institute of Mathematics, Technical University of Szczecin, al. Piastów 17, 70-310 Szczecin, POLAND, e-mail: ewert@arcadia.tuniv.szczecin.pl

19. **FEDORCHUK, Vasyl**, Institute of Mathematics, Pedagogical Academy, PL – 30–084 Kraków, ul. Podchorążych 2, Poland, Pidstryhach Institute of Applied Problems of Mechanics and Mathematics, National Ukrainian Academy of Sciences, Naukowa 3b, 79-053, L'viv, UKRAINE, e-mail: vas_fedorchuk@yahoo.com

20. **FERNANDES, Rui Loja**, Departamento de Matematica, Instituto Superior Tecnico, 1049-001 Lisbon, PORTUGAL, e-mail: rfern@math.ist.utl.pt, `http://www.math.ist.utl.pt/~rfern`

21. **GER, Roman**, Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, POLAND, `http://www.imi.wsp.czest.pl/pracownicy/ger.html`, e-mail: romanger@us.edu.pl

22. **GLANC, Barbara**, Institute of Mathematics, Technical University of Szczecin, aleja Piastów 17, 70-310 Szczecin, POLAND, e-mail: glanc@arcadia.tuniv.szczecin.pl

23. **GLAZEBROOK, James**, Eastern Illinois University/University of Illinois at Urbana-Champaign, e-mail: cfjfg@eiu.edu, glazebro@math.uiuc.edu

24. **GŁOGOWSKA, Małgorzata**, Department of Mathematics, Agricultural University of Wrocław, ul. Grunwaldzka 53, 50–357 Wrocław, POLAND, e-mail: mglog@ozi.ar.wroc.pl

25. **GRABOWSKI, Janusz**, Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, POLAND, e-mail: jagrab@mimuw.edu.pl

26. **HAJDUK, Bogusław**, Mathematical Institute, University of Wrocław, plac Grunwaldzki 2/4, 50–384 Wrocław, POLAND, e-mail: hajduk@math.uni.wroc.pl

27. **HALL, Graham**, Department of Mathematical Sciences, Meston Building, University of Aberdeen, Aberdeen AB24 3UE, Scotland, UNITED KINGDOM, e-mail: g.hall@maths.abdn.ac.uk

28. **HOTLOŚ, Marian**, Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, POLAND, e-mail: hotlos@im.pwr.wroc.pl

29. **HUEBSCHMANN, Johannes**, Université des Sciences et Technologies de Lille, UFR de Mathématiques, F–59 655 Villeneuve d'Ascq Cédex, FRANCE, e-mail: Johannes.Huebschmann@agat.univ-lille1.fr

30. **KELECHI, Obi**, Department of Mathematics, University of Lagos, Akoka, Cio Jide Joshua, 18 Okunola Street, Egbeda, Lagos State, NIGERIA, e-mail: batlimited@yahoo.com

31. **KLEIN, Dariusz**, Zakad Algebry, Pomorska Akademia Pedagogiczna, ul. Arciszewskiego 22 b, 76–200 Słupsk, POLAND, e-mail: klein@wsp.slupsk.pl

32. **KOCK, Anders**, Department of Mathematical Science, University of Aarhus, DK 8000 Aarhus C, DENMARK, e-mail: kock@imf.au.dk, `http://home.imf.au.dk/kock/`

33. **KONDERAK, Jerzy**, Dipartimento di Matematica, Università di Bari, Via Orabona 4, 70125 Bari, ITALY, e-mail: konderak@dm.uniba.it

34. **KORBAŠ, Július**, Faculty of Mathematics and Physics, Comenius University, Mlynská dolina, SK–842 48 Bratislava 4, SLOVAKIA, e-mail: korbas@fmph.uniba.sk

35. **KOWALSKI, Oldřich**, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, CZECH REPUBLIC, e-mail: kowalski@karlin.mff.cuni.cz, `http://www.karlin.mff.cuni.cz/~kowalski/`

36. **KROT, Ewa**, Institute of Computer Science, Białystok University, PL–15–887 Białystok, ul.Sosnowa 64, POLAND, e-mail: ewakrot@wp.pl

37. **KUBARSKI, Jan**, Institute of Mathematics, Technical University of Łódź, al. Politechniki 11, 90–924 Łódź, POLAND, e-mail: kubarski@ck-sg.p.lodz.pl, `http://im0.p.lodz.pl/~kubarski`

38. **KWAŚNIEWSKI, Andrzej Krzysztof**, Institute of Computer Science, University of Białystok, ul. Sosnowa 64, PL–15–887 Białystok, POLAND, e-mail: kwandr@uwb.edu.pl

39. **LECOMTE, Pierre**, Departement of Mathematics of the University of Liege, Grande Traverse 12 (B37), B-4000 Liege (Sart Tilman), BELGIUM, e-mail: plecomte@ulg.ac.be, `http://www.forum.math.ulg.ac.be`, `http://www.ulg.ac.be/geothalg`

40. **LISIECKI, Krzysztof**, Institute of Mathematics, Technical University of Łódź, aleja Politechniki 11, 90–924 Łódź, POLAND, e-mail: lisiecki@ck-sg.p.lodz.pl

41. **LUCZYSZYN, Dorota**, Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, POLAND, e-mail: luczysz@im.pwr.wroc.pl

42. **MIKULSKI, Włodzimierz**, Institute of Mathematics, Jagiellonian University, ul. W. Reymonta 4, 30–059 Kraków, POLAND, e-mail: mikulski@im.uj.edu.pl

43. **MISHCHENKO, Alexandr**, Moscow State University, Department of Mathematics, 119 992 GSP–2 Moscow, RUSSIA, `http://higeom.math.msu.su/asmish`, e-mail: asmish@higeom.math.msu.su

44. **MORMUL, Piotr**, Institute of Mathematics, Warsaw University, Banacha 2, 02–097 Warszawa, POLAND, e-mail: mormul@mimuw.edu.pl

45. **MYKYTYUK, Ihor**, Applied Mathematics Department, State University "Lviv Polytechnica", S. Bandera St. 12, 79013, Lvi'v, UKRAINE, e-mail: viva@iapmm.lviv.ua

46. **NAGY, Péter Tibor**, Debrecen University, Institute of Mathematics and Informatics, Department of Geometry, H-4010 Debrecen, P.O.B. 12, HUNGARY, e-mail: nagypeti@math.klte.hu

47. **NGUIFFO BOYOM, Michel**, Departement de Mathématiques, Université Montpellier2, FRANCE, e-mail: boyom@math.univ-montp2.fr

48. **NIKITIN, Anatoly**, Department of Applied Research, Institute of Mathematics of Nat. Acad. Sci. of Ukraine, 3 Tereshchenkivska St, 01601 Kiev–4, UKRAINE, e-mail: nikitin@imath.kiev.ua, `http://www.imath.kiev.ua/~nikitin/`

49. **NUNES DA COSTA, Joana Margarida**, Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001–454 Coimbra, PORTUGAL, e-mail: jmcosta@mat.uc.pt

50. **NYONG, Godwin**, GSA LIMITED, 12 Onasanya Street off Ishaga Road Surulere, Lagos State, NIGERIA, e-mail: gsalimitedng@hotmail.com

51. **OLSZAK, Zbigniew**, Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, POLAND, e-mail: olszak@im.pwr.wroc.pl

52. **PIĄTKOWSKI, Andrzej**, Institute of Mathematics, Technical University of Łódź, aleja Politechniki 11, 90–924 Łódź, POLAND, e-mail: andpiat@ck-sg.p.lodz.pl

53. **PLACHTA, Leonid**,Institute of Applied Problems of Mechanics and Mathematics of NAS of Ukraine, UKRAINE, e-mail: igor@iapmm.lviv.ua

54. **POPESCU, Paul**, Department of Algebra and Geometry, University of Craiova, 13, 'Al.I.Cuza' st., Craiova, 1100, ROMANIA, e-mail: paulpopescu@email.com

55. **PRADINES, Jean**, Toulouse, FRANCE, e-mail: jpradines@wanadoo.fr, pradines@picard.ups-tlse.fr

56. **PRYKARPATSKY, Anatolij K.**, Deptartment of Applied Mathematics at the AGH, al. Mickiewicza 30, bl. A4, 30–059 Kraków, POLAND, Deptartment of Nonlinear Math. Analysis at the Institute APMM of the Nat. Acad. of Sciences, Lviv, 79601 UKRAINE e-mail: prykanat@cybergal.com

57. **RYBICKI, Tomasz**, Faculty of Applied Mathematics of the University of Mining and Metallurgy (AGH), Cracow, al. A. Mickiewicza 30, 30–059 Kraków, POLAND, e-mail: tomasz@uci.agh.edu.pl

58. **SAKALOS, Stefan**, Faculty of Mathematics and Physics, Comenius University, Mlynská dolina, SK–842 48 Bratislava 4, SLOVAKIA, e-mail: sakalos@fmph.uniba.sk

59. **SAWICZ, Katarzyna**, Politechnika Częstochowska, Wydział Zarządzania, Instytut Ekonometrii i Informatyki, ul. Armii Krajowej 19B, 42–200 Częstochowa, POLAND, e-mail: k_sawicz@poczta.onet.pl

60. **SCÁRDUA**, **Bruno,** Institute of Mathematics, Federal University of Rio de Janeiro, Caixa Postal 68530, Rio de Janeiro – RJ, CEP. 21945-970, BRASIL, e-mail: scardua@im.ufrj.br

61. **SŁUKA, Karina**, Institute of Mathematics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, POLAND, e-mail: sluka@mazur.im.pwr.wroc.pl

62. **SOBOLEWSKA, Kamila**, Zakad Algebry, Pomorska Akademia Pedagogiczna, ul. Arc6iszewskiego 22 b, 76–200 Słupsk, POLAND, e-mail: Sobolewska@wsp.slupsk.pl

63. **SPIRO, Andrea**, Dipartimento di Matematica, Università di Camerino, Via Madonna delle Carceri, I-62032 Camerino (Macerata), ITALY, e-mail: andrea.spiro@unicam.it

64. **STĘPIEŃ, Zofia**, Institute of Mathematics, Technical University of Szczecin, al. Piastów 17, 70-310 Szczecin, POLAND, e-mail: stepien@arcadia.tuniv.szczecin.pl

65. **SZENTHE, János**, Department of Geometry, Eötvös University, 1518 Budapest, Pf. 120, HUNGARY, e-mail: szenthe@ludens.elte.hu

66. **TOHAN CHRIS OKENYI,** College of Continuing Education, Benin Study Centre, Upper Lawani Quarters, P. O. Box 14517, Benin City Edo State, NIGERIA

67. **TOMAN, Henrietta**, Debrecen University, Institute of Mathematics and Informatics, H-4010 Debrecen, P.O.B. 12, HUNGARY, e-mail: toman@math.klte.hu

68. **TULCZYJEW, Włodzimierz**, Camerino, ITALY, e-mail: tulczy@camserv.unicam.it

69. **URBAŃSKI, Paweł**, Faculty of Physics, University of Warsaw, Hoża 74, 00–682 Warszawa, POLAND, e-mail: urbanski@fuw.edu.pl

70. **VASSILIOU, Efstathios**, Department of Mathematics, University of Athens, Panepistimiopolis, Athens 157 84, GREECE, e-mail: evassil@cc.uoa.gr

71. **WALAS, Witold**, Institute of Mathematics, Technical University of Łódź, aleja Politechniki 11, 90–924 Łódź, POLAND, e-mail: walwit@ck-sg.p.lodz.pl

72. **WALISZEWSKI, Włodzimierz**, Faculty of Mathematics, University of Łódź, Banacha 22, 90–238 Łódź, POLAND, e-mail: faculty@math.uni.lodz.pl

73. **WEBER, Andrzej**, Institute of Mathematics, Warsaw University, ul. Banacha 2, 02–097 Warszawa, POLAND, e-mail: aweber@mimuw.edu.pl, `http://www.mimuw.edu.pl/~aweber`

74. **WOLAK, Robert**, Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, 30–059 Kraków, POLAND, e-mail: wolak@im.uj.edu.pl

75. **ZAJTZ, Andrzej**, Institute of Mathematics, Pedagogical Academy of Cracow, ul. Podchorążych 2, 30–084 Kraków, POLAND, e-mail: smzajtz@cyf-kr.edu.pl

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ABSTRACTS

QUANTUM HOLONOMIC COMPUTING VIA LAX TYPE FLOWS ON GRASSMANNIAN MANIFOLDS AND DUAL MOMENTUM MAPPINGS

D. L. BLACKMORE, ANATOLIJ PRYKARPATSKY, UFUK TANERI

Abstract

A general approach to holonomic quantum computing via Lax–type flows on Grassmann manifolds, based on the momentum mapping reduction and connection techniques is developed. It is shown that the associated holonomy groups can be effectively used in quantum computations of diverse practical problems.

Contents:

References

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D. L. BLACKMORE
Deptartment of Mathematical Sciences at the NJIT
Newark NJ 07102
USA

Anatolij PRYKARPATSKY
Deptartment of Applied Mathematics
at the Academy of Mining and Metallurgy
al. Mickiewicza 30, bl. A4, 30–059 Kraków
POLAND, e-mail: prykanat@cybergal.com
and
Deptartment of Nonlinear Math. Analysis
at the Institute APMM of the Nat. Acad. of Sciences
Lviv,79601 UKRAINE

Ufuk TANERI
Deptartment of Mathematics at the Eastern Mediterranean University
Famagusta, via Mersin 10
TURKEY

RIEMANN–HILBERT PROBLEM: K–THEORY AND BORDISMS[1]

BOGDAN BOJARSKI and ANDRZEJ WEBER

Abstract

We present classical and generalized Riemann–Hilbert problem in several contexts arising from K–theory and bordism theory. The language of Fredholm pairs turns out to be useful and unavoidable. We propose an abstract formulation of a notion of bordism in the context of Hilbert spaces equipped with splittings.

Bogdan BOJARSKI
Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8, P.O.Box 137
00-950 Warszawa
POLAND
e-mail: bojarski@impan.gov.pl

Andrzej WEBER
Institute of Mathematics
Warsaw University
ul. Banacha 2, 02–097 Warszawa
POLAND, e-mail: aweber@mimuw.edu.pl
`http://www.mimuw.edu.pl/~aweber`

[1] `http://www.mimuw.edu.pl/~aweber/ps/bowe4.ps`

MULTIPLE GRUPOIDS:
A NON ABELIAN TOOL FOR LOCAL–TO–GLOBAL PROBLEMS

RONALD BROWN

Abstract

A 1–fold groupoid is just a groupoid and and n–fold groupoid is a set with n groupoid structures any two of which satisfy an interchange law. The aim of the talk is to show the intuitions behind how types of n–fold groupoid have been used in homotopy theory and the cohomology of groups to give new calculations, with a view to encouraging their use in differential situations.

School of Informatics, Mathematics Division,
University of Wales, Bangor
Dean St., Bangor, Gwynedd LL57 1UT
UNITED KINGDOM
e-mail: r.brown@bangor.ac.uk
http://www.bangor.ac.uk/~mas010/

CURVATURE PROPERTIES OF SOME HYPERSURFACES IN SPACES OF CONSTANT CURVATURE

RYSZARD DESZCZ and MAŁGORZATA GŁOGOWSKA

Abstract

We present curvature properties of pseudosymmetry type of some hypersurfaces in semi-Riemannian spaces of constant curvature. Results related to this talk are contained in the following papers: [1], [2], [3] and [4].

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Ryszard DESZCZ
Department of Mathematics, Agricultural University of Wrocław
ul. Grunwaldzka 53
PL - 50-357 Wrocław
POLAND
e-mail: rysz@ozi.ar.wroc.pl

Małgorzata GŁOGOWSKA
Department of Mathematics, Agricultural University of Wrocław
ul. Grunwaldzka 53
PL - 50-357 Wrocław
POLAND
e-mail: mglog@ozi.ar.wroc.pl

THE INTEGRABLE GEOMETRIC DISCRETIZATION OF THE KOENIGS NETS

ADAM DOLIWA

Abstract

We introduce the Koenigs lattice, which is new integrable reduction of the quadrilateral lattice (discrete conjugate net). The discretization is performed by the natural extension of the basic geometric properties of the Koenigs net to the discrete level. We construct also the Darboux-type transformation of the Koenigs lattice and we show permutability of superpositions of such transformations, thus proving integrability of the Koenigs lattice. Details can be found in `nlin.SI/0203011`.

Institute of Theoretical Physics
Warsaw University
00–681 Warszawa, ul. Hoża 69
POLAND
e-mail: doliwa@fuw.edu.pl
`http://www.fuw.edu.pl/~doliwa`

ON LOCAL CLASSIFICATION OF SINGULAR SYMPLECTIC FORMS

WOJCIECH DOMITRZ

Abstract

Let ω be a closed 2-form on $\mathbb{R}^{2n}$, ω is called the singular form if $\omega^n = f\Omega$, where Ω is the volume form and f is a smooth function $f : \mathbb{R}^{2n} \to \mathbb{R}$ such that $\Sigma_2 = f^{-1}(0) \neq \emptyset$. In [3], [5] and [2] it was shown that there are five generic types of singularities for germs of 2-forms on $\mathbb{R}^4$. All these generic types, among which, one is unstable and has moduli parameters, are determined by the geometry of the pullback $i^*\omega$ to the hypersurface of degeneration $i : \Sigma_2 \to \mathbb{R}^4$.

If germs of singular symplectic forms are equivalent then their pullbacks to the hypersurfaces of degeneration are equivalent. We consider the following problem :

Does the pullback of the singular symplectic form to its hypersurface of degeneration determine the equivalence class of the form?

