Ideal convergence versus matrix convergence

Rafał Filipów
(joint work with Jacek Tryba)

University of Gdańsk

Cesáro convergence

Definition

A sequence of reals (x_n) is Cesáro convergent to x if the sequence of means

$$x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \dots, \frac{x_1+\dots+x_n}{n}, \dots$$

is ordinary convergent to x.

Theorem

If (x_n) is ordinary convergent to x, then (x_n) is Cesáro convergent to x.

Example

$(0, 1, 0, 1, 0, 1, \dots)$ is not ordinary convergent but it is Cesáro convergent to $1/2$.

Applications of Cesáro convergence

Theorem (Cesáro, 1890)

Cauchy product $\sum_n c_n$ of convergent series $\sum_n a_n$ and $\sum_n b_n$ is Cesáro convergent and

$$\sum_n c_n = (\sum_n a_n) \cdot (\sum_n b_n).$$

Theorem (Frejér, 1900)

Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

of a 2π-periodic continuous function f is uniformly Cesáro convergent to f.

Asymptotic density

Definition

For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$ we define

$$d_n(A) = \frac{|A \cap \{1, \ldots, n\}|}{n}$$

and (provided the limit exists)

$$d(A) = \lim_n d_n(A)$$

and call it the asymptotic density of A.

Remark

$d(A) = \alpha \iff$ the sequence $(\chi_A(n))_n$ is Cesáro convergent to α

Proof

$\frac{\chi_A(1)+\chi_A(2)+\cdots+\chi_A(n)}{n} = \frac{|A\cap\{1,\ldots,n\}|}{n}$

Asymptotic convergence

Definition(Mazur 1935, Zygmund 1935/1979, Steinhaus-Fast 1951)

A sequence of reals (x_n) is statistically convergent to x if there is a set F of asymptotic density 1 such that the subsequence $(x_n)_{n\in F}$ is ordinary convergent to x.

Remark

There is no relationship between Cesáro convergence and statistical convergence in general:

- $(0, 1, 0, 1, 0, 1, \dots)$ is Cesáro conv. but is not statistically;
- If $x_n = n^2$ for $n = 2^k$ and $x_n = 0$ otherwise, then (x_n) is statistically convergent but is not Cesáro convergent

Theorem (Schoenberg, 1959)

If (x_n) is bounded and statistically convergent to x, then (x_n) is Cesáro convergent to x.

Matrix convergence

Notation

For a matrix $A = (a_{i,k})$ and a sequence (x_n) we write

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots \\ \vdots & \vdots & \\ a_{i,1} & a_{i,2} & \dots \\ \vdots & \vdots & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \dots \\ \vdots \\ a_{i,1}x_1 + a_{i,2}x_2 + \dots \\ \vdots \end{pmatrix} = \begin{pmatrix} A_1(x) \\ \vdots \\ A_i(x) \\ \vdots \end{pmatrix}$$

Definition (Toeplitz, 1913)

A sequence of reals (x_n) is A-convergent to x if the sequence $A_i(x)$ is ordinary convergent to x.

Remark

A-convergence of a sequence (x_n) is usually called A-sumability of (x_n).

Cesáro convergence is a matrix convergence

Remark

Cesáro convergence is a matrix convergence with respect to the Cesáro matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \vdots & & & & \end{pmatrix}$$

Proof

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \dots \\ \vdots & & & & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 + 0x_3 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 + 0x_3 \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{x_1+x_2}{2} \\ \frac{x_1+x_2+x_3}{3} \\ \vdots \end{pmatrix}$$

The Scottish Book

Theorem (Mazur, 1935)

Statistical convergence *is not equal* to any matrix convergence in the realm of all sequences.

Problem 5 of the Scottish Book (Mazur, 1935)

Is statistical convergence equal to some matrix convergence in the realm of all *bounded sequences*?

Commentary to Problem 5 (Buck, 1981 and 2015)

Problem 5 remains unsolved.

