On a viability result for differential inclusions with Stieltjes derivative

by

Bianca Satco
Stefan cel Mare University of Suceava (Romania)
with George Smyrlis
NTUA Athens (Greece)

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Abstract.

Let $g : [0,1] \to \mathbb{R}$ be a left-continuous nondecreasing function and μ_g the **Lebesgue-Stieltjes measure** generated by g on $[0,1]$.

We prove a **viability result** for the following Initial Value Problem which corresponds to a **Stieltjes differential inclusion:**

$$\begin{cases} u_g'(t) \in F(t,u(t)),\ \mu_g - a.e.\ t \in [0,1) \\ u(t) \in K(t), \quad \forall\, t \in [0,1] \\ u(0) = x_0 \in K(0) \end{cases}$$

where

$$K : [0,1] \to \mathcal{P}_{cc}(\mathbb{R}^d), \quad F : [0,1] \times \mathbb{R}^d \to \mathcal{P}_{cc}(\mathbb{R}^d) \ \ (d \geq 1)$$

are convex compact set valued maps.

Based on the above result, we obtain a **Filippov type lemma** for Stieltjes inclusions.

Preliminaries.

Let $g : [0,1] \to \mathbb{R}$ be a left-continuous nondecreasing function.

- The measurability with respect to the σ-algebra (containing Borel sets) defined by g on $[0,1]$ will be called g-**measurability**.

- μ_g stands for the **Lebesgue-Stieltjes measure** generated by g on $[0,1]$. In particular, for $0 \leq a < b \leq 1$, we have

$$\mu_g([a,b)) = g(b) - g(a), \;\; \mu_g(\{a\}) = g(a^+) - g(a).$$

- The **Lebesgue-Stieltjes (LS) integrability w.r.t.** g means the abstract Lebesgue integrability w.r.t. the Stieltjes measure μ_g.

- Let $L^1_g([0,1], \mu_g)$ be the space of LS-integrable functions $u : [0,1] \to \mathbb{R}^d$ $(d \in \mathbb{N},\ d \geq 1)$ w.r.t. μ_g.

- The $g-$ **topology** on $[0,1]$ is the topology with basis the class of all sets

$$V_{g,\delta}(t) = \{t' \in [0,1] : |g(t') - g(t)| < \delta\}, \;\; \delta > 0, \;\; t \in [0,1].$$

Stieltjes derivative

• *R. López Pouso, A. Rodriguez, A new unification of continuous, discrete and impulsive calculus through Stieltjes derivatives. Real Anal. Exch., 2015.*

• *Young, W.H., On integrals and derivatives with respect to a function, Proc. London Math. Soc., 1(1917), s2-15 35 –63.*

Definition: The derivative with respect to g (the g–derivative) of $f:[0,1]\to\mathbb{R}^d$ at a point $t\in[0,1]$ is

$$\begin{aligned} f'_g(t) &= \lim_{s\to t}\frac{f(s)-f(t)}{g(s)-g(t)} \quad \text{if } g \text{ is continuous at } t,\\ f'_g(t) &= \lim_{s\to t^+}\frac{f(s)-f(t)}{g(s)-g(t)} \quad \text{if } g \text{ is discontinuous at } t, \end{aligned}$$

provided the limit exists.

Remark: If t is a point of discontinuity of g, the g-derivative $f'_g(t)$ exists if and only if the sided limit $f(t+)$ exists, and in this case

$$f'_g(t)=\frac{f(t+)-f(t)}{g(t+)-g(t)}.$$

The definition has no meaning in the parts of the domain in which g is constant; denote such a region by C_g (this doesn't provoke problems since $\mu_g(C_g) = 0$).

However, it has been generalized so as to include the points of C_g as well.

• *F.J. Fernández, I. Márquez Albés, F.A.F. Tojo, On first and second order linear Stieltjes differential equations, J. Math. Anal. Appl.* **511***(2022), No. 1, 126010.*

Let

$$N_g = \{u_n, v_n : n \in \mathbb{N}\} \setminus D_g,$$

where $C_g = \bigcup_{n\in\mathbb{N}} (u_n, v_n)$ with $(u_n, v_n)_n$ pairwise disjoint.