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Faculty of Mathematics and Information Science
Warsaw University of Techology
plac Politechniki 1, 00–661 Warszawa
POLAND
e-mail: wdomitrz@ise.pw.edu.pl, domitrz@mini.pw.edu.pl

NOTES ON EXTENDED RECURRENT AND EXTENDED QUASI-RECURRENT MANIFOLDS

STANISŁAW EWERT–KRZEMIENIEWSKI

Abstract

Recently M. Prvanović ([P]) introduced a type of manifold (M, g) whose $(0,4)$ curvarture tensor R satisfies

$$\begin{aligned}&\nabla_Z R(X,Y,U,V) = \\ &A(Z)\left[R(X,Y,U,V) + (\beta - \psi)\, G(X,Y,U,V)\right] + \\ &\tfrac{\beta}{2}\left[A(X)G(Z,Y,U,V) + A(Y)G(X,Z,U,V) + \right. \\ &\left. A(U)G(X,Y,Z,V) + A(V)G(X,Y,U,Z)\right],\end{aligned} \tag{1}$$

where β, ψ are functions on M, A is a closed form satisfying $\beta A(\frac{\partial}{\partial x^r}) = \frac{\partial \psi}{\partial x^r}$ and $G(X,Y,U,V) = g(Y,U)g(X,V) - g(Y,V)g(X,U)$. She proved that in a neighbourhood of a generic point the associated 1-form A is concircular, i. e.

$$(\nabla A)(X,Y) = Fg(X,Y) + HA(X)A(Y)$$

holds for some function F, H, and found the local form of the metric ([P]). The condition (1) can be considered as a generalisation of the well known notion of a recurrent manifold ($\nabla_Z R = a(Z)R$, ([W])) as well as of generalized recurrent manifold introduced by Dubey ([D])

$$\nabla_Z R(X,Y,U,V) = K(Z)R(X,Y,U,V) + L(Z)G(X,Y,U,V). \tag{2}$$

On the other hand Chaki ([Ch]) introduced and studied a type of manifolds satisfying

$$\begin{aligned}&\nabla_Z R(X,Y,U,V) = \\ &2a(Z)R(X,Y,U,V) + a(X)R(Z,Y,U,V) + \\ &a(Y)R(X,Z,U,V) + a(U)R(X,Y,Z,V) + a(V)R(X,Y,U,Z)\end{aligned}$$

known as a pseudo-symmetric (in the sense of Chaki) or quasi-recurrent.

We discuss the condition including the above ones.This can be reduce to the form

$$\begin{aligned}&\nabla_Z R(X,Y,U,V) = \\ &2a(Z)R(X,Y,U,V) + a(X)R(Z,Y,U,V) + \\ &a(Y)R(X,Z,U,V) + a(U)R(X,Y,Z,V) + a(V)R(X,Y,U,Z) \\ &2b(Z)G(X,Y,U,V) + b(X)G(Z,Y,U,V) + \\ &b(Y)G(X,Z,U,V) + b(U)G(X,Y,Z,V) + b(V)G(X,Y,U,Z).\end{aligned} \tag{3}$$

We derive the formulas for the Ricci tensor, its covariant derivative and the Riemann curvature tensor. It appears that the manifold is of almost constant curvature. Making use of tensor charcterisation of subprojective spaces ([K]) we prove

Theorem 1. *Let M be a conformally flat manifold, $dimM > 4$, whose curvature satisfy (3). If the curvature tensor R and the 1-form a do not vanish on any dense subset of M, then M is subprojective manifold.*

This enables finding the local form of the metric of manifolds under consideration.

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Institute of Mathematics
Technical University of Szczecin
al. Piastów 17, 70-310 Szczecin
POLAND
e-mail: ewert@arcadia.tuniv.szczecin.pl

FIRST–ORDER DIFFERENTIAL INVARIANTS OF THE SPLITTING SUBGROUPS OF THE POINCARÉ GROUP $P(1,4)$

V. M. FEDORCHUK and V. I. FEDORCHUK

Abstract

The differential invariants of the Lie groups of the point transformations play an important role in geometry (see, for example, [1]), group analysis of differential equations (see, for example, [1–3]), etc.

Our communication is devoted to the construction of the differential invariants for non-conjugate subgroups of the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is a group of rotations and translations of the five–dimensional Minkowski space $M(1,4)$. For all splitting subgroups of the group $P(1,4)$, the first–order differential invariants have been found.

The results obtained have been used for the construction of the first-order differential equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$.

Since the Lie algebra of the group $P(1,4)$ contains as subalgebras the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [4], the obtained differential equations can be used in relativistic and non–relativistic physics.

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V. M. FEDORCHUK
Institute of Mathematics, Pedagogical Academy
PL – 30–084 Kraków, ul. Podchorążych 2
POLAND
and
Pidstryhach Institute of Applied Problems of Mechanics and Mathematics
National Ukrainian Academy of Sciences
3b Naukowa St, 79–053, Lviv
UKRAINE
e-mail: vas_fedorchuk@yahoo.com

V. I. FEDORCHUK
Ivan Franko Lviv National University
1 Universytetska St, Lviv, 7900
UKRAINE

LIE BRACKET ON AFFINE BUNDLES

JANUSZ GRABOWSKI, KATARZYNA GRABOWSKA, PAWEŁ URBAŃSKI

Abstract

Natural analogs of Lie brackets on affine bundles are studied. In particular, a close relation to Lie algebroids and duality with certain affine analog of Poisson structures is established as well as affine versions of the complete lift and the Cartan exterior calculus.

Janusz GRABOWSKI
Mathematical Institute
Polish Academy of Sciences
ul. Śniadeckich 8, P.O.Box 137
00-950 Warszawa
POLAND
e-mail: jagrab@mimuw.edu.pl

Katarzyna GRABOWSKA
Faculty of Physics
University of Warsaw
Hoża 74, 00–682 Warszawa
POLAND
e-mail: koniecz@fuw.edu.pl

Paweł URBAŃSKI
Faculty of Physics
University of Warsaw
Hoża 74, 00–682 Warszawa
POLAND
e-mail: urbanski@fuw.edu.pl

THE WORKS OF PROFESSOR STANISŁAW GOŁĄB IN THE FIELD OF FUNCTIONAL EQUATIONS

ROMAN GER

Abstract

Although the impressive scientific output of Professor Stanisław Gołąb uniquely qualifies him as the very prominent Polish geometrist even an extremely brief (just one sentence altogether) biographic note in *Nowa encyklopedia powszechna PWN*, vol. 2, p. 567, states that S. Gołąb's papers deal also with functional equations. Actually, approximately 20% of his scientific publications might be classified among this theory. The list of 50 papers presented below (I will be frequently referring the listeners to this list during my talk) contains the items that are more or less directly related to functional equations. Among them the monograph [26] written jointly with János Aczél, the world-wide leader in this field, belongs to one of the first books devoted entirely to functional equations. The fundamental one among those considered in [26] is the translation equation

$$F(F(x,s),t) = F(x,ts)$$

intensively studied throughout several decades and nowadays. This equation, occurring while studying the problem of determination of geometric objects, yields also the subject of numerous papers listed below.

Another equation of basic importance is commonly known as the Gołąb-Schinzel functional equation

$$f(x + yf(x)) = f(x)f(y)$$

which was obtained in [24] while looking for some special subgroups or subsemigroups of a given continuous group of transformations. Enormous number of mathematicians were and are inspired by that idea which led them to substantial generalizations of this equation studied in various classes of functions and pretty general spaces.

Equally inspiring was the paper [21] written jointly with Mieczysław Kucharzewski who, like his teacher S. Gołąb, had great achievements in applying functional equation methods to geometry. Studying homomorphisms of the differential groups L_n^s and, in particular, the multiplicative Cauchy equation involving matrix arguments is extremely vivid up to now.

The other topics that will be discussed and reported on may be framed as follows:

- characterizations of mappings,
- other functional equations occurring in geometry,
- functional equations stemming from algebra,
- functional equations stemming from analysis,
- the significance of the domain of functional equation and the idea of local solutions,
- the functional equation of brigade;

plainly, all of them dealt with by Professor Stanisław Gołąb in his papers presented below.

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Institute of Mathematics
Silesian University
Bankowa 14, 40-007 Katowice
POLAND
`http://www.imi.wsp.czest.pl/pracownicy/ger.html`
e-mail: romanger@us.edu.pl

SYMPLECTIC STRUCTURES AND DIFFEOMORPHISMS

BOGUSŁAW HAJDUK

Abstract

I discuss the problem of existence of symplectic structures on tori with different differential structures and related question of calculating the action of the group of diffeomorphisms on the homotopy classes of almost complex structures. I show that diffeomorphisms which are topologically isotopic to the identity may act nontrivially.

Mathematical Institute
University of Wrocław
plac Grunwaldzki 2/4
50–384 Wrocław
POLAND
e-mail: hajduk@math.uni.wroc.pl

DUALITY OF SYMMETRIES IN 4–DIMENSIONAL LORENTZ MANIFOLDS[2]

GRAHAM S. HALL

Abstract

This paper presents a brief discussion of the description of symmetries in 4-dimensional Lorentz manifolds (with a view to the space-time of general relativity). The orbit structure in terms of foliations is particularly stressed. The main symmetry discussed is local isometry, but other symmetries are briefly mentioned.

1 Introduction

The aim of this paper is to present a brief, reasonably modern approach to the study of symmetry in general relativity theory, that is, on a 4-dimensional manifold admitting a Lorentz metric. Throughout, M will be a smooth, connected, Hausdorff manifold admitting a smooth, Lorentz metric g of signature $(-,+,+,+)$ (and hence M is paracompact [1]). If $m \in M, T_mM$ will denote the tangent space to M at m. A Lie derivative is denoted by $\mathcal{L}$. When component notation is used, a partial derivative and a covariant derivative with respect to the Levi-Civita connection Γ associated with g are denoted, respectively, by a comma and a semi-colon.

In Einstein's general relativity theory, M plays the role of the space-time and the geometrical objects g, Γ and the curvature tensor on M derived from Γ collectively describe the gravitational field. Einstein's equations provide the physical restrictions on these objects. However, they will not be required in this paper.

Of course, there are many different types of symmetry studied in general relativity, for example, (local) isometries, homotheties, conformal isometries, affine and projective collineations and symmetries of the curvature and related tensors (for reviews see [2, 3]). The purpose of this paper, however, is more general, and will concentrate on techniques rather than the specific symmetry involved. Nevertheless, local isometries will finally be studied as an application. So far as the present author is aware, the mathematical study of symmetry in general relativity theory has not taken into account the progress made in the recent studies of the integrability of vector fields and foliations. The main purpose of this paper is to attempt a small step in this direction and to set on a more rigorous basis the general theory of symmetries and their associated orbits.

2 Space-Time Geometry and Decomposition

Let $m \in M$ and $0 \neq v \in T_mM$. Then v is called *spacelike* (respectively, *timelike* or *null*) if $g(v,v) > 0$ (respectively, $g(v,v) < 0$ or $g(v,v) = 0$). A 1-dimensional subspace of T_mM is called a *direction* (at m), and is referred to as a *spacelike* (respectively, *timelike* or *null*) *direction* if it is spanned by a spacelike (respectively, timelike or null) vector at m. If U is a 2-dimensional subspace of T_mM, then U is called *spacelike* (respectively, *timelike* or *null*)

[2]This is the text of an invited lecture given at the 3rd Conference GEOMETRY and TOPOLOGY of MANIFOLDS held in Krynica, Poland, May 2001.

if all non-zero members of U are spacelike (respectively, if U contains exactly two distinct null directions or if U contains exactly one null direction). If U is a 3-dimensional subspace of T_mM, the same definitions as in the 2-dimensional case apply except that in the timelike case, one insists that at least two (or, equivalently, infinitely many) distinct null directions are contained in U. These definitions are exclusive and exhaustive of all non-zero members of T_mM and all 1-, 2- and 3-dimensional subspaces of T_mM. A (smooth) submanifold N of M of dimension $1, 2$ or 3 is called *spacelike at* $m \in M$ if its tangent space is a spacelike direction or subspace of T_mM and *spacelike* if it is spacelike *at each* $m \in M$ (and similarly for timelike and null). If N is a spacelike (respectively, timelike) submanifold of M, then g induces a positive definite (respectively, Lorentz) metric on N.

It should be pointed out here that the term (smooth) *submanifold* of M means what is sometimes referred to as a (smooth) *immersed* submanifold of M. Thus, if M' is a submanifold of M, then M' is a subset of M which has a manifold structure, and is such that the inclusion map $i : M' \to M$ is a (smooth) immersion. If, in addition, the manifold topology (from the manifold structure) on M' equals its subspace topology as a subspace of M when the latter has its manifold topology, then M is called a *regular* or *embedded* submanifold. One of the advantages of regular submanifolds is that if M_1 and M_2 are smooth manifolds and $f : M_1 \to M_2$ is a smooth map whose range $f(M_1)$ lies inside a smooth *regular* submanifold N_2 of M_2, then the map $f : M_1 \to N_2$ is also smooth. If N_2 is not regular, this latter map may not even be continuous (but if it is continuous then $f : M_1 \to N_2$ is smooth). There is a type of submanifold introduced, as far as the author is aware, by Stefan [4, 5], and which is intermediate between submanifolds and regular submanifolds. A *leaf* of M is a connected (immersed) submanifold N of M with the additional property that, if T is any locally connected topological space, and $f : T \to M$ is a continuous map whose range lies inside N, then the map $f : T \to N$ is continuous. It follows [4] that if M_1 and M_2 are smooth manifolds and N_2 is a leaf of M_2, and $f : M_1 \to M_2$ is a smooth map whose range lies in N_2, then the map $f : M_1 \to N_2$ is continuous, and hence smooth. If N is a subset of M admitting two structures N_1 and N_2 as smooth *regular* submanifolds of M, then, from earlier remarks in this paragraph, the identity maps $N_1 \to N_2$ and $N_2 \to N_1$ are each smooth and so $N_1 = N_2$ and the regular submanifold structure is unique (see, e.g. [6]). The same uniqueness conclusion also holds if *regular submanifold* is replaced by *leaf* [4]. Clearly, every connected regular submanifold is a leaf, but the three types of (connected) submanifold structures (immersed, embedded and leaf) are distinct since the *irrational wrap* on the torus is a leaf which is not regular [4], whilst the well known *figure of eight* in $\mathbb{R}^2$ (see, e.g. [6]) is a connected submanifold which is easily shown not to be a leaf.

Now let A be a vector space of global, smooth vector fields on M and define the *distribution* Δ *on* M *associated with* A by [7]

$$m \to \Delta(m) = \{X(m) : X \in A\} \subseteq T_mM. \tag{2.1}$$

Then, for $i = 0, 1, 2, 3, 4$ and $p = 1, 2, 3$, define subsets V_i, S_p, T_p and N_p by

$$V_i \quad = \quad \{m \in M : \dim \Delta(m) = i\} \tag{2.2}$$

$$
\begin{aligned}
S_p &= \{m \in M : \dim \Delta(m) = p \textit{ and } \Delta(m) \textit{ is spacelike}\} \\
T_p &= \{m \in M : \dim \Delta(m) = p \textit{ and } \Delta(m) \textit{ is timelike}\} \\
N_p &= \{m \in M : \dim \Delta(m) = p \textit{ and } \Delta(m) \textit{ is null}\}
\end{aligned}
\tag{2.3}
$$

Thus, $M = \cup_{i=0}^{4} V_i$ and $V_p = S_p \cup T_p \cup N_p$ $(p = 1, 2, 3)$. This decomposition of M can be refined topologically by appealing to the rank theorem to see that $M = \cup_{i=k}^{4} V_i$ is open in M for $k = 0, \ldots, 4$. This can then be used to reveal the following disjoint decompositions of M [7]

$$M = V_4 \cup \bigcup_{i=0}^{3} V_i \cup Z_1 \tag{2.4}$$

$$M = V_4 \cup \bigcup_{p=1}^{3} S_p \cup \bigcup_{p=1}^{3} T_p \cup \bigcup_{p=1}^{3} N_p \cup V_0 \cup Z \tag{2.5}$$

where denotes the topological interior (and $V_4 = V_4$) and where Z and Z_1 are closed subsets of M *each with empty interior.*

3 Local Space-Time Symmetries

With A as in the last section, let $A_1, \ldots, A_k \in A$ and let $\phi_{t_1}^1, \ldots, \phi_{t_k}^k$ be the smooth, local diffeomorphisms associated with them, for appropriate values of t. Then consider the set of all such local diffeomorphisms (where defined) of the form

$$m \to \phi_{t_1}^1(\phi_{t_2}^2(\cdots \phi_{t_k}^k(m) \cdots)), \qquad (m \in M) \tag{3.1}$$

for each choice of $k, X_1, \cdots, X_k$ and admissible $(t_1, \cdots, t_k) \in \mathbb{R}^k$. There is an equivalence relation on M given by $m_1 \sim m_2$ if some local diffeomorphism of the form (3.1) maps m_1 into m_2. The associated equivalence classes in M are called the *orbits* of A and it is known that *these orbits can each be given the structure of a connected, smooth submanifold of* M [8, 4, 5]. In fact, Stefan has shown that these submanifolds constitute a *foliation with singularities, so that each has the extra property of being a leaf.* He also showed that if O is any such leaf and $m \in O$, then the tangent space to O at m is the subspace $\{f_* v : v \in \Delta(m')\}$ of $T_m M$ for each f of the form (3.1) and each $m' \in M$ such that $f(m') = m$. This subspace need not equal $\Delta(m)$. The condition that it does so for each $m \in M$ is equivalent to the condition that the orbits are *integral manifolds* of the set A and then A is *integrable* [8, 4, 5].