Mazur's remark from the Scottish Book

If every A-convergent bounded sequence is statistically convergent then there is a set F of density 1 such that for every A-convergent sequence (x_n) the subsequence $(x_n)_{n\in F}$ is ordinary convergent. It implies that statistical convergence is not equal to any matrix convergence.

The Scottish Book

Theorem (Khan–Orhan, 2007)

Statistical convergence *is equal* to some matrix convergence in the realm of all bounded sequences.

Remark

Khan and Orhan didn't know that they solved the problem from the Scottish Book.

Plan of research

- Change statistical convergence to ideal convergence.
  - Characterize ideals for which ideal convergence is equal to some matrix convergence.
  - The same problem in the realm of bounded sequences.
- Characterize ideals having "Mazur's property": If every A-convergent bounded sequence is $\mathcal{I}$-convergent then there is a set $F \in \mathcal{I}^*$ such that for every A-convergent sequence (x_n) the subsequence $(x_n)_{n \in F}$ is ordinary convergent.

Ideal convergence

Definition

$\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if

- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$
- $A \in \mathcal{I} \wedge B \subset A \Rightarrow B \in \mathcal{I}$

Example

Ideal of finite sets: $\mathrm{FIN} = \{A : A \text{ is finite}\}$
Ideal of sets of asymptotic density zero: $\mathcal{I}_d = \{A : d(A) = 0\}$

Definition

(x_n) is *$\mathcal{I}$-convergent to x* if
$\forall \varepsilon > 0 \, \exists A_\varepsilon \in \mathcal{I} \, \forall n \in \mathbb{N} \setminus A_\varepsilon \, |x_n - x| < \varepsilon$

Example

FIN-convergence is equal to ordinary convergence
$\mathcal{I}_d$-convergence is equal to statistical convergence

Ideal convergence versus matrix convergence in the realm of all sequences

Definition

$\mathsf{FIN} \oplus \mathcal{P}(\mathbb{N}) = \{B \subset \mathbb{N} : B \cap \{1, 3, 5, \ldots\} \text{ is finite}\}$

Theorem

$\mathcal{I}$-convergence is equal to some matrix convergence $\iff$ $\mathcal{I} = \mathsf{FIN}$ or $\mathcal{I} = \mathsf{FIN} \oplus \mathcal{P}(\mathbb{N})$.

Ideal convergence versus matrix convergence in the realm of all bounded sequences

$d(A) = 0 \iff$ the sequence $(\chi_A(n))_n$ is Cesáro convergent to 0 (i.e. it is matrix convergent to 0 with respect to Cesáro matrix C)

Definition

For a matrix A we define a matrix ideal by
$\mathcal{I}(A) = \{B \subset \mathbb{N} : (\chi_B(n))_n$ is A-convergence to $0\}$

Example

Yes: $\mathcal{I}$(identity matrix) $=$ FIN, $\mathcal{I}$(Cesáro matrix) $= \mathcal{I}_d$, all Erdős-Ulam ideals
No: dense summable ideals

Theorem (Khan–Orhan, 2007)

$\mathcal{I}$-convergence is equal to some matrix convergence in the realm of bounded sequences $\iff$ $\mathcal{I}$ is a matrix ideal.

Ideals with Mazur's property

Definition

An ideal $\mathcal{I}$ has the property (M) if for every matrix A such that every A-convergent bounded sequence is $\mathcal{I}$-convergent then there is a set $F \in \mathcal{I}^*$ such that for every A-convergent sequence (x_n) the subsequence $(x_n)_{n\in F}$ is ordinary convergent.

Theorem (Essentially Khan–Orhan)

No matrix ideal has the property (M). In particular $\mathcal{I}_d$ does not have the property (M).

Theorem

FIN and FIN $\oplus \mathcal{P}(\mathbb{N})$ have the property (M).

Question

Does there exist an ideal with the property (M) which is not isomorphic to FIN nor FIN $\oplus \mathcal{P}(\mathbb{N})$?