Let $N_g^- = \{u_n : n \in \mathbb{N}\} \setminus D_g$ and $N_g^+ = \{v_n : n \in \mathbb{N}\} \setminus D_g$.

The derivative with respect to g of $f : [0,1] \to \mathbb{R}^d$ at $\bar{t} \in [0,1]$ is defined by

$$
\begin{aligned}
f'_g(\bar{t}) &= \lim_{t \to \bar{t}} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } \bar{t} \notin D_g \cup C_g, \\
f'_g(\bar{t}) &= \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } \bar{t} \in D_g, \\
f'_g(\bar{t}) &= \lim_{t \to v_n+} \frac{f(t) - f(v_n)}{g(t) - g(v_n)} \quad \text{if } \bar{t} \in (u_n, v_n) \subseteq C_g,
\end{aligned}
$$

if the limits exist.
The points of N_g must be approached in the following manner:

$$
\begin{aligned}
f'_g(\bar{t}) &= \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } \bar{t} \in N_g^+, \\
f'_g(\bar{t}) &= \lim_{t \to \bar{t}-} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } \bar{t} \in N_g^-.
\end{aligned}
$$

Classical Viability Theory.

Let K be a closed subset of $\mathbb{R}^d$ ($d \geq 1$).

Definition 2.1. (G. Bouligand (1932)). Let $x \in K$. The **contingent cone** of K at x is defined as follows:

$$T_K(x) = \left\{ y \in \mathbb{R}^d : \liminf_{h \to 0^+} \frac{d(x+hy, K)}{h} = 0 \right\}.$$

- $y \in T_K(x)$ iff there there exist sequences

  $$(h_k)_k \subseteq \mathbb{R}_+ , \quad (x_k)_k \subseteq K$$

  such that

  $$\lim_{k \to \infty} h_k = 0, \quad y = \lim_{k \to \infty} \frac{x_k - x}{h_k}.$$

- $T_K(x)$ is a closed cone. If K is convex, $T_K(x)$ is convex too.

- for $x \in \mathrm{int}K$, $T_K(x) = \mathbb{R}^d$.

Definition 2.2. A function $u : [0, 1] \to \mathbb{R}^d$ is called $K-$**viable** iff $u(t) \in K$, $\forall\, t \in [0, 1]$.

Proposition 2.1. *Let $u : [0, 1] \to K$ be $K-$ viable and differentiable. Then*

$$u'(t) \in T_K(u(t)), \quad \forall\ t \in [0, 1).$$

Theorem 2.1.(M. Nagumo (1942)). *Let $f : K \to \mathbb{R}^d$ be a bounded continuous map. Then the following are equivalent:*

(i) for each $x_0 \in K$, the Initial Value Problem

$$u'(t) = f(u(t)), \quad t \in [0, 1], \quad u(0) = x_0$$

has at least one $K-$ viable solution.

(ii) $f(x) \in T_K(x)$, $\forall\, x \in K$.

The above theorem is extended to the case of differential inclusions as follows:

Theorem 2.2. (J.-P. Aubin - A. Cellina (1984)). *Let $F : K \to \mathcal{P}(\mathbb{R}^d)$ be upper semicontinuous with compact convex values . Then the following are equivalent:*

(i) for each $x_0 \in K$, the Initial Value Problem

$$u'(t) \in F(u(t)), \quad t \in [0,1], \quad u(0) = x_0$$

has at least one $K-$ viable solution.

(ii) $F(x) \cap T_K(x) \neq \emptyset$, $\forall\, x \in K$.

Next, we consider the more general case where K **depends on the time** $t \in [0,1]$.

Namely, let $K : [0,1] \to \mathcal{P}(\mathbb{R}^d)$ be a set-valued map with closed graph.

Definition 2.3. A function $u : [0,1] \to \mathbb{R}^d$ is called $K(\cdot)-$**viable** iff $u(t) \in K(t)$, $\forall\, t \in [0,1]$.