In general relativity, the situations of interest occur when A is a Lie algebra (under the Lie bracket operator) of global, smooth vector fields on M and then attention is directed to the nature of the *orbits* of the symmetries represented by A and whether they are integral manifolds of A. If $\dim \Delta(m)$ is constant on M, the Fröbenius theorem (see e.g. [6]) guarantees that the orbits are submanifolds and, in fact, integral manifolds of A. The work of Stefan then ensures that the orbits are leaves of a foliation on M. If $\dim \Delta(m)$ is not constant, then integrability need not follow. If, however, A satisfies the *locally finitely generated* condition (i.e. that each $m \in M$ has an open neighbourhood U and a finite subset A' of A such that each $X \in A$, when restricted to U, is a combination of members of A' (restricted to U) with coefficients which are smooth maps $U \to \mathbb{R}$), then Hermann [9] has shown that A is integrable (in fact, he showed more than this). *Thus, if A is a finite-dimensional Lie algebra, it is integrable and, again [4, 5], the orbits are leaves of a foliation with singularities.*

The symmetries usually studied in general relativity are described by a Lie algebra of global, smooth vector fields on the space-time M, with each particular symmetry being characterised by insisting upon the appropriate property being possessed by the resulting local diffeomorphisms of the type (3.1) (see, e.g. [2, 3]). Thus, *projective symmetry* is defined by insisting that each map (3.1) takes geodesics to geodesics and the resulting Lie algebra A, now labelled $P(M)$, is the set of all global, smooth vector fields on M with this property. The vector fields in $P(M)$ are called *projective* and are characterised by the condition that, in any chart of M

$$\begin{aligned} X_{a;b} &= \frac{1}{2}h_{ab} + F_{ab} \qquad (h_{ab} = h_{ba},\ F_{ab} = -F_{ba}) \qquad (3.2) \\ h_{ab;c} &= 2g_{ab}\psi_c + g_{ac}\psi_b + g_{bc}\psi_a \end{aligned}$$

for some closed 1-form field ψ and 2-form field F on M. Special cases are the *affine* vector fields (for which $\psi \equiv 0$ on M and whose associated maps (3.1) preserve also the geodesic affine parameter), the *homothetic* vector fields (which are affine and satisfy $h_{ab} = cg_{ab}, c \in \mathbb{R}$) and the *Killing* vector fields which are homothetic with $c = 0$ and so $\mathcal{L}_X g = 0$ (and for which each map (3.1) is a local isometry). The sets of all affine, homothetic and Killing vector fields on M are labelled $A(M), H(M)$ and $K(M)$ respectively, and $K(M) \subseteq H(M) \subseteq A(M) \subseteq P(M)$, with each being a subalgebra of $P(M)$. *Conformal symmetry* is defined by insisting that each map f in (3.1) is a local conformal diffeomorphism, that is, $f^*g = \alpha g$ for some appropriate local, smooth real valued function α. The resulting set of all global, smooth vector fields on M with this property is labelled $C(M)$ and its members are called *conformal.* Then $X \in C(M)$ is characterised in any chart of M by

$$X_{a;b} = \phi g_{ab} + F_{ab} \qquad (F_{ab} = -F_{ba}) \qquad (3.3)$$

where $\phi : M \to \mathbb{R}$ and F is a 2-form field on M. The set $C(M)$ is a Lie algebra and $H(M)$ and $K(M)$ above are subalgebras of it. Now it is well-known that $P(M)$ and $C(M)$ are finite-dimensional with $\dim P(M) \leq 24$ and $\dim C(M) \leq 15$ and so it follows from the discussion above that *the orbits of $P(M)$ and $C(M)$ are each foliations with singularities and are integral manifolds of $P(M)$ and $C(M)$, respectively, and similarly for their subalgebras mentioned above.* [It is remarked that the local action on M provided by the local diffeomorphisms described in the above Lie algebras need not lead to a *global* Lie group action on M. This occurs if and only if each vector field in the Lie algebra is *complete* [10].]

4 The Killing Algebra K(M)

Consider the finite-dimensional Lie algebra of Killing vector fields $K(M)$ on M. The material of section 3 shows that *the orbits associated with $K(M)$ are leaves of a foliation with singularities* and are integral manifolds of $K(M)$. It also shows that, if O is any orbit of $K(M)$, and f any associated local isometry of $K(M)$ whose domain and range are the open subsets U and U' of M, then f gives rise to a smooth map $U \cap O \to U'$ whose range lies in the leaf O. Hence, it gives rise to a smooth map $U \cap O \to U' \cap O$, since $U' \cap O$ is an open and hence, regular submanifold of O. Then if $m \in U \cap O$, $f_*(T_m O) = T_{f(m)} O$. The definitions at the beginning of section 2 then show that, since f is a local isometry, O *is either spacelike, timelike or null.* If O is spacelike (respectively, timelike), then g induces a metric $h = i^* g$ on O which is positive-definite (respectively, Lorentz). If $X \in K(M)$ then X is tangent to O and so there is a unique smooth, global vector field $\tilde{X}$ on O such that $i_* \tilde{X} = X$. If O is non-null with induced metric h, then the condition that $X \in K(M)$, that is $\mathcal{L}_X g = 0$, is easily shown to imply that $\mathcal{L}_{\tilde{X}} h = 0$ and so $\tilde{X}$ is a Killing vector field on O with metric h, that is, $\tilde{X} \in K(O)$. In fact, the map $k : X \to \tilde{X}$ is a Lie algebra homomorphism $K(M) \to K(O)$.

In general, the map k is neither injective nor surjective. That the map k is not surjective can be seen from the space-time metric given in a global chart on $\{(x, y, z, t) \in \mathbb{R}^4 : t > 0\} \equiv M$ by

$$ds^2 = -dt^2 + t dx^2 + e^{2t} dy^2 + e^{3t} dz^2. \tag{4.1}$$

Here $K(M)$ is 3-dimensional, being spanned by the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. However, each subset O of constant t is an orbit of $K(M)$ and is, with its induced metric, flat Euclidean 3-space and so $\dim K(O) = 6$.

To investigate whether k is injective or not, let $0 \neq X \in K(M)$ and let $m \in M$ with $X(m) = 0$. Then the local isometries ϕ_t associated with X satisfy $\phi_t(m) = m$ and m is called a *zero* of X (or a *fixed point* of each ϕ_t). If U is a coordinate neighbourhood of m with coordinates y^a, then the linear isomorphism $\phi_{t*} : T_m M \to T_m M$ is represented in the basis $(\frac{\partial}{\partial y^a})_m$ by the matrix

$$e^{tB} = \exp t \Big(\frac{\partial X^a}{\partial y^b}\Big)_m \tag{4.2}$$

where $B^a_b \equiv \Big(\frac{\partial X^a}{\partial y^b}\Big)_m$ is the *linearisation* of X at m. Thus, since $X \in K(M)$, it follows from (3.2) that $B^a_b = (F^a{}_b)_m$. Also, since X is affine, if χ is the usual exponential diffeomorphism from some open neighbourhood of $0 \in T_m M$ onto some open neighbourhood V of m, then [11]

$$\phi_t \circ \chi = \chi \circ \phi_{t*}. \tag{4.3}$$

It is easily checked from this that, in the resulting normal coordinate system x^a with domain V about m, the components X^a of X are *linear functions* of the coordinates x^a. Since $B^a_b = (F^a{}_b)_m$ is skew self- adjoint with respect to $g(m)$, it follows that the rank of B is *even.* If $B = 0$ then $X \equiv 0$ on M and so B has rank 2 or 4. The zeros of X in V have coordinates satisfying $B^a{}_b x^b = 0$ and so, if $B = 4$, the zero m is isolated, whereas if $B = 2$, the zeros of X in V can be given the structure of a 2-dimensional, *regular* submanifold N of the open submanifold V [12, 13]. Now return to the map k and suppose it is *not* injective. Let O be the orbit of $K(M)$ through m. Then there exists $X \in K(M), X \not\equiv 0$, such that X vanishes on O, that is, $\tilde{X} = 0$. Since m is thus not isolated, $A = 2$, and so the zeros of X *in* V are exactly the points on the 2-dimensional regular submanifold N of V. Let $O' = O \cap V$. Then O' is an open subset (and hence an open submanifold) of O. It follows that O' is a submanifold of M contained in the open (hence regular) submanifold V of M and hence O' is a submanifold of V [6]. But then $O' \subseteq N \subseteq V$, with O' and N submanifolds of V with N regular. It follows that O' is a submanifold of N and so $\dim O' \leq \dim N$ and hence $\dim O (= \dim O') \leq 2$. *Hence, if* $\dim O$ *is* 3 *or* 4, k *is injective.* If, however, $\dim O \leq 2$, k can fail to be injective, as the following example shows. Let M_1 and M_2 be 2-dimensional, connected, smooth manifolds with $M_2 = \mathbb{R}^2$. Let g_1 be a positive definite metric on M_1 with $K(M_1)$ 1-dimensional and spanned by a Killing vector field with a single zero at $m \in M_1$. Let g_2 be the usual Minkowski metric on M_2, so that $\dim K(M_2) = 3$. Then the space-time $M_1 \times M_2$ with metric $g_1 \otimes g_2$ is such that $\dim K(M) = 4$ and $O = \{m_1\} \times M_2$ is a 2-dimensional, timelike orbit of $K(M)$ with $\dim K(O) = 3$. Thus, the map $K(M) \to K(O)$ is *not* injective.

If O is an orbit of $K(M)$, it was pointed out above that O is either spacelike, timelike or null. Thus, if $\dim O = p$ $(1 \leq p \leq 3)$ and $O \cap S_p \neq \emptyset$, then $O \subseteq S_p$ (and similarly for T_p and N_p). It is convenient at this point to distinguish between orbits which are, in some sense, *stable* with respect to their type and dimension and those which are not. Thus, an orbit is called *stable* if it is contained in one of the subsets S_p, T_p or $N_p (1 \leq p \leq 3)$. Actually, since the inner product of a Killing vector field and the tangent vector to an affinely parameterised geodesic is constant along the geodesic, an argument based on the normal geodesics to orbits contained in S_3 and T_3 and an appeal to the rank theorem similar to that made at the end of section 2 shows that S_3 *and* T_3 *are open. Thus, all orbits in* S_3 *and* T_3 *are stable.* Regarding the stability of orbits, it is easy to show that, if O is any orbit of $K(M)$ such that $O \cap S_p \neq \emptyset$ $(1 \leq p \leq 3)$, then $O \subseteq S_p$ (and similarly for T_p and N_p). It is now possible to prove a number of results about how the existence of a certain type of *stable* orbit restricts the dimension of $K(M)$. These results are often used in the relativistic literature without justification. Some similar (but, as yet, incomplete) results are available in a similar context for *unstable* orbits [14].

In summary then (see [14, 15] for further discussion), the Lie algebra $K(M)$ of global, smooth Killing vector fields on a space-time M with smooth, Lorentz metric g is finite-dimensional and the orbits resulting from the maps (3.1) constitute a foliation with singularities. The maps (3.1) are smooth (local) maps $M \to M$ (and also $O \to O$, for any orbit O) and give rise to a Lie group (global) action on M if and only if each member of $K(M)$ is complete. A convenient decomposition of M with respect to the Lorentz metric g on M is provided by (2.2)–(2.5). The tangency of the members of $K(M)$ to an orbit leads to a natural Lie algebra homomorphism $K(M) \to K(O)$ which is easily seen to be not necessarily surjective and which is, perhaps less obviously, not necessarily injective, but is injective if $\dim O \geq 3$. This latter remark stems from a study of the zeros of the members of $K(M)$. The orbits of $K(M)$ were then divided into stable and unstable ones and the known (and used) results in orbit theory in general relativity can then be shown to apply to the stable orbits.

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Department of Mathematical Sciences
University of Aberdeen
Aberdeen AB24 3UE
Scotland
UNITED KINGDOM
e-mail: g.hall@maths.abdn.ac.uk

HOLONOMY THEORY AND 4–DIMENSIONAL LORENTZ MANIFOLDS

GRAHAM S. HALL

Abstract

Let M be a smooth, 4–dimensional, connected, Hausdorff manifold with smooth Lorentz metric g. Let Φ be the holonomy group of M associated with the Levi–Civita connection from g with holonomy algebra ϕ. The purpose of this talk is to give some results regarding the possibilities for Φ. Let $\mathcal{L}$ be the Lorentz group with connected component $\mathcal{L}_0$ and with Lie algebra L. Clearly, Φ is a subgroup of $\mathcal{L}$ and the well-known classification of L is thus important in this work. Φ is a Lie group and, if M is simply connected, Φ is a connected subgroup of $\mathcal{L}_0$. In this case, since ϕ is a subalgebra of L, there exists a one–to–one correspondence between the possibilities for Φ and the subalgebras of L. This gives a classification up to isomorphism of the possible holonomy groups of M. A coarser classification can (and will) be given in terms of covariantly constant and recurrent vector fields on M. Another useful approach (thinking of the pair (M, g) as the space–time of Einstein's general relativity) is to seek the possibilities for Φ when the energy–momentum tensor representing the physics of space–time is given. This will be discussed for the more commonly used energy–momentum tensors. Some remarks will also be made concerning the use of holonomy theory in the study of space–time curvature structure and space–time symmetries and in the theory of the Petrov classification of the Weyl tensor for gravitational fields.

Department of Mathematical Sciences
University of Aberdeen
Aberdeen AB24 3UE
Scotland
UNITED KINGDOM
e-mail: g.hall@maths.abdn.ac.uk

ON HYPERSURFACES WITH TYPE NUMBER TWO IN SPACE FORMS

MARIAN HOTLOŚ

Abstract

1. Basic notations

Let (M,g), $n \geq 3$, be a connected semi-Riemannian manifold of class C^∞. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where $X, Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Next, we define the endomorphisms $\mathcal{R}(X,Y)$, $\mathcal{C}(X,Y)$ and $X \wedge_A Y$ of $\Xi(M)$ by

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y\,, \\
\mathcal{R}(X,Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z\,, \\
\mathcal{C}(X,Y)Z &= \mathcal{R}(X,Y)Z - \frac{1}{n-2}\,(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y \\
&\quad - \frac{\kappa}{n-1}\,X \wedge_g Y)Z\,,
\end{aligned}
$$

respectively, where A is a symmetric $(0,2)$-tensor and $X, Y, Z \in \Xi(M)$. The Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the $(0,4)$-tensor G of (M,g) are defined by

$$
\begin{aligned}
R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4)\,, \\
C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4)\,, \\
G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3,X_4)\,,
\end{aligned}
$$

respectively. For a $(0,k)$-tensor field T, $k \geq 1$, and a $(0,2)$-tensor field A on (M,g) we define the tensors $R \cdot T$ and $Q(A,T)$ by

$$
\begin{aligned}
(R \cdot T)(X_1,\dots,X_k;X,Y) &= -T(\mathcal{R}(X,Y)X_1,X_2,\dots,X_k) \\
&\quad - \cdots - T(X_1,\dots,X_{k-1},\mathcal{R}(X,Y)X_k)\,, \\
Q(A,T)(X_1,\dots,X_k;X,Y) &= -T((X \wedge_A Y)X_1,X_2,\dots,X_k) \\
&\quad - \cdots - T(X_1,\dots,X_{k-1},(X \wedge_A Y)X_k)\,,
\end{aligned}
$$

respectively. Putting in the above formulas $T = R$, $T = S$ or $T = C$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $Q(g,R)$, $Q(S,R)$, $R \cdot S$, $Q(g,S)$, $R \cdot C$, $Q(g,C)$ and $Q(S,C)$. We define the following subsets of M: $\mathcal{U}_R = \{x \in M \,|\, R - \frac{\kappa}{(n-1)n}\,G \neq 0 \text{ at } x\}$, $\mathcal{U}_S = \{x \in M \,|\, S - \frac{\kappa}{n}\,g \neq 0 \text{ at } x\}$, $\mathcal{U}_C = \{x \in M \,|\, C \neq 0 \text{ at } x\}$ and $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C$. We note that $\mathcal{U} \subset \mathcal{U}_R$.

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be pseudosymmetric if at every point of M the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent. This is equivalent to

$$R \cdot R = L_R\, Q(g,R) \tag{1}$$

on $\mathcal{U}_R$, where L_R is some function on $\mathcal{U}_R$. The class of pseudosymmetric manifolds is an extension of the class of semisymmetric manifolds ($R \cdot R = 0$). Some geometrical considerations show that (1) is a more natural curvature condition than the condition of semisymmetry. For a presentation of facts related to this statement and other conditions of pseudosymmetry type we refer to a recent review paper [1].

2. Main resuts

Let (M, g), $n = \dim M \geq 4$, be a semi-Riemannian manifold satisfying at every point the following curvature condition of pseudosymmetry type: the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent. This is equivalent on the set U consisting of all points of M at which $Q(S, C) \neq 0$ to

$$R \cdot C = L\, Q(S, C)\,, \tag{2}$$

where L is some function on U. In this paper, without loss of generality, we restrict our investigations to the set $\mathcal{U}_L \subset U$ defined by $\mathcal{U}_L = \{x \in U \,|\, L \neq 0 \text{ at } x\}$.