More matrices

Definition

For an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ we define
$\mathcal{M}(\mathcal{I}) = \{A : A \text{ is a matrix such that } \mathcal{I} \subset \mathcal{I}(A)\}$

Theorem (Fridy–Miller, 1991)

A bounded sequence (x_n) is $\mathcal{I}_d$-convergent to $x \iff (x_n)$ is A-convergent to x for every $A \in \mathcal{M}(\mathcal{I}_d)$.

Remark

- Fridy and Miller remarked that the same holds for any matrix ideal instead of $\mathcal{I}_d$.
- Using Khan–Orhan Theorem the proof of the their remark is easy.

More matrices

$\mathcal{M}(\mathcal{I}) = \{A : A \text{ is a matrix such that } \mathcal{I} \subset \mathcal{I}(A)\}$

Definition

$\mathcal{I}_{1/n} = \{A \subset \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty\}$

Theorem (Gogola–Mačaj–Visnyai, 2011)

A bounded sequence (x_n) is $\mathcal{I}_{1/n}$-convergent to $x \iff (x_n)$ is A-convergent to x for every $A \in \mathcal{M}(\mathcal{I}_{1/n})$.

Problem (Gogola–Mačaj–Visnyai, 2011)

Is the theorem true for any ideal $\mathcal{I}$ instead of $\mathcal{I}_{1/n}$?

More matrices

$\mathcal{M}(\mathcal{I}) = \{A : A \text{ is a matrix such that } \mathcal{I} \subset \mathcal{I}(A)\}$

Definition

An ideal $\mathcal{I}$ has the property GMV if every bounded sequence (x_n) is $\mathcal{I}$-convergent to $x \iff (x_n)$ is A-convergent to x for every $A \in \mathcal{M}(\mathcal{I})$.

Theorem

If $\mathcal{M}(\mathcal{I}) = \emptyset$, then $\mathcal{I}$ does not have the property GMV

Corollary

Any maximal ideal does not have the property GMV.

Theorem

There is F_σ ideal $\mathcal{I}$ such that $\mathcal{M}(\mathcal{I}) = \emptyset$. Hence it does not have GMV.

More matrices

Theorem

There is F_σ ideal $\mathcal{I}$ such that

- $\mathcal{M}(\mathcal{I}) \neq \emptyset$ and
- $\mathcal{I}$ does not have the property GMV.

Proof

- If $\mathcal{I}$ does not have GMV then the ideal $\mathcal{I} \oplus \mathcal{J}$ does not have GMV for any $\mathcal{J}$
  - If $\mathcal{M}(\mathcal{I}) \neq \emptyset$, then
    $\mathcal{I}$ has GMV $\iff \mathcal{I} = \bigcap\{\mathcal{I}(A) : A \in \mathcal{M}(A)\}$
- If $\mathcal{M}(\mathcal{I}) = \emptyset$ and $\mathcal{M}(\mathcal{J}) \neq \emptyset$, then $\mathcal{M}(\mathcal{I} \oplus \mathcal{J}) \neq \emptyset$
- Let
  - $\mathcal{I}$ be F_σ ideal such that $\mathcal{M}(\mathcal{I}) = \emptyset$
  - $\mathcal{J}$ be F_σ ideal such that $\mathcal{M}(\mathcal{I}) \neq \emptyset$ (say $\mathcal{J} = \mathrm{FIN}$)
- Then $\mathcal{I} \oplus \mathcal{J}$ is the required ideal

More matrices

Remark

All known examples of ideals with the property GMV are Borel (for instance FIN, $\mathcal{I}_d$, $\mathcal{I}_{1/n}$).

Theorem

If $\mathcal{I}$ has the property GMV, then $\mathcal{I}$ has the Baire property.

Question

Does there exist a non-Borel (non-analytic) ideal with the property GMV? In particular, does the ideal generated by a maximal almost disjoint family have the property GMV?