Definition 2.4. Let $t \in [0,1)$, $x \in K(t)$. The **contingent derivative** of the map K at (t,x) is defined as follows:

$$DK(t,x) = \left\{ y \in \mathbb{R}^d : \liminf_{h \to 0^+} \frac{d(x+hy, K(t+h))}{h} = 0 \right\}.$$

Remark 2.1: $y \in DK(t,x)$ iff there exist sequences

$$(h_k)_k \subseteq \mathbb{R}_+ , \quad (x_k)_k \subseteq \mathbb{R}^d$$

such that

$$\lim_{k\to\infty} h_k = 0, \;\; x_k \in K(t+h_k), \;\; k \in \mathbb{N}, \quad y = \lim_{k\to\infty} \frac{x_k - x}{h_k}.$$

Remark 2.2: Assume that K is upper semicontinuous. Then for all $t \in [0,1)$, $x \in K(t)$,

$$y \in DK(t,x) \quad \Longleftrightarrow \quad (1,y) \in T_{GraphK}(t,x).$$

In particular, if $K(t) = K$, $t \in [0,1]$, then

$$DK(t,x) = T_K(x), \quad \text{for all } t \in [0,1), \, x \in K.$$

Proposition 2.2. *Let* $u : [0,1] \to \mathbb{R}^d$ *be* $K(\cdot)-$ *viable and differentiable. Then*

$$u'(t) \in DK(t, u(t)), \quad \forall\ t \in [0,1).$$

Theorem 2.3. (J.-P. Aubin - A. Cellina (1984)). *Let* F : *Graph*$K \to \mathcal{P}(\mathbb{R}^d)$ *be upper semicontinuous with compact convex values. The following are equivalent:*

(i) for each $x_0 \in K$, *the Initial Value Problem*

$$u'(t) \in F(t, u(t)), \quad t \in [0,1], \quad u(0) = x_0$$

has at least one $K(\cdot)-$ *viable AC solution.*

(ii) $F(t,x) \cap DK(t,x) \neq \emptyset$, $\forall\, t \in [0,1)$, $\forall\, x \in K(t)$.

Now we consider the much more demanding case of **measurable viability** problems.

Theorem 2.4.(H. Frankowska - S. Plaszkacz -T. Tzezuchowski (1995)).

Assume that K **is absolutely continuous from the left** *in the following sense :*

for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ satisfying, for any set $\{(t'_\gamma, t''_\gamma)\}_{\gamma\in\Gamma}$ of non-overlapping subintervals of $[0,1]$ with Γ at most countable index set,

$$\sum_{\gamma\in\Gamma}(t''_\gamma - t'_\gamma) < \delta_\varepsilon \;\Rightarrow\; \sum_{\gamma\in\Gamma} e(K(t'_\gamma), K(t''_\gamma)) < \varepsilon.$$

Moreover, let $F : [0,1]\times\mathbb{R}^d \to \mathcal{P}_{kc}(\mathbb{R}^d)$ be such that

- *F is of* **Carathéodory type**, *i.e.,*

  – for $\mu-$ a.e. $t\in[0,1]$, $x\mapsto F(t,x)$ is continuous.

  –for each $x\in\mathbb{R}^d$, $t\mapsto F(t,x)$ is measurable.

- *for $\mu-$ a.e. $t\in[0,1]$ and all $x\in\mathbb{R}^d$,*

  $$|F(t,x)| \;\le\; h(t),$$

  where $h\in L^1([0,1],\mu)$, $h\ge 0$.

Then the following are equivalent:

(i) for each $x_0 \in K(0)$, *the Initial Value Problem*

$$\begin{cases} u'(t) \in F(t, u(t)), \ \mu - a.e. \ t \in [0,1] \\ \\ u(0) = x_0 \in K(0), \end{cases}$$

has at least one $K(\cdot)-$ *viable absolutely continuous (AC) solution.*

(ii) there exists $E \subset [0,1]$ *with* $\mu(E) = 0$ *such that for every* $t \in [0,1] \setminus E$ *and every* $x \in K(t)$,

$$F(t,x) \cap DK(t,x) \neq \emptyset.$$

A viability result for Stieltjes differential inclusions.