Our aim is to investigate hypersurfaces in semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$, with signature $(s, n+1-s)$. Let M be a such hypersurface. We denote by g the metric tensor of M induced from the metric of the ambient space and by H the second fundamental tensor of (M, g). Let $\mathcal{U}_H$ be the subset of M consisting of all points at which the tensor H^2 is not a linear combination of H and g. In a few earlier papers we considered hypersurfaces in semi-Euclidean spaces $\mathbb{E}_s^{n+1}$, $n \geq 4$, fulfilling (2). For instance, in [4] (see Theorem 4.1 and Theorem 4.2) it was shown that if M is a hypersurface in $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying (2) then on $\mathcal{U}_H \cap \mathcal{U}_L$ we have: $rank\,(S - \frac{\kappa}{n-1}\, g) = 1$ and $R \cdot C = Q(S, C)$, i.e. (2) with $L = 1$.
On the other hand we have

Proposition 1 ([3], Proposition 4.3). *Let M be a hypersurface in $N_s^{n+1}(c)$ satisfying $R \cdot C = Q(S, C)$. If $\mathcal{U}_H \subset M$ is nonempty then the ambient space must be semi-Euclidean.*

Therefore we investigate hypersurfaces M in $N_s^{n+1}(c)$ with nonzero sectional curvature c satisfying (2). The main result is the following

Theorem 2 ([3], Theorem 4.1). *Let M be a hypersurface in $N_s^{n+1}(c)$, $c \neq 0$, satisfying (2). Then at every point $x \in \mathcal{U}_H \cap \mathcal{U}_L \subset M$ we have*

$$\frac{\kappa}{n-1} = \frac{\widetilde{\kappa}}{n+1}\,, \quad S - \frac{\kappa}{n}\, g = \beta\, w \otimes w\,, \quad \beta \in, \quad w \in T_x^* M$$

and

$$L \;=\; \frac{1}{n-1}\,, \quad R \cdot R = \frac{\kappa}{n(n-1)}\, Q(g, R)\,, \qquad R \cdot R \;=\; Q(S, R) - \frac{(n-2)\kappa}{n(n-1)}\, Q(g, C)\,.$$

From Theorem 5.1 of [2] it follows: (1) holds at a point $x \in \mathcal{U}_H$ if and only if at this point $rank\, H \;=\; 2$, i.e. the type number of M at this point is equal to 2. We have the following

Proposition 2 ([3], Proposition 5.1). *Let M be a pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*
(i) At every point $x \in \mathcal{U}_H \subset M$ we have

$$(a)\;\; rank\,(H^2 - tr(H)\, H) \;=\; 1\,, \quad or \quad (b)\;\; rank\,(H^2 - tr(H)\, H) \;=\; 2\,, \tag{3}$$

(ii) If $c \neq 0$ and at $x \in \mathcal{U}_H$ we have (3)(b) then (2) cannot be satisfied at this point.
(iii) If $c \neq 0$ and at a point $x \in \mathcal{U}_H$ we have (3)(a), i.e.

$$H^2_{ij} - tr(H) H_{ij} = \frac{1}{\rho}\, a_i a_j\,, \qquad \rho \in$$

then the following relations are fulfilled at x: (2) with $L = 1/(n-1)$, $\frac{\kappa}{n-1} = \frac{\widetilde{\kappa}}{n+1}$ and

$$a^k a_k \;=\; 0\,, \qquad a^k \;=\; g^{jk} a_j\,, \qquad S_{ij} - \frac{\kappa}{n}\, g_{ij} = \frac{\varepsilon}{\rho}\, a_i a_j\,,$$

Basing on this result we construct two examples of hypersurfaces with type number two. The first one satisfies the equality $R \cdot C = \frac{1}{n-1}\, Q(S, C)$ and the second one does not.

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Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50–370 Wrocław
POLAND
e-mail: hotlos@im.pwr.wroc.pl

COMBINATORICS OF THE FIRST NEIGHBOURHOOD OF THE DIAGONAL

ANDERS KOCK

Abstract

The consideration of the k'th neighbourhood of the diagonal of a manifold M

$$M_{(k)} \subseteq M \times M$$

was initiated by Grothendieck to import notions from differential geometry into the realm of algebraic geometry. These notions were re-imported into differential geometry by Malgrange. Grothendieck and Malgrange utilzed the notion of *ringed* space (a space equipped with a *structure sheaf* of functions). The only points of the (underlying space of) $M_{(k)}$ are the diagonal points (x, x) with $x \in M$. But it is worthwhile to describe mappings to and from $M_{(k)}$ *as if* it consisted of "pairs of k-neighbour points (x, y)" (write $x \sim_k y$ for such a pair; such x and y are "point proches" in the terminology of A. Weil). The introduction of topos theoretic methods has put this "synthetic" way of speaking onto a rigourous basis, and we shall freely use it.

A differential 1-form ω on M is thus defined to be a map $\omega : M_{(1)} \to \mathbf{R}$, vanishing on the diagonal. So $\omega(x, y)$ makes sense whenever $x \sim_1 y$; and $\omega(x, x) = 0$ for all $x \in M$. Unravelling the definition of $M_{(1)}$ in terms of its structure sheaf almost immediately reveals that such an ω is an element of the Kähler differentials $\Omega^1(M) = I/I^2$. There is no linearity requirement on ω; and $\omega(x, y) = -\omega(y, x)$ is automatic.

One can go on and define a k-form on M as an element of the k'th exterior power of Ω^1. This is the classical approach in algebraic geometry. But there is an alternative, more geometric/simplicial approach to the theory of differential forms, which we shall expound.

It is based on the consideration of the space

$$M_{[k]} \subseteq M^{k+1}$$

of *"infinitesimal k-simplices"*. It is the "set" of $k+1$-tuples $(x_0, \dots, x_k)$ with $x_i \sim_1 x_j$ for all $i, j = 0, \dots, k$. We shall call the simplex *degenerate* if two of its vertices x_i and x_j are equal. Then the geometric/synthetic/combinatorial approach to differential forms is based on

Theorem *There is a bijective correspondence between functions $\omega : M_{[k]} \to \mathbf{R}$ vanishing on degenarate simplices, and classical differential k-forms on M.*

Note that there is no multilinearity or alternating requirement on ω. – Since the $M_{[k]}$'s jointly form a simplicial "set", a differential k-form may be seen as a k-cochain, and there is therefore a formula for its coboundary; for instance, if ω is a 1-form, $d\omega$ is the 2-form given by

$$d\omega(x, y, z) = \omega(x, y) + \omega(y, z) + \omega(z, x). \tag{1}$$

It can be proved to correspond to the classical exterior derivative, under the correspondence of the theorem. But it generalizes in a more seamless way to differential forms with values in Lie groups G more general than $\mathbf{R}$. For instance, (1) is replaced by

$$d\omega(x,y,z) = \omega(x,y)\cdot\omega(y,z)\cdot\omega(z,x). \tag{2}$$

This aspect of the theory is well suited for a treatment of the theory of *connections* in principal G-bundles P or groupoids PP^{-1}. A *connection* in a bundle $E \to M$ is, in the synthetic theory, a law ∇ which to a pair of 1-neighbour points $x \sim_1 y$ in M associates a "parallel transport" map ∇_{yx} from the x-fibre of E to its y-fibre.

We shall describe the relationship between gauge-P valued forms, connections, and curvature, in synthetic/combinatorial terms. The curvature R_∇ of ∇ is a 2-form (in the above sense) on M with values in a suitable group bundle on M, and is given by

$$R_\nabla(x,y,z) = \nabla(x,y)\circ\nabla(y,z)\circ\nabla(z,x),$$

– We shall utilize these descriptions to provide an explicit construction of a connection in a principal G-bundle $P \to M$, out of the data of a G-valued Čech-cocycle for the bundle P, and an $\mathbf{R}$-valued partition of unity on M; this is based on the possibility of forming arbitrary affine combinations of the vertices of an infinitesimal simplex.

Department of Mathematical Science
University of Aarhus
DK 8000 Aarhus C
DENMARK
e-mail: kock@imf.au.dk
http://home.imf.au.dk/kock/

ON FIBRATION WITH GRASSMANN MANIFOLDS AS FIBRES

JÚLIUS KORBAŠ

Abstract

The aim of this talk is to present some results on $\mathbf{Z_2}$-cohomology properties of smooth fiber bundles and of Serre fibrations having complex or real Grassmann manifolds as their fibers.

We mainly concentrate on describing situations when the fiber inclusion (into the total space) induces an epimorphism in $\mathbf{Z_2}$-cohomology. This is related to other topological questions.

The talk will be based on the paper *On fibrations with Grassmannian fibers*, Bull. Belgian Math. Soc. 8 (2001), 119–130; more recent results will also be included.

Department of Algebra
Faculty of Mathematics and Physics
Comenius University
Mlynská dolina
SK-842 48 Bratislava 4
SLOVAKIA
e-mail: korbas@fmph.uniba.sk

TENSOR PRODUCT OF MODULES OVER NON–UNITAL ALGEBRAS AND LIE–RINEHART ALGEBRAS

JAN KUBARSKI

Abstract

In 1990–99 J. Huebschmann wrote a series of papers relating to Lie–Rinehart algebras. In

J. Huebschmann, *Poisson cohomology and quantization*, J. für die Reine und Angew. Math. 408 (1990), 57–113

the base of the series – the author wrote [p. 70] "Let A be an algebra over R, not necessarily with 1" and repeated this sentence in the context of commutative algebras on next pages. However, all the technical tools which were used, are appropriate in the case of unital algebras only. If we consider the typical situation where the base ring R is unital, then non-unitality of the R–algebra A means that there is no homomorphism of rings $l : R \to A$ such that $l(r)a = ra = al(r)$. There are some simple anomalies in the theory of A–modules over non-unital R–algebra A which caused that the planned researches on Lie–Rinehart algebras for algebras not necessarily with 1 failed. Moreover, the construction of the Picard group for a ring A with a unit can not be adapted to the non–unital case. The reason is that there is a difference between projective modules in the category of non–unital and unital modules. The aim of this paper is to construct the notion of a tensor product of modules over non–unital algebras which does not possesses the anomalies and its applications for the Picard group of non–unital algebras and Lie–Rinehart algebras.

Institute of Mathematics
Technical University of Łódź
al. Politechniki 11, 90–924 Łódź
POLAND
e-mail: kubarski@ck-sg.p.lodz.pl

SPACES OF DIFFERENTIAL OPERATORS AS MODULES OVER THE LIE ALGEBRA OF VECTOR FIELDS

PIERRE LECOMTE

Abstract

The space $\mathcal{D}_\lambda$ of differential operators acting on the λ-densities over a smooth manifold M is filtered by the order of differentiation. The associated graded space is the space $\mathcal{S}(M)$ of smooth functions on T^*M that are polynomial on the fibers. Both $\mathcal{D}_\lambda$ and $\mathcal{S}(M)$ are modules over the Lie algebra of vector fields $Vect(M)$ of M and the projection map $\mathcal{D}_\lambda \to \mathcal{S}(M)$, called the *principal symbol map*, is equivariant. It is onto. Moreover, although as vector spaces, $\mathcal{D}_\lambda$ and $\mathcal{S}(M)$ are isomorphic, there is no equivariant bijection between them.

Viewing T^*M as the phase space of some mechanical system, $\mathcal{S}(M)$ is then the Poisson algebra of the classical observables. A bijection from $\mathcal{S}(M)$ onto $\mathcal{D}_\lambda$ that preserves the principal symbol could then be interpreted as a quantification procedure. From the infinitesimal point of view, symmetries of the system lead to vector fields leaving that procedure equivariant.

It is known that there is no natural quantization, that is no quantization procedure that is equivariant under $Vect(M)$. On the other hand, quantization procedures have been constructed on $\mathbb{R}^m$ that are equivariant under the projective embedding sl_{m+1} of $sl(m+1, \mathbb{R})$ and the conformal embedding of $so(p+1, q+1), p+q = m$. Moreover some uniqueness properties have been shown for these quantizations. In particular, the sl_{m+1}-equivariant quantization is unique. This has been used to study various questions about $Vect(M)$-modules of differential operators over arbitrary manifolds, the strategy being as follows: first study them over $\mathbb{R}^m$, filtering $Vect(\mathbb{R}^m)$ by sl_{m+1} then glue the local informations collected on the various domains of chart by this means to get a global result. The power of the method comes from the fact that the filtering algebra is finite dimensional and simple, simplifying for instance cohomological considerations.

The above algebras of symmetries turn out to be *maximal subalgebras* of the Lie algebra of polynomial vector fields of $\mathbb{R}^m$. Because of that the family of these maximal subalgebras has been studied since then as well as the corresponding existence and uniqueness problem for the corresponding quantization procedures. The main result about the maximal subalgebras is that they coincide with the well known filtered algebras studied by Kobayashi and Nagano, that are related to geometries of order 2. Besides, algorithms have been found to decide wether or not the corresponding quantization procedures exist and are unique, using the ressources of the represntation theory of the semisimple Lie algebras. 3 The existence problem of quantization procedures is a particular case of the more general problem of classification of the spaces of differential operators as modules over Lie subalgebras of the algebra of vector fields. These questions involved some cohomological considerations that have also been in vestigated, leading to some nice universal cocycles.

On the other hand, going from vector space to curved manifold in order to get coordinate free expression of these quantization procedures also poses nice questions that one has started to study.

My goal would be to present a landscape of all that stuff, presenting the main results, the main methods and tools and the main contributors in the field.

Departement of Mathematics of the University of Liege
Grande Traverse 12 (B37), B-4000 Liege (Sart Tilman)
BELGIUM
e-mail: plecomte@ulg.ac.be
http://www.forum.math.ulg.ac.be
http://www.ulg.ac.be/geothalg

SHORTLY ON NATURAL BUNDLES

WŁODZIMIERZ MIKULSKI

Abstract

In this year it is celebrated centenary of Polish mathematician Professor **Stanisław Gołąb** at Jagellonian University, who formulated the definition of geometric objects and created "Polish Shool" investigating these objects. On this occasion a short survey on so called natural bundles and bundle functors is presented. A new result is presented, too. Namely, a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of infinite order in the first factor is constructed

Introduction

The modern development of the differential geometry clarified that geometric objects (see [1]) are sections of fiber bundles over manifolds. (For example, the vector fields on a manifold M are sections of the tangent bundle $T(M)$ of M, the 1-forms on M are sections of the cotangent bundle $T^*(M)$, etc.) Each type of geometric objects can be interpreted as a rule F transforming every m-dimensional manifold M into a fibered manifold $F(M) \to M$ over M and every local diffeomorphism $f : M \to N$ into a fibered manifold morphism $F(f) : F(M) \to F(N)$ over f. The geometric character of F is then expressed by the functoriality condition $F(g \circ f) = F(g) \circ F(f)$. Hence the classical bundles of geometric objects are now studied in the form of the so called natural bundles on the category $\mathcal{M}f_m$ of all m-dimensional manifolds and their local diffeomorphism.

All manifolds are assumed to be finite dimensional, paracompact, second countable, without boundaries and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

I. Natural bundles over m-manifolds

Let us present a definition of natural bundles over m-manifolds.

Definition 1. (Nijenhuis, [22]) A *natural bundle over m-manifolds* is a covariant functor $F : \mathcal{M}f_m \to \mathcal{FM}$ from the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms into the category $\mathcal{FM}$ of fibered manifolds and fibered maps satisfying the following conditions:

(i) (Prolongation) For every m-manifold M $F(M)$ is a fibered manifold over M with projection $p_M : F(M) \to M$ and for every local diffeomorphism $f : M \to N$ of two m-manifolds $F(f) : F(M) \to F(N)$ is a fibered map covering f;

(ii) (Locality) If $i : U \to M$ is an inclusion of an open submanifold, then $F(U) = p_M^{-1}(U)$ and $F(i)$ is the inclusion of $p_M^{-1}(U)$ into $F(M)$.

(iii) (Regularity) F transform smoothly parametrized systems of $\mathcal{M}f_m$-morphisms into smoothly parametrized systems of $\mathcal{FM}$-morphisms.

Simple examples of natural bundles are following.

Example 1. *The tangent bundle.* $T(M)$ denotes the tangent bundle of M and $T(f) : T(M) \to T(N)$ is the tangent map of $f : M \to N$. The functor $T : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds.

[3] *Key words:* natural bundles, gauge natural bundles, bundle functors, jets, Weil algebras, Weil bundles
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Example 2. *The cotangent bundle.* $T^*(M)$ denotes the cotangent bundle of M, and for a local diffeomorphism $f : M \to N$ and a point $x \in M$ let $(T_x(f))^* : T^*_{f(x)}(N) \to T^*_x(M)$ be the transposed mapping of the differential $T_x(f) : T_x(M) \to T_{f(x)}(N)$ of f at x. Then $T_x(f)$ is a linear isomorphism, and we define $T^*_x(f) = ((T_x(f))^*)^{-1} : T^*_x(M) \to T^*_{f(x)}(N)$ and the induced mapping $T^*(f) : T^*(M) \to T^*(N)$ by $(T^*(f))_{|T^*_x(M)} = T^*_x(f)$. The functor $T^* : \mathcal{M}f_m \to \mathcal{VB}$ is a natural bundle over m-manifolds.

Example 3. *The bundle of frames of order* r. Let m and r be natural numbers. We denote by G^r_m *the differential group of order* r *in dimension* m, it means

$$G^r_m = \left\{ j^r_0(\xi) \mid \xi : U \to \mathbf{R}^m,\ U \subset \mathbf{R}^m \text{ is open, } \xi \text{ is a local diffeom., } \xi(0) = 0 \right\},$$

where $j^r_0(\xi)$ is the r-jet of ξ at 0. G^r_m is a Lie group with respect to r-jets multiplication $j^r_0(\xi_1) \cdot j^r_0(\xi_2) = j^r_0(\xi_1 \circ \xi_2)$.

Let M be an m-manifold. We denote by $P^r(M)$ *the* r*-th order frame bundle of* M over M. It means

$$P^r(M) = \left\{ j^r_0(\gamma) \mid \gamma : U \to M,\ U \subset \mathbf{R}^m \text{ is open, } \gamma \text{ is a local diffeomorphism} \right\}.$$

The group G^r_m acts on $P^r(M)$ on the right by the formula $j^r_0(\gamma) \cdot j^r_0(\xi) = j^r_0(\gamma \circ \xi)$. $P^r(M)$ is a principal fibre bundle with the structure group G^r_m. For a local diffeomorphism $f : M \to N$ of two m-manifolds the induced mapping $P^r(f) : P^r(M) \to P^r(N)$ is given by $P^r(f)(j^r_0(\gamma)) = j^r_0(f \circ \gamma)$. The functor $P^r : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds.

The most general example of natural bundles is following.

Example 4. *The associated bundle.* Let S be an G^r_m-space, it means that S is a manifold and G^r_m acts on S on the left by an action α. For an m-manifold M we put

$$E^S(M) = P^r(M)[S, \alpha] ,$$

where $P^r(M)[S, \alpha]$ is the associated bundle with $P^r(M)$ and standard fibre S. We recall that $P^r(M)[S, \alpha]$ is the set of all orbits in $P^r(M) \times S$ with respect to the action of G^r_m given by $(j^r_0(\gamma), y) \cdot j^r_0(\xi) = (j^r_0(\gamma \circ \xi), j^r_0(\xi^{-1}) \cdot y)$. We denote by $< j^r_0(\gamma), y >$ the orbit of $(j^r_0(\gamma), y)$ and we define $\pi : E^S(M) \to M$, $\pi(< j^r_0(\gamma), y >) = \gamma(0)$.

For a local diffeomorphism $f : M \to M$ of two m-manifolds we define $E^S(f) : E^S(M) \to E^S(N)$ by

$$E^S(f)(< j^r_0(\gamma), y >) =< j^r_0(f \circ \gamma), y > .$$

The functor $E^S : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds.

Definition 2. Let F be a natural bundle over m-manifolds. We say that F is *of finite order* if there is a number r such that from $j^r_x(f) = j^r_x(g)$ it follows $F_x(f) = F_x(g)$ for every local diffeomorphisms $f, g : M \to N$ between m-manifolds and every point $x \in M$, where (obviously) $F_x(f) : F_x(M) \to F_{f(x)}(N)$ is the restriction and corestriction of $F(f) : F(M) \to F(N)$ to the fibres $F_x(M)$ and $F_{f(x)}(N)$ of $F(M)$ and $F(N)$ over $x \in M$ and $f(x) \in N$. The smallest number r satisfying the above property is called *the order of* F.

The tangent bundle T is of order 1, the cotangent bundle T^* is of order 1, the bundle P^r of frames of order r is of order r and the associated (with P^r) bundle E^S is at most of order r.

Theorem 1. (Palais and Terng, [23]) *Every natural bundle* F *over* m*-manifolds has a finite order. The order of* F *is less than or equal to* $2^f + 1$*, where* $f = dim(F_0(\mathbf{R}^m))$*.*

In [16], Masuda investigated homomorphisms of Lie algebras of vector fields. From the results of Masuda one can very easily deduce the above order theorem (even with a better estimation). Namely, from the results of [16] it follows that the order of a natural bundle F over m-manifolds is less than or equal $2(f^2+f)+1$, see [17]. The paper of Masuda was published one year ealier than the paper of Palais and Terng. However Masuda has not formulated the order theorem explicitly.

Theorem 2. (Epstein and Thurston, [5]) *If $F : \mathcal{M}f_m \to \mathcal{FM}$ is a covariant functor from Definition 1 satisfying the conditions (i) and (ii), then F satisfies also (iii) and F has a finite order less than or equal to $2f+1$, where $f = dim(F_0(\mathbf{R}^m))$. If $m = 1$, the estimation of the order is sharp.*

Theorem 3. (Zajtz, [26]) *Let F be a natural bundle over m-dimensional manifolds, $m \geq 2$, with a standard fibre of dimension f. Then the order r of F satisfies the inequality*

$$r \leq \max(\frac{f}{m-1}, \frac{f}{m} + 1) \ .$$

This estimation of the order is sharp.

Let F be a natural bundle over m-manifolds. According to Theorem 1 F has a finite order r. Let $S_F = F_0(\mathbf{R}^m)$ be the standard fibre of F. We can define an action α_F on the left of the r-th order differential group G^r_m on S_F by the formula

$$j^r_0(\xi) \cdot y = F(\xi)(y) \,,$$

where $y \in S_F$ and $j^r_0(\xi) \in G^r_m$. This action is well-defined because of the order argument and it is smooth because of the regularity condition (iii) of Definition 1.

The above action is called *the standard action of G^r_m on the standard fibre S_F of F.*

Now we can consider the associated bundle $E^{S_F} : \mathcal{M}f_m \to \mathcal{FM}$.

Theorem 4. (Palais and Terng, [23]) *Every natural bundle F over m-manifolds is equivalent with the associated bundle E^{S_F}, where $S_F = F_0(\mathbf{R}^m)$ is the standard fibre of F with the standard action of G^r_m on S_F, where r is the order of F.*

Proof. (see [6]) A canonical diffeomorphism $I_M : E^{S_F}(M) \longrightarrow F(M)$ is given by $I_M(< j^r_0(\gamma), y >) = F(\gamma)(y)$.

In particular case when $F(M)$ is a principal bundle Theorem 4 was proved independently by Krupka, [13].

II. Bundle functors on manifolds

Some "natural bundles" (as T) can be defined on the whole category $\mathcal{M}f$ of all manifolds and maps.

Definition 3. A *bundle functor on Mf* is a covariant functor $F : \mathcal{M}f \to \mathcal{FM}$ satisfying the following conditions (i)—(iii) of the definition of natural bundles with $\mathcal{M}f$ instead of $\mathcal{M}f_m$.

We have the following examples of bundle functors on $\mathcal{M}f$:

Example 5. *The tangent bundle $T : \mathcal{M}f \to \mathcal{FM}$* is a bundle functor on $\mathcal{M}f$.

Example 6. *The bundle T^r_p of p^r-velocities. The r-tangent bundle T^r.* Let M be a manifold. Let $T^r_p(M) = J^r_0(\mathbf{R}^p, M)$ be the bundle of r-jets at $0 \in \mathbf{R}^p$ of smooth functions $\mathbf{R}^p \to M$. $T^r_p(M)$ is called the bundle of p^r-velocities over M. Every mapping $f : M \to N$

of two manifolds is extended to a fibre bundle mapping $T^r_p(f) : T^r_p(M) \to T^r_p(N)$, $j^r_0(\gamma) \to j^r_0(f \circ \gamma)$. The functor $T^r_k : \mathcal{M}f \to \mathcal{FM}$ is a bundle functor.

For $p = 1$ the functor $T^r = T^r_1$ is called the r-tangent bundle functor.

Example 7. *The r-th order vector tangent bundle $T^{(r)}$.* Let M be a manifold. Let $T^{(r)}(M) = (T^{r*}(M))^*$ be the dual vector bundle, where $T^{r*}(M) = J^r(M, \mathbf{R})_0$. Vector bundle $T^{(r)}(M)$ is called *the r-th order vector tangent bundle of M.* Every smooth mapping $f : M \to N$ of two manifolds induces vector bundle mapping $T^{(r)}(f) : T^{(r)}(M) \to T^{(r)}(N)$ as follows. We denote the fibres of $T^{(r)}(M)$ and $T^{r*}(M)$ over a point $x \in M$ by $T^{(r)}_x(M)$ and $T^{r*}_x(M)$ respectively. For any $x \in M$ we have the linear mapping $f^{r*}_{|x} : T^{r*}_{f(x)}(N) \to T^{r*}_x(M)$ defined by $f^{r*}_{|x}(j^r_{f(x)}(\gamma)) = j^r_x(\gamma \circ f)$. We define $T^{(r)}(f) : T^{(r)}(M) \to T^{(r)}(N)$ by the condition that the restriction $T^{(r)}(f)_{|T^{(r)}_x(M)} : T^{(r)}_x(M) \to T^{(r)}_{f(x)}(N)$ is the transposed mapping to $f^{r*}_{|x}$. The functor $T^{(r)} : \mathcal{M}f \to \mathcal{VB}$ is a vector bundle functor on $\mathcal{M}f$.

To present a next example we need a preparation.

Let $L^r = \{L^r_{m,n}\}_{m,n=0,1,2,\ldots}$, where

$$L^r_{m,n} = J^r_0(\mathbf{R}^m, \mathbf{R}^n)_0$$

is the space of r-jets at $0 \in \mathbf{R}^m$ of maps $\mathbf{R}^m \to \mathbf{R}^n$ sending 0 into 0.

Let $S = \{S_0, S_1, ...\}$ be a system of manifolds. An action α of L^r on S is a system of smooth maps $\alpha_{m,n} : L^r_{m,n} \times S_m \to S_n$ satisfying

$$\alpha_{m,p}(B \circ A, s) = \alpha_{n,p}(B, \alpha_{m,n}(A, s))$$

for every $A \in L^r_{m,n}$, $B \in L^r_{n,p}$ and $s \in S_m$.

Example 8. *The associated bundle.* Consider a system of manifolds $S = \{S_0, S_1, ...\}$ and an action α of L^r on S. We shall construct a bundle functor E^α determined by this action. The restriction α_m of the maps $\alpha_{m,m}$ to invertible jets form actions of the jet group G^r_m on S_m. For every m-manifold M we put

$$E^\alpha(M) = P^r(M)[S_m, \alpha_m] \ .$$

For every map $f : M \to N$ we define $E^\alpha(f) : E^\alpha(M) \to E^\alpha(N)$ by

$$E^\alpha(f)(< u, s >) =< v, \alpha_{m,n}(v^{-1} \circ A \circ u, s) > \ ,$$

where $m = dim(M)$, $n = dim(n)$, $u \in P^r_x(M)$, $A = j^r_x(f)$, $s \in S_m$ and $v \in P^r_{f(x)}(N)$ is an arbitrary element. One can easily seen that this is a corect definition and the correspondence $E^\alpha : \mathcal{M}f \to \mathcal{FM}$ is a bundle functor on $\mathcal{M}f$.

Definition 4. Let r be a non-negative integer or infinity. Let F be a bundle functor on $\mathcal{M}f$. We say that F is *of order less or equal to r* if from $j^r_x(f) = j^r_x(g)$ it follows $F_x(f) = F_x(g)$ for every mappings $f, g : M \to N$ and every point $x \in M$. The smallest number r satisfying the above property is called *the order of F*.

Theorem 5. (Mikulski, [19]) *Every bundle functor F on $\mathcal{M}f$ has locally a finite order. More precisely, for any manifolds M, N, any point $x \in M$ and any mappings $f, g : M \to N$ the condition $j^{r(dim(M)+1)}_x(f) = j^{r(dim(M)+1)}_x(g)$ implies $F_x(f) = F_x(g)$, where $r(m)$ denotes the order of the natural bundle obtained by the restriction of F to the category of m-dimensional manifolds and their local diffeomorphisms.*

An open problem. Whether the condition $j_x^{r(dim(M))}(f) = j_x^{r(dim(M))}(g)$ implies $F_x(f) = F_x(g)$ for any manifolds M, N, any point $x \in M$ and any mappings $f, g : M \to N$, i.e. is the estimation in Theorem 5 sharp.

Example 9. (Mikulski, [18]) *A bundle functor on $\mathcal{M}f$ of infinite order.* We recall that $T^{(r)}(M)$ denotes the r-th order vector tangent bundle (see Example 7). Let $d_r = dim(T_0^{(r)}(\mathbf{R}^r))$. We set

$$F(M) = \bigoplus_{k=1}^{+\infty} (\bigwedge^{d_k} T^{(k)}(M)) \,.$$

For every manifold M, $F(M)$ is a finite dimensional manifold because for $k > dim(M)$ the bundle $\bigwedge^{d_k} T^{(k)}(M)$ is zero-bundle. Hence the direct sum in the definition of $F(M)$ is in reality a finite sum. For a mapping $f : M \to N$ the induced mapping $F(f) : F(M) \to F(N)$ is defined in the natural way from $\bigwedge^{d_k} T^{(k)}(f)$.

The bundle functor F is of infinite order because its restriction to the category of k-dimensional manifolds is at least of order k.

Theorem 6. (Kolář and Slovák, [12]) *If F is a covariant functor from Definition 3 satisfying the conditions (i) and (ii), then F satisfies also (iii).*

Let F be a bundle functor on $\mathcal{M}f$ of finite order r. Put $S^F = \{S_0^F, S_1^F, ...,\}$, where $S_m^F = F_0(\mathbf{R}^m)$ for $m = 0, 1, ...$ are the standard fibers of F. We can define an action $\alpha^F = \{\alpha_{m,n}^F\}$ of L^r on S^F by the formula

$$j_0^r(h) \cdot s = F(h)(s)\,,$$

where $s \in S_m^F$ and $j_0^r(h) \in J_0^r(\mathbf{R}^m, \mathbf{R}^n)_0$. This action is well defined because of the order argument, and it is smooth because of the regularity condition (iii).

The above action is called *the standard action of L^r on the standard fibres S^F of F.*

Now we can consider the associated bundle $E^{\alpha^F} : \mathcal{M}f \to \mathcal{FM}$.

Theorem 7. (Janyška, [7]) *Every bundle functor F on $\mathcal{M}f$ of finite order r is equivalent with the associated bundle E^{α^F}.*

Proof. A canonical diffeomorphism $I_M : E^{\alpha^F}(M) \longrightarrow F(M)$ is given by $I_M(< j_0^r(\gamma), s >) = F(\gamma)(s)$.

III. Product preserving bundle functors on manifolds

Many bundle functors on $\mathcal{M}f$ are product preserving.

Definition 5. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor. For two manifolds M_1, M_2 we denote the standard projection onto i-th factor by $pr_i : M_1 \times M_2 \to M_i$, where $i = 1, 2$. F is called *product preserving* if the mapping

$$(F(pr_1), F(pr_2)) : F(M_1 \times M_2) \to F(M_1) \times F(M_2)$$

is a diffeomorphism for all manifolds M_1, M_2.

Roughly speaking, F is product preserving if $F(M_1 \times M_2) = F(M_1) \times F(M_2)$ for any manifolds M_1 and M_2.

Example 10. The tangent bundle $T : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor.

Example 11. The p^r-velocities bundle $T_p^r : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor.

To present a next example of product preserving bundle functors on $\mathcal{M}f$ we need a preparation.

Definition 6. A finite dimensional real commutative associative unital algebra A is called a *Weil algebra* if it is of the form $A = \mathbf{R} \cdot 1 \oplus N$, where N is a nilpotent ideal.

Remark 1. Let $\mathcal{C}_0^\infty(\mathbf{R}^p)$ be the algebra of all germs at 0 of maps $\mathbf{R}^p \to \mathbf{R}$. It is a local algebra and the ideal m of all germs $\mathbf{R}^p \to \mathbf{R}$ vanishing at 0 is its maximal ideal. Let $\underline{A} \subset \mathcal{C}_0^\infty(\mathbf{R}^p)$ be an ideal such that $m \supset \underline{A} \supset m^{\kappa+1}$ for some natural number κ, where $m^{\kappa+1}$ is the algebraic $\kappa+1$ power of m. The factor algebra $A = \mathcal{C}_0^\infty(\mathbf{R}^p)/\underline{A}$ is a Weil algebra. Any Weil algebra is isomoprhic to some $A = \mathcal{C}_0^\infty(\mathbf{R}^p)/\underline{A}$.

Example 12. (Weil, [25]) *The Weil bundle $T^A = Hom(.,A)$ of near A-points.* Let A be a Weil algebra. Let M be a manifold. Let $\mathcal{C}_x^\infty(M)$ be the algebra of all germs at $x \in M$ of mappings $M \to \mathbf{R}$. An unital algebra homomorphism $v : \mathcal{C}_x^\infty(M) \to A$ is called *a near A-point over $x \in M$.* The space of all near A-points over x will be denoted by $Hom(\mathcal{C}_x^\infty(M), A)$. Let

$$T^A(M) = \bigcup_{x \in M} Hom(\mathcal{C}_x^\infty(M), A) \ .$$

$T^A(M)$ is a smooth fiber bundle over M with the obvious projection. Every map $f : M \to N$ of two manifolds is extended to a fibre bundle map $T^A(f) : T^A(M) \to T^A(N)$ over f by

$$T^A(f)(v)(germ_{f(x)}(g)) = v(germ_x(g \circ f)) \ ,$$

$x \in M$, $v \in Hom(\mathcal{C}_x^\infty(M), A)$, $g : N \to \mathbf{R}$. The functor $T^A : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor. It is of finite order less or equal to the nilpotency order of A.

Let us denote the addition and the multiplication on $\mathbf{R}$ by $+, \cdot : \mathbf{R}^2 \to \mathbf{R}$ and for $\lambda \in \mathbf{R}$ denote $m_\lambda : \mathbf{R} \to \mathbf{R}$ the scalar multiplication by λ. If we apply a product preserving bundle functor F we obtain $F(+) : F(\mathbf{R}) \times F(\mathbf{R}) = F(\mathbf{R} \times \mathbf{R}) \to F(\mathbf{R})$, $F(\cdot) : F(\mathbf{R}) \times F(\mathbf{R}) \to F(\mathbf{R})$ and $F(m_\lambda) : F(\mathbf{R}) \to F(\mathbf{R})$. Now, we can formulate the key lemma.

Lemma 1 (Kainz and Michor, [8], Eck, [3], Luciano, [15]) *If F is a product preserving bundle functor on $\mathcal{M}f$, then $A^F = F(\mathbf{R})$ is a Weil algebra with operations $F(+)$, $F(\cdot)$, $F(m_\lambda)$, zero $F(0)$ and unit $F(1)$, and $F_0(\mathbf{R})$ is its nilpotent ideal.*

Proof. Let r be the order of the restriction of F to the category $\mathcal{M}f_1$. Then r is finite because of Theorem 1. The map $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x + x^{r+1}$, satisfies $j_0^r(f) = j_0^r(id_R)$. Then $F_0(f) = id_{F_0(\mathbf{R})}$. Hence $a + a^{r+1} = F_0(f)(a) = a$, i.e. $a^{r+1} = 0$ for any $a \in F_0(\mathbf{R})$.

We call A^F *the Weil algebra of F.*

Then we have Weil bundle T^{A^F} of near A^F-points.

Theorem 8. (Kainz and Michor, [8], Eck, [3], Luciano, [15]) *Every product preserving bundle functor F on $\mathcal{M}f$ is equivalent to the Weil bundle T^{A^F} of near A^F point, where $A^F = F(\mathbf{R})$ is the Weil algebra of F.*

Proof. A canonical diffeomorphism $I_M : F(M) \to T^{A^F}(M)$ can be constructed as follows. If $v \in F_x(M)$ we have an algebra homomorphism $\tilde{v} : \mathcal{C}_x^\infty(M) \to A^F$ by $\tilde{v}(germ_x(g)) = F(g)(v)$ for any $g : M \to \mathbf{R}$, and we put $I_M(v) = \tilde{v}$.