We will need the notion of **contingent** $g-$ **derivative** which extends the notion of contingent derivative mentioned above and it is naturally used in studying viability problems for **Stieltjes differential inclusions.**

Definition 3.1 (R. López Pouso - I. Márquez Albés - J. Rodriguez-Lopez (2020)). Let $t \in [0,1)$, $x \in K(t)$. The **contingent** $g-$ **derivative** of K at (t,x) is the set $D_gK(t,x)$ containing all $y \in \mathbb{R}^d$ with the property

$$\liminf_{h \to 0^+} \frac{d(x + (g(t+h) - g(t))y,\ K(t+h))}{g(t+h) - g(t)} = 0.$$

It is easily checked that $y \in D_gK(t,x)$ iff there exist sequences

$$(h_k)_k \subseteq \mathbb{R}_+ \ , \quad (x_k)_k \subseteq \mathbb{R}^d$$

such that

$$\lim_{k \to \infty} h_k = 0, \ \ x_k \in K(t+h_k), \ \ k \in \mathbb{N}, \quad y = \lim_{k \to \infty} \frac{x_k - x}{g(t+h_k) - g(t)}.$$

Let $g : [0,1] \to \mathbb{R}$ be a left-continuous nondecreasing function and μ_g the Lebesgue-Stieltjes measure generated by g on $[0,1]$.

We aim to prove a **viability result** for the following Initial Value Problem which corresponds to a **Stieltjes differential inclusion:**

$$\begin{cases} u'_g(t) \in F(t, u(t)),\ \mu_g - a.e.\ t \in [0,1) \\ u(t) \in K(t), \quad \forall\, t \in [0,1] \\ u(0) = x_0 \in K(0). \end{cases} \tag{1}$$

Hypotheses:

1). $K : [0,1] \to \mathcal{P}_{cc}(\mathbb{R}^d)$ is g-**absolutely continuous from the left w.r.t. the** $g-$ **topology** in the following sense :

for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ satisfying, for any set $\{(t'_\gamma, t''_\gamma)\}_{\gamma\in\Gamma}$ of non-overlapping subintervals of $[0,1]$ with Γ at most countable index set,

$$\sum_{\gamma\in\Gamma}(g(t''_\gamma) - g(t'_\gamma)) < \delta_\varepsilon \ \Rightarrow\ \sum_{\gamma\in\Gamma} e(K(t'_\gamma), K(t''_\gamma)) < \varepsilon.$$

Moreover, we assume that **at every discontinuity point** $t_1 \in [0,1)$ of g, K has **limit at the right** for the usual topology in the sense of **Hausdorff distance**:

there exists $K(t_1^+) \in \mathcal{P}_{cc}(\mathbb{R}^d)$ such that for every $\varepsilon > 0$ there exists $\delta_{t_1,\varepsilon} > 0$ satisfying, for any $t \in [0,1]$ with $t_1 < t < t_1 + \delta_{t_1,\varepsilon}$,

$$K(t) \subset K(t_1^+) + B_\varepsilon(0) \quad \text{and} \quad K(t_1^+) \subset K(t) + B_\varepsilon(0).$$

2). The map $F : [0,1] \times \mathbb{R}^d \to \mathcal{P}_{cc}(\mathbb{R}^d)$ is **upper semicontinuous** for the product of the g-**topology** of $[0,1]$ with the usual topology of $\mathbb{R}^d$, i.e.,

for each (t,x) and $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that for all (t',x'),

if $|g(t) - g(t')| < \delta_\varepsilon$ and $||x' - x|| < \delta_\varepsilon$, then

$$F(t',x') \subset F(t,x) + B_\varepsilon(0).$$

3). There is $M > 0$ such that for every $(t,x) \in GraphK$, $|F(t,x)| \leq M$.

4). There exists $E \subset [0,1]$ with $\mu_g(E) = 0$ such that for every $t \in [0,1] \setminus E$ and every $x \in K(t)$,

$$F(t,x) \cap D_g K(t,x) \neq \emptyset.$$

The first **key auxiliary result** is the following

Lemma 3.1. *Let* $t_0 \in [0,1) \setminus E$ *and* $\overline{x_0} \in K(t_0)$. *Then for each* $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $v : [t_0, t_0+\delta] \to \mathbb{R}^d$ *a* g*-absolutely continuous function with*

$$v(t_0) = \overline{x_0}\ ;$$

$$v(t) \in K(V^l_{g,\varepsilon}(t)) + B_{(2+M)\varepsilon}(0),\ \forall t \in [t_0, t_0+\delta];$$

$$v(t_0+\delta) \in K(t_0+\delta);$$

$$v'_g(t) \in F((V_{g,\varepsilon}(t) \times (v(t) + B_{(2+M)\varepsilon}(0))) \cap GraphK) + B_\varepsilon(0),$$

$$\mu_g - a.e. \text{ on } [t_0, t_0+\delta).$$

Here

$$V^l_{g,\varepsilon}(t) = \{t' \in [0,t] : g(t) - g(t') < \varepsilon\}$$

and

$$V_{g,\varepsilon}(t) = \{t' \in [0,1] : |g(t) - g(t')| < \varepsilon\}.$$

Sketch of the proof:

We will present the proof only in the case where

- t_0 is a discontinuity point of g
- g is not constant at the right of t_0 .

Due to **hypothesis 4)**, we may find

$$y \in F(t_0, \overline{x}_0) \cap D_g K(t_0, \overline{x}_0),$$

i.e.,

$$y = \lim_{k \to \infty} \frac{x_k - \overline{x_0}}{g(t_0 + h_k) - g(t_0)},$$

for some $h_k \downarrow 0$ and $x_k \in K(t_0 + h_k)$ for all $k \in \mathbb{N}$.

It follows that there exists the limit $\lim\limits_{k \to \infty} x_k$ and

$$y = \frac{\lim\limits_{k \to \infty} x_k - \overline{x_0}}{g(t_0^+) - g(t_0)} .$$

Moreover, **hypothesis 1)** implies that $\lim_{k\to\infty} x_k \in K(t_0^+)$.

Choose $y_k \in F(t_0 + h_k, x_k),\ k \in \mathbb{N}$ and k_ε such that

$$g(t_0 + h_{k_\varepsilon}) - g(t_0^+) < \varepsilon \cdot \min\left(1, \frac{g(t_0^+) - g(t_0)}{1 + M}\right),$$

$$\|x_{k_\varepsilon} - \lim_{k\to\infty} x_k\| < \varepsilon \cdot \min\left(1, \frac{g(t_0^+) - g(t_0)}{1 + M}\right),$$

where $M > 0$ is as postulated in **hypothesis 3)**, i.e.,

$$|F(t,x)| \leq M, \quad \text{for all } (t,x) \in GraphK.$$

Define $v : [t_0, t_0 + h_{k_\varepsilon}] \to \mathbb{R}^d$ as follows:

$$v(t) = \begin{cases} \overline{x_0}, \text{ if } t = t_0 \\ x_{k_\varepsilon} + y_{k_\varepsilon}(g(t) - g(t_0 + h_{k_\varepsilon})), \text{ if } t \in (t_0, t_0 + h_{k_\varepsilon}]. \end{cases}$$

Fix $t \in (t_0, t_0 + h_{k_\varepsilon}]$. Then

$$\|v(t) - \lim_{k\to\infty} x_k\| \leq \varepsilon + \|v(t) - x_{k_\varepsilon}\| \leq$$

$$\leq \quad \varepsilon + \|(g(t) - g(t_0 + h_{k_\varepsilon}))y_{k_\varepsilon}\| \leq \varepsilon + \varepsilon M.$$

Besides, **hypothesis 1)** implies that

$$K(t_0^+) \subset K(V_{g,\varepsilon}^l(t)) + B_\varepsilon(0).$$

Consequently,

$$v(t) \in K(t_0^+) + B_{\varepsilon+\varepsilon M}(0) \subset K(V_{g,\varepsilon}^l(t)) + B_{2\varepsilon+\varepsilon M}(0).$$

Moreover, $v(t_0 + h_{k_\varepsilon}) = x_{k_\varepsilon} \in K(t_0 + h_{k_\varepsilon})$.