Very interesting is also the following characterization:

Theorem 9. (Kolář and Slovák, [12]) *A bundle functor F on $\mathcal{M}f$ is product preserving if and only if $card(F(\mathbf{R}^0)) = 1$ and $dim(F(\mathbf{R}^n)) = n \cdot dim(F(\mathbf{R}))$ for any $n = 0, 1, 2, ...$*

IV. Bundle functors on fibered manifolds and on some other local categories over manifolds

Some importrant "bundle functors" are defined on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional basis and n-dimensional fibers and their local fibered diffeomorphisms. Some of them are defined on a larger category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered maps covering local diffeomorphisms. Some are defined on the whole category $\mathcal{FM}$ of all fibered manifolds and fibered maps. The definitions of such bundle functors are obvious modifications of the definition of natural bundles or bundle functors. Similarly we define the order of such bundle functors.

Example 13. *The r-jet prolongation bundle functor.* For an $\mathcal{FM}_m$-object $Y \to M$ we set

$$J^r(Y) = \left\{ \, j^r_x(\sigma) \mid \sigma \text{ is a (locally defined) section of } Y \ , \ x \in M \right\}$$

Then $J^r(Y)$ is a fibered manifold over Y with respect to the target projection. $J^r(Y)$ is called *the r-jet prolongation* of Y. For an $\mathcal{FM}_m$-morphism $f : Y \to Z$ covering a local diffeomorphism $\underline{f} : M \to N$ we have a fibered map $J^r(f) : J^r(Y) \to J^r(Z)$ covering f by the formula $J^r(f)(j^r_x(\sigma)) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$. Functor $J^r : \mathcal{FM}_m \to \mathcal{FM}$ is a bundle functor on $\mathcal{FM}_m$.

Let r, s, q be natural numbers such that $q \geq r \leq s$. Let $Y \to M$ and $Z \to N$ be fibered manifolds. Two fibered maps $f, g : Y \to Z$ with base maps $\underline{f}, \underline{g} : M \to N$ determine the same (r, s, q)-jet $j^{r,s,q}_y(f) = j^{r,s,q}_y(g)$ at $y \in Y_x$, $x \in M$, if

$$j^r_y(f) = j^r_y(g) \ , \ j^s_y(f|Y_x) = j^s_y(g|Y_x) \text{ and } j^q_x(\underline{f}) = j^q_x(\underline{g}) \ .$$

The space of all (r, s, q)-jets of Y into Z is denoted by $J^{r,s,q}(Y, Z)$, see [10].

Example 14. *The bundle of $(k, l)^{r,s,q}$-velocities.* Let r, s, q, k, l be natural numbers with $q \geq r \leq s$. For any fibered manifold Y we define $T^{r,s,q}_{k,l}(Y) = J^{r,s,q}_0(\mathbf{R}^{k,l}, Y)$. It is a fiber bundle over Y with respect to the target projection. If $f : Y \to Z$ is a fibered map we have fibered map $T^{r,s,q}_{k,l}(f) : T^{r,s,q}_{k,l}(Y) \to T^{r,s,q}_{k,l}(Z)$ covering f by $T^{r,s,q}_{k,l}(f)(j^{r,s,q}_0(\gamma)) = j^{r,s,q}_0(f \circ \gamma)$. The correspondence $T^{r,s,q}_{k,l} : \mathcal{FM} \to \mathcal{FM}$ is a bundle functor on $\mathcal{FM}$.

For bundle functors on $\mathcal{FM}_{m,n}$ we have a similar characterization as in Theorem 4 (we construct the associated bundle in similar way as for m-manifolds by using $\mathcal{FM}_{m,n}$-maps instead of $\mathcal{FM}f_m$-maps). By Example 9, one can construct bundle functors on $\mathcal{FM}_m$ (or $\mathcal{FM}$) of infinite order. Every bundle functor of $\mathcal{FM}_m$ is of locally finite order. For finite order bundle functors on $\mathcal{FM}_m$ (or $\mathcal{FM}$) one can obtain similar characterization as in Theorem 7. In particular we have the following order theorem similar to Theorems 2, 3 and 5.

Theorem 10. (Slovák, [24]) (i) *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a functor from the definition of bundle functors satisfying conditions (i) and (ii) of the definition and let $m \geq 1$ and $n \geq 0$. Then F satisfies the regularity condition (iii) and it is of finite order $r \leq 2f + 1$, where f is the dimension of the standad fiber $S_F = F_0(\mathbf{R}^{m,n})$ of F. If moreover $m > 1$, $n = 0$, then*

$$r \leq max(\frac{f}{m-1}, \frac{f}{m} + 1) \ ,$$

and if $m > 1$ and $n > 1$, then

$$r \leq max(\frac{f}{m-1}, \frac{f}{m} + 1, \frac{f}{n-1}, \frac{f}{n} + 1) \ .$$

All these estimations are sharp.

(ii) *Every bundle functor F on $\mathcal{FM}_m$ has locally a finite order. More precisely, for any fibered $\mathcal{FM}_{m,n}$-manifold Y, any $\mathcal{FM}_m$-object Z, any point $y \in Y$ and any $\mathcal{FM}_m$-maps $f, g : Y \to Z$ the condition $j_y^{r(n+1)}(f) = j_y^{r(n+1)}(g)$ implies $F_y(f) = F_y(g)$, where $r(n)$ denotes the order of the bundle functor obtained by the restriction of F to the category $\mathcal{FM}_{m,n}$.*

Some bundle functors F (as $T^{r,s,q}_{k,l}$) on $\mathcal{FM}$ are product preserving, i.e. $F(Y_1 \times Y_2) = F(Y_1) \times F(Y_2)$. We have the following characterization of such functors.

Theorem 11. (Mikulski, [20]) (i) *Every algebra homomorphism $\mu : A \to B$ of Weil algebras determines (explicitely) a product preserving bundle functor $T^\mu : \mathcal{FM} \to \mathcal{FM}$.* (ii) *Conversely, any product preserving bundle functor $F : \mathcal{FM} \to \mathcal{FM}$ determines (explicitely) an algebra homomorphism $\mu^F : A^F \to B^F$ of Weil algebras, and $F \tilde{=} T^{\mu^F}$.*

A simple proof of Theorem 11 one can find also in [2].

Some bundle functors F (as J^r) on $\mathcal{FM}_m$ are fiber product preserving, i.e. $F(Y_1 \times_M Y_2) = F(Y_1) \times_M F(Y_2)$. We have the following characterization of such functors.

Theorem 11. (Kolář and Mikulski, [11]) (i) *Every fiber product preserving bundle functor $F : \mathcal{FM}_m \to \mathcal{FM}$ is of finite order.* (ii) *Every triple (A, H, t), where A is a Weil algebra of nipotency order less or equal to a finite number r, H is a group homomorphism from the differential group G^r_m into the group $Aut(A)$ of all automorphisms of A and t is a G^r_m-invariant algebra homomorphism from the Weil algebra $\mathcal{D}^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into A, determines (explicitely) a fiber product preserving bundle functor $T^{(A,H,t)} : \mathcal{FM}_m \to \mathcal{FM}$.* (iii) *Conversely, every fiber product preserving bundle functor $F : \mathcal{FM}_m \to \mathcal{FM}$ of order r determines (explicitely) a triple (A^F, H^F, t^F), where A^F is a Weil algebra of nipotency order less or equal to a finite number r, H^F is a group homomorphism from the differential group G^r_m into the group $Aut(A^F)$ of all automorphisms of A^F and t^F is a G^r_m-invariant algebra homomorphism from the Weil algebra $\mathcal{D}^r_m = J^r_0(\mathbf{R}^m, \mathbf{R})$ into A^F, and $F \tilde{=} T^{(A^F, H^F, t^F)}$.*

Remark 2. A fiber product preserving bundle functors $F : \mathcal{FM}_m \to \mathcal{FM}$ such that $J^r(Y) \subset F(Y) \subset \tilde{J}^r(Y)$ (the non holonomic r-jet prolongation of Y) is called of jet type. In [9], Kolář described completely all such bundle functors of jet type.

Remark 3. Roughly speaking, a local category over manifolds is a category $\mathcal{C}$, whose objects can be interpreted as manifolds with "structures" and morphisms as "maps preserving structures", such that "open subsets of $\mathcal{C}$-objects" are $\mathcal{C}$-object and "restrictions of $\mathcal{C}$-morphisms to open subsets" are $\mathcal{C}$-morphisms. For example, $\mathcal{M}f_m$, $\mathcal{M}f$, $\mathcal{FM}_{m,n}$, $\mathcal{FM}_m$ and $\mathcal{FM}$ are local categories over manifolds. (For example $\mathcal{FM}$ is local because an open subset of a fibered manifold is again a fibered manifold and the restriction of a fibered map to an open subset is again a fibered map). In contrast, the category $\mathcal{VB}$ of vector bundles and their maps is not local (not every open subset $U \subset E$ of a vector bundle E is a vector bundle). The concept of bundle functors $F : \mathcal{C} \to \mathcal{FM}$ on an arbitrary local category $\mathcal{C}$ over manifolds is presented in [10]. Roughly speaking a bundle functor on a local category $\mathcal{C}$ over manifolds is a covariant functor $F : \mathcal{C} \to \mathcal{FM}$ satisfying the conditions (i)—(iii) of Definition 1 (with $\mathcal{C}$ instad of $\mathcal{M}f_m$). In [10], under some assumptions on $\mathcal{C}$ the authors proved that condition (iii) for $F : \mathcal{C} \to \mathcal{FM}$ is a consequence of functoriality of F and conditions (i) and (ii) for F (i.e. a theorem similar to the first part of Theorem 2), and under some assumptions on $\mathcal{C}$ the authors presented a characterization of bundle functors $F : \mathcal{C} \to \mathcal{FM}$ of finite order (similar to the one given in Theorem 7). The mentioned above assumptions are satisfied for $\mathcal{C} = \mathcal{M}f_m,\ \mathcal{M}f,\ \mathcal{FM},\ \mathcal{FM}_{m,n},\ \mathcal{FM}_m$.

The category $\mathcal{C} = Symp\mathcal{M}f_{2m}$ of symplectic $2m$-manifolds and their local symplectic diffeomorphisms is a local category over manifolds. The category $\mathcal{C} = Vol\mathcal{M}f_m$ of m-manifolds with locally integrable volume forms and their local diffeomorphisms preserving volume forms is local, too.

Theorem 12. (Lubaś and Zajtz, [14]) (i) *Let $F : Symp\mathcal{M}f_{2m} \to \mathcal{FM}$ be a covariant functor satisfying (i), (ii) and some stronger than (iii) regularity condition. If $m \geq 2$, then F is of finite order*

$$r \leq \frac{f}{m} \ ,$$

where $f = dim(F_0(\mathbf{R}^{2m})$ and $\mathbf{R}^{2m}$ is with the standard symplectic structure. The estimation is sharp.

(ii) *Let $F : Vol\mathcal{M}f_m \to \mathcal{FM}$ be a covariant functor satisfying (i), (ii) and some stronger than (iii) regularity condition. If $m \geq 2$, then F is of finite order*

$$r \leq \frac{f}{m-1} \ ,$$

where $f = dim(F_0(\mathbf{R}^m)$ and $\mathbf{R}^m$ is with the standard volume form. The estimation is sharp.

V. Bundle functors on some not local categories over manifolds

Fix a Lie group G and consider the category $\mathcal{PB}_m(G)$, whose objects are principal G-bundles over m-manifolds and whose morphisms are the morphisms of principal G-bundles $f : P \to \overline{P}$ with the base map lying in $\mathcal{M}f_m$ and with homomorphisms of G equal to id_G. Since $\mathcal{PB}_m(G)$ is not local (not every open subset in a principal G-bundle P is again a principal G-bundle), there is a problem with the locality condition (ii) of bundle functors. To introduce the locality condition (ii) we must modify the prolongation condition (i).

Definition 7. (Eck, [4]) A *gauge natural bundle on $\mathcal{PB}_m(G)$* is a covariant functor $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ satisfying the following conditions:

(i) (Prolongation) For every $\mathcal{PB}_m(G)$-object $P \to M$ $F(P)$ is a fibered manifold over M (not over P) with projection $p_P : F(P) \to M$ and for every $\mathcal{PB}_m(G)$-map $f : P \to \overline{P}$ covering $\underline{f} : M \to \overline{M}$ $F(f) : F(P) \to F(\overline{P})$ is a fibered map covering $\underline{f}$;

(ii) (Locality) If $i : P|U \to P$ is an inclusion, where $U \subset M$ is open, then $F(P|U) = p_P^{-1}(U)$ and $F(i)$ is the inclusion of $p_P^{-1}(U)$ into $F(P)$.

(iii) (Regularity) F transform smoothly parametrized systems of $\mathcal{PB}_m(G)$-morphisms into smoothly parametrized systems of $\mathcal{FM}$-morphisms.

Example 15. *The bundle of connection.* Let $P \to M$ be an $\mathcal{PB}_m(G)$-object with the right action $r : P \times G \to P$. We shall also denote by r the canonical right action $r : J^1(P) \times G \to J^1(P)$ given by $r^g(j^1_x(s)) = j^1_x(r^g \circ s)$ for all $g \in G$ and $j^1_x(s) \in J^1(P)$. Then

$$Q(P) = J^1(P)/G \ ,$$

where $J^1(P)/G$ is the set of orbits of the action r, is a bundle over M with projection $[j^1_x(s)] \to x$. It is called *the bundle of principal connections on P.* (The smooth sections $M \to Q(P)$ are in bijection with principal connections on P.) Every $\mathcal{PB}_m(G)$-map $f : P \to \overline{P}$ covering $\underline{f} : M \to \overline{M}$ induces a fibered map $Q(f) : Q(P) \to Q(\overline{P})$ over $\underline{f}$ by

$$Q(f)([j^1_x(s)]) = [j^1_{\underline{f}(x)}(f \circ s \circ \underline{f}^{-1})] \ .$$

The correspondence $Q : \mathcal{PB}_m(G) \to \mathcal{FM}$ is a gauge natural bundle.

If two $\mathcal{PB}_m(G)$-maps $f, g : P \to \overline{P}$ satisfy $j^r_y(f) = j^r_y(g)$ at a point $x \in P_x$ of the fiber of P over $x \in M$, then the fact that the raight translations of principal bundles are diffeomorphisms implies $j^r_z(f) = j^r_z(g)$ for every $z \in P_x$. In this case we write $j^r_x(f) = j^r_x(g)$.

Definition 8. (Eck, [4]) Let F be a gauge natural bundle on $\mathcal{PB}_m(G)$. We say that F is *of finite order* if there is a number r such that from $j^r_x(f) = j^r_x(g)$ it follows $F_x(f) = F_x(g)$ for every $\mathcal{PB}_m(G)$-maps $f, g : P \to \overline{P}$ covering $\underline{f} : M \to \overline{M}$ and every point $x \in M$. The smallest number r satisfying the above property is called *the order of F*.

We have the following order theorem similar to Theorems 2 and 3. This theorem was firstly proved by Eck, [4], but without a sharp estimation.

Theorem 13. (Eck, [4], Kolář, Michor, Slovák, [10]) *Let $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ be a functor from Definition 7 satisfying condition (i) and (ii), $m \geq 1$. Then F satisfies condition (iii) and it is of finite order $r \leq 2f + 1$, where f is the dimension of the standad fiber $S_F = F_0(\mathbf{R}^m \times G)$ of F. If moreover $m > 1$, then*

$$r \leq max(\frac{f}{m-1}, \frac{f}{m} + 1) \ .$$

All these estimations are sharp.

Theorem 13 gives a complete characterization of gauge natural bundles on $\mathcal{PB}_m(G)$ similar to Theorem 4 (we construct the associated bundle in similar way as for m-manifolds by using $\mathcal{PB}_m(G)$-maps instead of $\mathcal{M}f_m$-maps). This characterization was obtained by Eck, [4].

Another not local category over manifolds is the category $\mathcal{VB}$ of all vector bundles and their vector bundle maps. Gauge bundle functors on $\mathcal{VB}$ we define similarly as gauge natural bundles on $\mathcal{PB}_m(G)$. We have the following complete characterization of product preserving gauge bundle functors on $\mathcal{VB}$.

Theorem 14. (Mikulski, [21]) (i) *Every pair (A, V) of a Weil algebra A and of an A-module V with $dim_R(V) < \infty$ determines (explicitely) a product preserving gauge bundle functor $T^{(A,V)} : \mathcal{VB} \to \mathcal{FM}$.* (ii) *Conversely, any product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$ determines (explicitely) a pair (A^F, V^F) of a Weil algebra A^F and of an A^F-module V^F with $dim_R(V^F) < \infty$, and $F \tilde{=} T^{(A^F,V^F)}$.*

Remark 4. Another not local category over manifolds is the product $\mathcal{M}f_m \times \mathcal{M}f$ of categories $\mathcal{M}f_m$ and $\mathcal{M}f$. The objects of $\mathcal{M}f_m \times \mathcal{M}f$ are pairs (M, N), where M is an m-manifold and N is a manifold. The morphisms $(M, N) \to (\overline{M}, \overline{N})$ are the pairs (φ, f), where $\varphi : M \to \overline{M}$ is a local diffeomorphism between m-manifolds and $f : N \to \overline{N}$ is a map. An example of a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ is the bundle functor $J^r(M, N)$ of holonomic r-jets. Every pair of a local diffeomorphism $\varphi : M \to \overline{M}$ and a map $f : N \to \overline{N}$ induces fibered map $J^r(\varphi, f) : J^r(M, N) \to J^r(\overline{M}, \overline{N})$ over $\varphi \times f$ by $J^r(\varphi, f)(j^r_x(g)) = j^r_{\varphi(x)}(f \circ g \circ \varphi^{-1})$. Taking a bundle functor $G : \mathcal{M}f \to \mathcal{FM}$ of infinite order we produce a bundle functor $F(M, N) = M \times G(N)$, $F(\varphi, f) = \varphi \times G(f)$ of infinite order in the second factor. In [11], we present a complete characterization of all bundle functors F on $\mathcal{M}f_m \times \mathcal{M}f$ which are of finite order in the first factor.

We have the following new unpublished example.

Example 16. *A bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of infinite order in the first factor.* For any $r = 1, 2, 3, ...$ we define a vector bundle functor $G^r : \mathcal{M}f \to \mathcal{VB}$ by $G^r(N) =$

$T(N) \times \mathcal{D}^r$ and $G^r(f) = T(f) \times id_{\mathcal{D}^r} : G^r(N) \to G^r(\overline{N})$ for any map $f : N \to \overline{N}$, where $\mathcal{D}^r = J^r_0(\mathbf{R}^m, \mathbf{R})$ is the vector space of r-jets at 0 of maps $\mathbf{R}^m \to \mathbf{R}$.

Let $d_r = dim(G^r_0(\mathbf{R}^r))$. We set

$$G(N) = \bigoplus_{k=1}^{+\infty} (\bigwedge^{d_k} G^k(N)) .$$

For every manifold N, $G(N)$ is a finite dimensional manifold because for $k > dim(N)$ the bundle $\bigwedge^{d_k} G^k(N)$ is zero-bundle. Hence the direct sum in the definition of $G(N)$ is in reality a finite sum. For a mapping $f : N \to \overline{N}$ the induced mapping $G(f) : G(N) \to G(\overline{N})$ is defined in the natural way from $\bigwedge^{d_k} G^k(f)$. In this way we obtain vector bundle functor G on $\mathcal{M}f$ of order 1.

The group $G^\infty_m = invJ^\infty_0(\mathbf{R}^m, \mathbf{R}^m)_0$ of invertible ∞-jets acts on $\mathcal{D}^r$ by the pull-back. Hence for $r = 1, 2, ...$ we have a group homomorphism $H^r : G^\infty_m \to \mathcal{NE}_{vb}(G^r)$ into the group of vector bundle natural equivalences of G by $H^r(\xi)_N : G^r(N) \to G^r(N)$, $H^r(\xi)_N(v, \eta) = (v, \xi \cdot \eta)$, $v \in T(N)$, $\xi \in G^\infty_m$, $\eta \in \mathcal{D}^r$, where the dot is the action. Hence for any $\xi \in G^\infty_m$ and a manifold N we have a vector natural equivalence $H(\xi)_N : G(N) \to G(N)$ which is defined in natural way from $\bigwedge^{d_k} H^k(\xi)_N : \bigwedge^{d_k} G^k(N) \to \bigwedge^{d_k} G^k(N)$. This defines a group homomorphism $H : G^\infty_m \to \mathcal{NE}_{vb}(G)$.

For any manifold N the corresponding action H of G^∞_m on $G(N)$ can be factorized by a smooth action of $G^{l_N}_m$ on $G(N)$ for some finite l_N. Hence we can proced similarly as in [11] for above G and H and obtain bundle functor (G, H) on $\mathcal{M}f_m \times \mathcal{M}f$. More precisely, we set

$$(G, H)(M, N) = P^\infty(M)[G(N), H^N] ,$$

the associated bundle with $P^\infty(M)$ and fibre $G(N)$. Then $(G, H)(M, N)$ is a finite dimensional manifold diffeomorphic to $P^{l_N}(M)[G(N)]$. We define the source and the target projections $\alpha : (G, H)(M, N) \to M$ by $\alpha(< j^\infty_0(\gamma), v >) = \gamma(0)$ and $\beta : (G, H)(M, N) \to N$ by $\beta(< j^\infty_0(\gamma), v >) = y$ for all $j^\infty_0(\gamma) \in P^\infty(M)$ and $v \in G_y(N)$, $y \in N$. Every pair of a local diffeomorphism $\varphi : M \to \overline{M}$ and a map $f : N \to \overline{N}$ induce a fibered map $(G, H)(\varphi, f) = (G, H)(M, N) \to (G, H)(\overline{M}, \overline{N})$ by

$$(G, H)(\varphi, f)(< j^\infty_0(\gamma), v >) =< j^\infty_0(\varphi \circ \gamma), G(f)(v) > .$$

The bundle functor (G, H) is of order 1 in the second factor. The bundle functor (G, H) is of infinite order in the first factor because the homomorphism H can not be factorized by a group homomorphism $G^l_m \to \mathcal{NE}(G)$ with finite l.

Similarly, starting from a bundle functor $\tilde{T} : \mathcal{M}f \to \mathcal{FM}$ of infinite order instead of T we can produce a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of infinite order in both factors.

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Institute of Mathematics
Jagiellonian University
30–059 Kraków, ul. W. Reymonta 4
POLAND
e-mail: mikulski@im.uj.edu.pl

POINCARÉ DUALITY OF TOPOLOGICAL MANIFOLDS

ALEXANDR MISHCHENKO

Abstract

The problem of writing the Hirzebruch formula (see [1]) for topological manifolds for families of representations of the fundamental group collides with two difficulties. The first is related to difinition of rational Pontryagin characteristic classes that demands in particular the Novikov theorem about topological invariance of the rational Pontryagin classes. The second difficulty consists of that the construction of the signature needs a modification of classical construction for topological manifolds. Really, it is impossible to define the Poincare duality as a homomorphism of finite generated differential module since homology by itself is defined using either singular chains or spectrum of open coverings of manifolds. In both cases one should deal with infinite generated modules although homology here turns out to be finite generated spaces.

Here the following construction is possible. Let $\mathcal{U} = \{U_\alpha\}$ be a finite covering of the compact manifold X. Let $N_{\mathcal{U}}$ be the nerve of the covering $\mathcal{U}$. The nerve $N_{\mathcal{U}}$ detemines a finite simplicial polyhedron and hence determines chain and cochain complexes with local system of coefficients defined by a finite dimensional representation ρ of the fundamental group $\pi_1(X)$.

Consider a refining sequence of covering $\mathcal{U}_n = \{U_\alpha^n\}$, $\mathcal{U}_{n+1} \succ \mathcal{U}_n$. This means that $U_\alpha^{n+1} \subset U_\beta^n$ for a proper $\beta = \beta(\alpha)$. Hence the function $\beta = \beta(\alpha)$ defines a simplicial mapping

$$\pi_n^{n+1} : N_{\mathcal{U}_{n+1}} \to N_{\mathcal{U}_n},$$

and defines the homomorphism of the homology and cohomology groups

$$\begin{array}{rcl} (\pi_n^{n+1})_* & : & H_*\left(N_{\mathcal{U}_{n+1}}\right) \to H_*\left(N_{\mathcal{U}_n}\right), \\ (\pi_n^{n+1})^* & : & H^*\left(N_{\mathcal{U}_n}\right) \to H_*\left(N_{\mathcal{U}_{n+1}}\right). \end{array}$$

Here homology and cohomology of the manifold X are defined as limits of homology and cohomology of nerves of the coverings:

$$H^*(X) = \varinjlim H^*\left(N_{\mathcal{U}_n}\right),$$
$$H_*(X) = \varprojlim H_*\left(N_{\mathcal{U}_n}\right).$$

Then the Poincare duality should defined as a homomorphism $D : H^*(X) \to H_*(X)$, generated by the intersection operator with the open cycle of dimension n.

Notice that one can choose the refining sequence of covering in such way that each covering had multiplicity equal to $N+1$, $N = \dim(X)$. The the Poincare duality can be defined as the intersection operator with the cycle $D_n \in C_n\left(\tilde{N}_{\mathcal{U}_n}\right)$, where $(\pi_n^{n+1})_*(D_{n+1}) = D_n$. There is the following simple argument due to the Mittag-Leffler condition: let $C_*^\infty(X) = \varprojlim C_*\left(N_{\mathcal{U}_n}\right)$, $C_\infty^*(X) = \varinjlim C^*\left(N_{\mathcal{U}_n}\right)$. Then $H_*(X) = H\left(C_*^\infty(X)\right)$, $H^*(X) = H\left(C_*^\infty(X)\right)$, where the Poincare homomorphism D is induced by the intersection operator with the cycle $D_\infty = \varprojlim(D_n)$, that is $D = H(\,\cap D_\infty)$.

On the other hand

$$C^{\infty}_{*}(X) = \hom\left(C^{*}_{\infty}(X); R\right),$$

that allows to define the signature on the level of a quadratic form defined on the cycle group.

To this end let consider the category $\mathcal{C}_0$ of countable generated vector spaces over real numbers. Each space V can be considered as a direct limit of countable sequence of finite dimensinal spaces with topology of the direct limit. Then the vector space $V^* = \hom(V, R)$ can be represented as a inverse limit of finite dimensional spaces with topology of inverse limit. Denote by $\hom^t(V, R)$ the space of continuous linear functionals. Then for $V \in \mathcal{C}_0$ one has $\hom(V, R) = \hom^t(V, R)$ where $\hom^t(V^*, R) = V$. Let $\mathcal{C}$ denote the category with objects being topological space of the form $V = V_1 \oplus V_2^*$ and morphisms being continuous linear mappings. A non degenerated quadratic form on the object of the category $\mathcal{C}$ is defined as a continuous isomorphism

$$\varphi : V^* \to V, \quad \varphi^* = \varphi.$$

If $V = V_1 \oplus V_1^*$ and the isomorphism φ defined by a hyperbolic matrix then the quadratic form is called trivial.

þ For non degenerated quadratic form there exists correctly defined invariant signφ which is additive with respect to direct sum, equals to zero on trivial quadratic forms and coincides with classical signature on finite dimensional vector spaces.

The invariant defined above is admissible for constructing of signature of topological manifold with local system of coefficients ([2]).

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Moscow State University
Department of Mathematics
119 992 GSP–2 Moscow
RUSSIA
http://higeom.math.msu.su/asmish
e-mail: asmish@higeom.math.msu.su

MINIMAL NILPOTENT ALGEBRAS IN GOURSAT FLAGS OF LENGTHS NOT EXCEEDING 6

PIOTR MORMUL

1 Nilpotent approximation of a distribution at a point.

For any distribution D of rank d on an n–dimensional, smooth or real analytic, manifold M (i. e., rank–d subbundle in the tangent bundle TM) its *small flag* is the nested sequence

$$V_1 \subset V_2 \subset V_3 \subset \cdots$$

of modules (or: presheaves of modules) of vector fields (of the same category as M) tangent to M: $V_1 = D,\ V_{j+1} = V_j + [D,\, V_j]$ for $j = 1, 2, \ldots$ The *small growth vector* at $p \in M$ is the sequence (n_j) of linear dimensions at p of the modules V_j: $n_j = \mathrm{dim} V_j(p)$. Naturally, $n_1 = d$ independently of p.

D is *completely nonholonomic* when at every point of M its small growth vector attains (sooner or later) the highest value $n = \mathrm{dim} M$. Once this value attained, we truncate the vector after the first appearance of n in it. The length d_{NH} of thus truncated vector is called the *nonholonomy degree* of D at p.

In the theory that we recall (cf. [1], [AGS], [BS], [B]; this list of references is not complete) important are the *weights* w_i related to the small flag at a point: $w_1 = \cdots = w_d = 1$, $w_{d+1} = \cdots = w_{n_2} = 2$ (no value 2 among them when $n_2 = d$), and generally $w_{n_j+1} = \cdots = w_{n_{j+1}} = j+1$ (no value $j+1$ among them when $n_j = n_{j+1}$) for $j = 1, 2, \ldots$

Definition 1. For a completely nonholonomic distribution D on M, coordinates $z_1,\, z_2, \ldots,\, z_n$ around $p \in M$ are *linearly adapted* at p when $D(p) = (\partial_1, \ldots, \partial_d)$, $V_2(p) = (\partial_1, \ldots, \partial_d, \ldots, \partial_{n_2})$, and so on until $V_{d_{\mathrm{NH}}}(p) = (\partial_1, \ldots, \partial_n) = T_pM$. Here and in the sequel we skip writing 'span' before a set of v. f. generators.
For such linearly adapted coordinates we define their *weights* $w(z_i) = w_i$, $i = 1, \ldots, n$.

On the other hand, having a completely nonholonomic D, every smooth function f on M near p has its *nonholonomic order* wrt D at p ($+\infty$ is not excluded). It is the minimal number of differentiations of f along the local generators of D that give at p a nonzero result.
It follows directly from the above definitions that, for linearly adapted coordinates, their nonholonomic orders **do not exceed** their weights. Linearly adapted coordinates $z_1, \ldots, z_n$ are *adapted* (or: *privileged*) when the nonholonomic order of z_i **equals** $w(z_i)$, $i = 1, \ldots, n$. (In particular, adapted coordinates must vanish at p; one says that they are centered at p.)

It is not so quick, but true, that adapted coordinates always exist, and can even be algorithmically constructed from any à priori given coordinates, even in a polynomial way, as is done for inst. in [B]. They are not unique, there remains plenty of liberty behind the requirement that the nonholonomic orders (of linearly adapted coordinates) be maximal possible. In adapted coordinates it is purposeful to attach quasihomogeneous weights also to monomial vector fields (this definition goes back to the 1970's, to the theory of differential operators),

$$w(z_{i_1} \cdots z_{i_k} \partial_j) = w(z_{i_1}) + \cdots + w(z_{i_k}) - w(z_j)\,. \tag{1}$$

Proposition 3. *Every smooth vector field X with values in D has in its Taylor expansion in arbitrary coordinates adapted for D only terms of weights not smaller than -1 that can be grouped in homogeneous summands* $X = X^{(-1)} + X^{(0)} + X^{(1)} + \cdots$

(superscripts mean the weights defined by (1)). We denote by $\widehat{X}$ the lowest ('nilpotent') summand $X^{(-1)}$. That is, $\widehat{X} = X^{(-1)}$.

When a distribution D has around p local generators (vector fields) $X_1, \ldots, X_d$, then

Definition 2. The distribution $\widehat{D} = \big(\widehat{X_1}, \ldots, \widehat{X_d}\big)$ is called the *nilpotent approximation* of D at p.

This object $\widehat{D}$ is invariantly defined, independently of the used adapted coordinates, see Prop. 5.20 in [B]. Its basic properties are

Proposition 4. *The nilpotent approximation $\widehat{D}$ of D has at p the same small growth vector as D (and hence the same nonholonomy degree d_{NH}). Moreover, the real Lie algebra generated by $\widehat{X_1}, \ldots, \widehat{X_d}$ is a nilpotent Lie algebra of nilpotency order* d_{NH}

(the *nilpotency order* of a Lie algebra is the minimal number of multiplications in that algebra yielding always zero).

Definition 3 ([AG]). Distribution D is *strongly nilpotent* at a point p when its germ at p is equivalent to its nilpotent approximation $\widehat{D}$ at p.

([HLS], [M4]) Distribution D is *weakly nilpotent* at a point p when D possesses, locally around p, a basis generating over $\mathbb{R}$ a nilpotent (real) Lie algebra of vector fields.

In the founding work [HLS], and posterior control theory literature, weakly nilpotent distributions are called *nilpotentizable.* Nilpotent approximations of distributions, and in particular strongly nilpotent distributions play a growing role in sub-riemannian geometry ([B], [AG], [5]).

2 Kumpera-Ruiz algebras of Goursat distributions.

In the sequel we deal with Goursat distributions – a rather restricted class of objects for which preliminary (local) polynomial normal forms of [KR] exist with real parameters only, and no functional moduli. Their definition requires that the sequence of consecutive Lie squares of the original rank–2 subbundle of TM consist of regular distributions of ranks $3, 4, \ldots$ until $n = \dim M$. In the abstract [M3] we recalled a basic partition of Goursat germs into disjoint *geometric classes* encoded by words of length $n-2$ over the alphabet G, S, T, with two first letters always G and such that never a T goes directly after a G. Their construction was done by Montgomery and Zhitomirskii; it is reproduced in Sec. 1.3 of [M2]. (Implicitly these classes are already present in a pioneering work [J], in which the author uses a trigonometric presentation of Goursat objects. Another way of constructing them has been proposed in [PR1] where they are called *singularity types.*)

In dimension 4 there is but one class GG, in dimension 5 – only GGG and GGS, in dimension 6 – GGGG, GGSG, GGST, GGSS, GGGS.

The union of all geometric classes of fixed length with letters S in *fixed* positions in the codes is called, after [MZ], a *Kumpera-Ruiz class* of Goursat germs of that corank. For inst., in dimension 6 the two geometric classes GGSG and GGST build one KR class $**S*$.

Passing to nonholonomy degrees, Jean was able to compute them for objects showing up in his trigonometric presentation. That, in view of Thm. 4.1 of [BH] (see also [MZ], [PR1]) and after putting geometric classes in relief in Jean's approach, does for all G. germs.

Proposition 5 ([J]). *In dimension n, the nonholonomy degree d_{NH} of any Goursat germ in the geometric class $\mathcal{C}$, equals the last term b_{n-2} in the sequence $b_1, b_2, \dots, b_{n-2}$ defined only in terms of $\mathcal{C}$: $b_1 = 2,\ b_2 = 3$,*
$b_{j+2} = b_j + b_{j+1}$ when the $(j+2)$-th letter in $\mathcal{C}$ is S,
$b_{j+2} = 2b_{j+1} - b_j$ when the $(j+2)$-th letter in $\mathcal{C}$ is T,
$b_{j+2} = 1 + b_{j+1}$ when the $(j+2)$-th letter in $\mathcal{C}$ is G.

The mentioned polynomial (local) presentations of G. objects were not used in [J], but what are they? The essence of the contribution [KR], given in the notation of vector fields and taking into account posterior works, is as follows. We construct a (not unique) rank–2 distribution on $(\mathbb{R}^n(x^1, \dots, x^n), 0)$ departing from the code of a geometric class $\mathcal{C}$.