To check the last assertion for $t = t_0$ we may write

$$v_g'(t_0) = \frac{v(t_0+) - v(t_0)}{g(t_0+) - g(t_0)} = \frac{x_{k_\varepsilon} - \overline{x_0} + y_{k_\varepsilon}(g(t_0+) - g(t_0 + h_{k_\varepsilon}))}{g(t_0+) - g(t_0)},$$

whence

$$\|v_g'(t_0) - y\| = \left\| \frac{x_{k_\varepsilon} - \lim\limits_{k\to\infty} x_k + y_{k_\varepsilon}(g(t_0+) - g(t_0 + h_{k_\varepsilon}))}{g(t_0+) - g(t_0)} \right\|$$

$$\leq \quad \frac{||x_{k_\varepsilon} - \lim\limits_{k\to\infty} x_k||}{g(t_0+) - g(t_0)} + \frac{||y_{k_\varepsilon}||(g(t_0 + h_{k_\varepsilon}) - g(t_0+))}{g(t_0+) - g(t_0)}$$

$$< \quad \frac{\varepsilon}{1+M} + \frac{\varepsilon M}{1+M} = \varepsilon.$$

Since $y \in F(t_0, \overline{x_0}) = F(t_0, v(t_0))$, the assertion is proved.

Finally, for $t \in (t_0, t_0 + h_{k_\varepsilon})$, we have

$$||v(t) - x_{k_\varepsilon}|| \leq \varepsilon M, \quad 0 \leq g(t_0 + h_{k_\varepsilon}) - g(t) \leq g(t_0 + h_{k_\varepsilon}) - g(t_0^+) < \varepsilon$$

and thus,

$$\begin{aligned} v_g'(t) &= y_{k_\varepsilon} \in F(t_0 + h_{k_\varepsilon}, x_{k_\varepsilon}) \subset \\ &\subset F((V_{g,\varepsilon}(t) \times (v(t) + B_{(2+M)\varepsilon}(0))) \cap GraphK). \end{aligned}$$

$\square$

Now fix $\varepsilon > 0$.

Since K is $g-$ **AC from the left (see hypothesis 1)**, we may choose $r_\varepsilon \in (0, \varepsilon)$ s.t.

for any set $\{(t'_\gamma, t''_\gamma)\}_{\gamma \in \Gamma}$ of non-overlapping subintervals of $[0, 1]$ with Γ at most countable index set,

$$\sum_{\gamma \in \Gamma} (g(t''_\gamma) - g(t'_\gamma)) < r_\varepsilon \;\Rightarrow\; \sum_{\gamma \in \Gamma} e(K(t'_\gamma), K(t''_\gamma)) < \varepsilon. \tag{2}$$

Moreover, since $\mu_g(E) = 0$ (see hypothesis 4), the **regularity** of the measure μ_g implies that there exists E_ε **open** with

$$E \subset E_\varepsilon, \quad \mu_g(E_\varepsilon) < r_\varepsilon. \tag{3}$$

Under the above circumstances, the **second key auxiliary result** is:

Theorem 3.1. *Let* $\varepsilon,\ r_\varepsilon,\ E_\varepsilon$ *be as in (2), (3). Then* $\exists\ v : [0,1] \to \mathbb{R}^d$ *a* $g-$ *AC function satisfying:*

$$v(0) = x_0;$$
$$v(t) \in K(V^l_{g,\varepsilon}(t)) + B_{(2+M)\varepsilon}(0),\ \forall t \in [0,1];$$

$[0,1) = \cup_{i\in I}[\alpha_i,\beta_i),\ \ [\alpha_i,\beta_i),\ i \in I$ *being pairwise disjoint ,*

where I *is at most countable such that for each* $i \in I$,

- *if* $\alpha_i \in E$, *then* $[\alpha_i,\beta_i) \subset E_\varepsilon$ and *either*

g *is constant on* $[\alpha_i,\beta_i)$ *or*

$$\int_{[\alpha_i,\beta_i)} \|v'_g(t)\| dg(t) \le e(K(\alpha_i), K(\beta_i));$$

- *otherwise*, $\mu_g - a.e.\ on\ [\alpha_i,\beta_i)$

$$v'_g(t) \in F((V_{g,\varepsilon}(t) \times (v(t) + B_{(2+M)\varepsilon}(0))) \cap GraphK) + B_\varepsilon(0).$$