When the code starts with precisely s letters G, one puts $\overset{1}{Y} = \partial_1,\ \overset{2}{Y} = \overset{1}{Y} + x^3\partial_2, \dots, \overset{s+1}{Y} = \overset{s}{Y} + x^{s+2}\partial_{s+1}$. When $s < n-2$, then the $(s+1)$th letter in $\mathcal{C}$ is S. More generally, if the mth letter in $\mathcal{C}$ is S, and $\overset{m}{Y}$ is already defined, then

$$\overset{m+1}{Y} = x^{m+2}\overset{m}{Y} + \partial_{m+1}\,.$$

But there can also be T's or G's after an S. If the mth letter in $\mathcal{C}$ is not S, and $\overset{m}{Y}$ is already defined, then

$$\overset{m+1}{Y} = \overset{m}{Y} + \big(c^{m+2} + x^{m+2}\big)\partial_{m+1}\,,$$

where a real constant c^{m+2} is not absolutely free but

- equal to 0 when the mth letter in $\mathcal{C}$ is T,
- not equal to 0 when the mth letter is G going directly after a string ST…T (or after a short string S).

Now, on putting $\mathbf{X} = \partial_n$ and $\mathbf{Y} = \overset{n-1}{Y}$, and understanding $(\mathbf{X}, \mathbf{Y})$ as the germ at $0 \in \mathbb{R}^n$,

Theorem 3 ([KR]). *Any Goursat germ D on a manifold of dimension n, sitting in a geometric class $\mathcal{C}$, can be put (in certain local coordinates) in a form $D = (\mathbf{X}, \mathbf{Y})$, with certain constants in the field $\mathbf{Y}$ corresponding to* G*'s past the first* S *in $\mathcal{C}$.*

Definition 4. The real Lie algebra generated by $\mathbf{X}$ and $\mathbf{Y}$ is called the *KR algebra* of the germ D. This algebra does not depend on the choice of coordinates in Thm. 3, although so do the constants in $\mathbf{Y}$. Its nilpotency order is denoted by O_{KR}.
(A short analysis shows that this algebra depends solely on the Kumpera-Ruiz class of D.)

In 2000 we proved (cf. Rem. 1 in [M1])

Theorem 4 ([M4]). *The KR algebra of any Goursat distribution in dimension n, locally around a point belonging to a geometric class $\mathcal{C}$, is nilpotent of nilpotency order d_{n-2}, where d_{n-2} is the last term in the sequence $d_1, d_2, \dots, d_{n-2}$ defined only in function of the Kumpera-Ruiz class subsuming $\mathcal{C}$: $d_1 = 2$, $d_2 = 3$, $d_{j+2} = d_j + d_{j+1}$ when the $(j+2)$-th letter in $\mathcal{C}$ is* S*, and otherwise $d_{j+2} = 2d_{j+1} - d_j$.*

Note that in [PR2] a result stating just the nilpotency of the same algebras (the *existence* of finite nilpotency orders for them) is given.

Definition 5. We call *tangential* geometric classes whose codes possess letters G only in the beginning, before the first S (if any) in the code. Tangential Goursat germs are those sitting in the tangential classes.

Example 1. Up to dimension 5 all geometric classes are tangential. The first, and unique non-tangential class in dimension 6 is GGSG. In dimension 7 there are eight tangential classes and five non-tangential: GGSGG, GGSTG, GGSGS, GGGSG, GGSSG. In dimension 8 there are sixteen tangential and eighteen non-tangential geometric classes (cf. Thm. 5 below).

Tangential classes become clearly visible when one uses the polynomial presentation.

Observation 1. *A Goursat germ D is tangential $\Longleftrightarrow$ in any KR presentation $(\mathbf{X}, \mathbf{Y})$ of D there is no non-zero constant*

(so tangential germs are easily given local models with no parameters).

3 Minimality of KR algebras in several geometric classes.

The comparison of the formulas for the nonholonomy degree d_{NH} (Prop. 5) with the ones for nilpotency order O_{KR} (Thm. 4) yields that in each fixed geometric class $\mathcal{C}$ a). $d_{\mathrm{NH}} \leq O_{\mathrm{KR}}$, and b). $d_{\mathrm{NH}} = O_{\mathrm{KR}}$ **only** when $\mathcal{C}$ is tangential. Thus within tangential classes the KR algebras cannot be improved in the sense of lowering nilpotency orders. (Also – as is visible in the proof of Thm. 4 in [M4] – the germs in tangential classes are all strongly nilpotent, and Kumpera-Ruiz coordinates of Thm. 3 are already adapted.)

How is it in non-tangential classes? Do there exist there Goursat germs with better nilpotent bases – with lower nilpotency orders?

This question makes sense from dimension 6 onwards (see Ex. 1). We announce below the full answer in dimensions 6 and 7 (addressing all non-tangential classes in these dimensions), and a partial one in dimension 8 (addressing eight out of eighteen non-tangential classes).

Theorem 5.
A. *In dimension* 6*, for germs in the class* GGSG *the nilpotency order* $O_{\mathrm{KR}} = 7$ *is minimal among all possible local nilpotent bases, despite the fact that* $d_{\mathrm{NH}} = 6$ *for these germs.*

B. *In dimension* 7*, for germs in the classes:*

geometric class	d_{NH}	O_{KR}
GGSGG	7	9
GGSTG	8	9
GGSSG	9	11
GGSGS	11	12
GGGSG	8	10

their respective KR algebras are of minimal possible nilpotency order.

C. *In dimension* 8*, for germs in the non-tangential classes:*

geometric class	d_{NH}	O_{KR}
GGSGGG	8	11
GGSTGG	9	11
GGSTTG	10	11
GGSSGS	17	19
GGSGSG	12	17
GGSTSG	13	17
GGSGST	16	17
GGGSGS	15	17

their respective KR algebras are of minimal possible nilpotency order, too.

We do not yet know the answer in the remaining ten non-tangential classes in dimension 8.

Corollary 1. *In dimension* 6 *and* 7*, for all Goursat germs in the non-tangential geometric classes, their nilpotent approximations are not equivalent to the departure germs. The same concerns the germs in dimension* 8 *sitting in the eight non-tangential classes listed in* Thm. 5, C*. It is so because of the second property of nilpotent approximations recalled in* Prop. 4*. Thus those germs are not strongly nilpotent.*

The weak form of nilpotency – possession of a nilpotent basis – appears thus much weaker than the strong form of nilpotency of a distribution germ. In [M4] it is conjectured that the pattern emerging from Thm. 5 is valid for all non-tangential geometric classes in all dimensions. Implying that, conjecturally, the notions 'tangential' and 'strongly nilpotent' simply coincide.

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Institute of Mathematics
Warsaw University
Banacha 2, 02–097 Warszawa
POLAND
e-mail: mormul@mimuw.edu.pl

INVARIANT KÄHLER POLARIZATIONS AND GEOMETRY OF VECTOR FIELDS

IHOR MYKYTYUK

Abstract

Let G/K be a Riemannian symmetric space. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the corresponding invariant splitting of the Lie algebra $\mathfrak{g}$ of G. Geometric constructions which come from geometric quantization naturally lead to complex structures defined on the punctured cotangent bundle $T_0^*(G/K) = T^*(G/K) - \{\text{zero section}\}$. Such structure J_S for the spheres was found by Souriau. Later it was observed by Rawnsley, that the length function is strictly plurisubharmonic with respect to the above complex structure J_S and defines the Kähler metric on $T_0^* S^n$ with the canonical symplectic form Ω as the Kähler form. He also observed that J_S is invariant with respect to the Hamiltonian flow of the length function (the normalized geodesic flow) and used the Kähler polarization J_S to quantize the geodesic flow on the spheres.

Our purpose is to describe G-invariant Kähler structures on $T^*(G/K)$. Denote by $\langle \ , \ \rangle$ the scalar product on $\mathfrak{m}$ defining the Riemannian metric on G/K. Let $S^+(\mathfrak{m}^{\mathbb{C}})$ be the set of all invertible symmetric (with respect to $\langle \ , \ \rangle$) operators $s : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ such that $\operatorname{Re} s > 0$ (positive definite).

Theorem. *There is one-to-one correspondence between the set of all G-invariant Kähler structures $\{(J, \Omega)\}$ on $T^*(G/K)$ and the set $\{P\}$ of all smooth mappings $P : \mathfrak{m} \to S^+(\mathfrak{m}^{\mathbb{C}})$, $w \mapsto P_w$ such that* 1) $(\operatorname{Ad}_k P \operatorname{Ad}_{k^{-1}})|\mathfrak{m} = P_{\operatorname{Ad}_k w}$, $\forall w \in \mathfrak{m}, k \in K$; 2) $\big[P_w(\xi), P_w(\eta)\big](w) = -[w, [\xi, \eta]]$ *on* $\mathfrak{m}$, $\forall \xi, \eta \in \mathfrak{m}$, *where we consider the maps* $w \mapsto P_w(\xi)$, $w \mapsto P_w(\eta)$ *as vector fields on* $\mathfrak{m}$.

It is described all mappings P satisfying conditions of Theorem in the following cases: (1) if rank $G/K = 1$, $\dim G/K \geq 3$; (2) if the corresponding Kähler structure (J, Ω) is invariant under the normalized geodesic flow. For Hermitian symmetric spaces G/K of rank ≥ 2 a family of Kähler structures (of mappings P) is constructed. As an application of these results we obtain 1) a description of the curvature tensor of the symmetric space $F_4/Spin(9)$ in terms of its $Spin(9)$-structure; 2) new results concerning the hyper-Kähler metric on $T^*(G/K)$ if G/K is a Hermitian symmetric space of rank ≥ 1 (in this case the natural holomorphic-symplectic structure underlies the hyper-Kähler metric, whose restriction to G/K is the given homogeneous metric).

Applied Mathematics Department
State University "Lvi'v Polytechnica"
S. Bandera St. 12
79013, Lviv
UKRAINE
e-mail: viva@iapmm.lviv.ua

LOCAL STRUCTURES OF KV-ALGEBROIDS AND LIE ALGEBROIDS

MICHEL NGUIFFO BOYOM and ROBERT WOLAK

Abstract

The word manifold is used for smooth or complex analytic manifold. We deal with connected and paracompact manifolds. For a given manifold M $F(M, K)$ stands for the associative K-algebra of K-valued smooth functions if $K = \mathbb{R}$ (resp. the sheaf of complex analytic functions if $K = \mathbb{C}$).Geometric objects considered on M are either smooth or complex analytic according to $K := \mathbb{R}$ or $\mathbb{C}$. Given a vector bundle A on M $sect(A)$ stands for the $F(M, K)$-module of sections of A. A Koszul-Vinberg algebroid (resp Lie algebroid) on M is couple (A, a) where A is a vector bundle on M and a is a vector bundle M-homomorphism from A to TM (TM is the holomorphic tangent bundle when $K = \mathbb{C}$) such that:

i) $sect(A)$ is endoved with a structure of Koszul-Vinberg algebra $(sect(A),\ .\ )$ (resp. structure of Lie algebra $(sect(A), [\ ,\ ])$;

ii) given S, S' in $sect(A)$ and f in $F(M, K)$ one has $(fS).S' = f(S.S')$ and $S.(fS') = f(S.S') + (a(S)f)S'$, (resp. $[fS, S'] = f[S, S'] + (a(S)f)S'$).

The purpose of the confernce is to expose a decomposition theorem (normal forms) of KV-algebroids (resp. Lie algebroids) and a local classification theorem as well.

Michel NGUIFFO BOYOM
Departement de Mathématiques
Université Montpellier 2
FRANCE
e-mail: boyom@math.univ-montp2.fr

Robert WOLAK
Institute of Mathematics
Jagiellonian University
ul. Reymonta 4, 30–059 Kraków
POLAND
e-mail: wolak@im.uj.edu.pl

EXTENDED POINCARÉ PARASUPERGROUP WITH CENTRAL CHARGES

ANATOLY NIKITIN

Abstract

Irreducible represemtations of thr extended Poincaré parasuperalgebra with non-trivial central charges are described. Linear and non-linear models invariant w.r.t. the related parasupergroup are discussed.

Institute of Mathematics of Nat. Acad. Sci. of Ukraine
3 Tereshchenkivska St, 01601 Kiev–4
UKRAINE
e-mail: nikitin@imath.kiev.ua
`http://www.imath.kiev.ua/~nikitin/`

THE HOLOMORPHIC PSEUDOSYMMETRY OF KÄHLER MANIFOLDS

ZBIGNIEW OLSZAK

Abstract

The holomorphic pseudosymmetry is a natural generalization of the semisymmetry, in particular, the local symmetry.

A Kähler manifold $M(J,g)$ is said to be holomorphically pseudosymmetric if its Riemann curvature R satisfies the condition

$$R \cdot R = f\,\tilde{R} \cdot R,$$

where f is a certain function on M and $\tilde{R}$ is the derivation of the tensor algebra generated by the curvature type endomorphism

$$\tilde{R}(X,Y) = X \wedge Y + (JX) \wedge (JY) - 2g(JX,Y)J.$$

Basing on the Lichnerowicz's integral formulas, we can investigate the holomorphic pseudosymmetry of compact Kähler manifolds. The main result is the following:

Let M be a compact Kähler manifold with constant scalar curvature. If M is holomorphically pseudosymmetric with $f \geq 0$, then it is locally symmetric.

Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50–370 Wrocław
POLAND
e-mail: olszak@im.pwr.wroc.pl

GEOMETRIC ASPECTS OF INVARIANTS OF FINITE TYPE OF KNOTS IN S^3

LEONID PLACHTA

Abstract

The knot invariants of finite type (or Vassiliev invariants) give a coherent and systematic way to study the polynomial invariants of knots in S^3. However the geometric nature and geometric understanding of both the polynomial and the Vassiliev invariants remain missing. That is, it is unclear what kind of geometric knot information is carried by these invariants. In the present talk, we briefly review the well-known facts on this subject and report some new results.

The two knots are called n-equivalent, where $n \geq 1$, if they cannot be distinguished by the Vasssiliev invariants of order $\leq n$, the invariants taking values in any abelian group. The ralation of n-equivalence on knots in S^3 has been characterized by Habiro [1], in terms of C_{n+1}-moves on knots, and Stanford [5], in terms of the pure braid $(n+1)$-commutators. A knot is called n-trivial, if it is n-equivalent to the trivial one. In [2], Kalfagianni and Lin initiated the study of knot invariants of finite type in S^3 via geometric methods. They interpret these invariants as obstructions to a knot's bounding a regular Seifert surface with certain properties. More precisely, in order to describe the classes of n-equivalent knots, they introduced for each $n > 1$ the classes of n-hyperbolic, n-elliptic and n-parabolic knots, and showed that all they are n-trivial. It is unknown however whether the class consisting of all n-hyperbolic, n-elliptic and n-parabolic knots coincides with the one of n-trivial knots. They asked if furthermore, $(n-1)$- and $(n-2)$-hyperbolic knots are n-trivial. We show that in the first case the answer to the question is affirmative, while in the second case this is negative. More precisely, we show that all $(n-1)$-hyperbolic, $n > 2$, are n-trivial and that for each odd integer $n > 3$ there is an $(n-2)$-hyperbolic knot which is not n-trivial. Note that any n-hyperbolic and n-elliptic knots have the trivial Alexander polynomial. We introduce the notion of n-band equivalent knots and study the relationship between the class of all n-hyperbolic and n-elliptic knots and the class of knots, n-band equivalent to the trivial one. We discuss also the problem of how can change the group of a knot, after an application of a simple C_n-move to it, or equivalently, after an insertion of a pure braid n-commutator p_σ [3] in the knot. We provide a sufficient condition under which a simple C_n-move on a knot does not affect the Alexander polynomial of it [4]. Several problems concerning the geometric properties of invariants of finite order are also raised.

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Institute of Applied Problems of Mechanics and Mathematics
of NAS of Ukraine
UKRAINE
e-mail: igor@iapmm.lviv.ua

ON HIGER GEOMETRY AND INDUCED OBJECTS ON SUBSPACES

MARCELA POPECSU and PAUL POPESCU

Abstract

Besides a theory of higher order Finsler and Lagrange spaces, a dual theory of higher order Hamilton spaces was only recentely studied using the bundles of accelerations. In this paper we investigate the possibility to use these ideas in a more general setting. A recursive definition of higher order bundles defined by an affine bundle E and a vector pseudo-field on E can be considered, obtaining the acceleration bundles as a particular case. The case of non-holonomic spaces is effectively studied. A dual theory between lagrangians and hamiltonians (via Legendre transformations) is considered using affine bundles. A canonical way to induce a hamiltonian on an affine subbundle is also given.

Department of Algebra and Geometry
University of Craiova
13, 'Al.I.Cuza' st., Craiova, 1100
ROMANIA
e-mail: marcelapopescu@email.com, paulpopescu@email.com

A SMOOTH STRUCTURE ON LOCALLY COMPACT TOPOLOGICAL GROUP

TOMASZ RYBICKI

Abstract

It is shown that any connected locally compact topological group (or, more generally, any PL-group) carries a smooth structure. Here a smooth structure is determined by a set of smooth curves with some conditions specified. It is observed that the Lie algebra of a locally compact group is identified with the Lie algebra assigned to a smooth group structure. Further properties are exhibited.

Faculty of Applied Mathematics
of the University of Mining and Metallurgy (AGH)
Cracow, al. A. Mickiewicza 30
30–059 Kraków
POLAND
e-mail: tomasz@uci.agh.edu.pl

PROPERTIES OF THE CURVATURE OF KÄHLER–NORDEN MANIFOLDS

KARINA SŁUKA

Abstract

Let M be a smooth $2n$-dimensional manifold with an almost complex structure J and a pseudo-Riemannian anti-Hermitian metric g, i.e.

$$J^2 = -I, \quad g(JX, JY) = -g(X, Y).$$

If additionaly $\nabla J = 0$, where ∇ is the Levi-Civita connection of g, then the triple (M, J, g) form a Kähler-Norden manifold. A structure of this kind is also called an almost complex structure with B-metric, or an anti-Kählerian structure.

We have considered the classes of Kähler-Norden manifolds with recurrent curvature, recurrent holomorphic projective curvature, or recurrent conformal curvature.

We are also interested in those Kähler-Norden manifolds satisfying the conditions of semisymmetry or more generally the conditions of pseudosymmetry type.

Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50–370 Wrocław
POLAND
e-mail: sluka@mazur.im.pwr.wroc.pl