Steps of the proof:

Step 1: We consider the set $\mathcal{M}$ of all triplets $(v,\ \delta,\ \{[\alpha_i,\beta_i) : i \in I\}\ )$ with the following properties:

$$\delta \in (0,1] \text{ and } v : [0,\delta] \to \mathbb{R}^d \text{ is } g-\text{AC with } v(0) = x_0;$$
$$v(t) \in K(V^l_{g,\varepsilon}(t)) + B_{(2+M)\varepsilon}(0),\ \forall t \in [0,\delta];$$
$$v(\delta) \in K(\delta);$$
$$[0,\delta) = \cup_{i\in I}[\alpha_i,\beta_i),\ \ [\alpha_i,\beta_i),\ i \in I \text{ being pairwise disjoint },$$
where I is at most countable such that for each $i \in I$,

- if $\alpha_i \in E$, then $[\alpha_i,\beta_i) \subset E_\varepsilon$ and either g is constant on $[\alpha_i,\beta_i)$ or
$$\int_{[\alpha_i,\beta_i)} \|v'_g(t)\| dg(t) \le e(K(\alpha_i), K(\beta_i));$$
- if $\alpha_i \notin E$, $\mu_g - a.e.$ on $[\alpha_i,\beta_i)$
$$v'_g(t) \in F((V_{g,\varepsilon}(t) \times (v(t) + B_{(2+M)\varepsilon}(0))) \cap Graph K) + B_\varepsilon(0).$$

By using (2) combined with **Lemma 3.1** we may prove that the set $\mathcal{M}$ defined above is **nonempty.**

Step 2: We consider on $\mathcal{M}$ a **partial ordering** ($\preceq$) in the following way:

$$\Big(v_1,\ \delta_1,\ \big\{[\alpha_i^1, \beta_i^1) : i \in I_1\big\}\ \Big) \quad \preceq \quad \Big(v_2,\ \delta_2,\ \big\{[\alpha_i^2, \beta_i^2) : i \in I_2\big\}\ \Big)$$

iff

- $\delta_1 \leq \delta_2$ and $v_2\,|_{[0,\delta_1]} = v_1$.
- $\big\{[\alpha_i^1, \beta_i^1) : i \in I_1\big\} \subset \big\{[\alpha_i^2, \beta_i^2) : i \in I_2\big\}$.

By using **Zorn's Lemma** combined with the g - **absolute continuity of** K **from the left**, we prove that the set $\mathcal{M}$ admits a **maximal element**

$$\Big(\overline{v},\ \overline{\delta},\ \{[\alpha_i, \beta_i) : i \in I\}\ \Big) .$$

Step 3: Arguing by contradiction and exploiting **Lemma 3.1**, we prove that $\overline{\delta} = 1$.

$\square$

In the proof of our main result we will also need the following

Lemma 3.2. (R. López Pouso - I. Márquez Albés - J. Rodriguez-Lopez (2020)).

Let

$$v_n : [0,1] \to \mathbb{R}^d, \quad n \in \mathbb{N},$$

be a sequence of $g-$ AC functions, pointwisely convergent to some $v : [0,1] \to \mathbb{R}^d$.

Assume that $\exists\, C > 0$ s.t. $\forall\, n \in \mathbb{N}$,

$$\|(v_n)'_g(t)\| \leq C, \quad \mu_g - \text{a.e. on } [0,1).$$

Then v is $g-$ AC and

$$v'_g(t) \in \bigcap_{p\in\mathbb{N}} \overline{co} \bigcup_{n=p}^{\infty} \{(v_n)'_g(t)\}, \quad \mu_g - \text{a.e. on } [0,1).$$

Moreover, the sequence $(v_n)_n$ has a subsequence uniformly convergent to v.

Now our main viability existence result is the following

Theorem 3.2. *Under assumptions* $1) - 4)$*, the problem (1) has* g*-absolutely continuous solutions on* $[0,1]$*.*

Steps of the proof:

Step 1: Choose $(\varepsilon_n)_n$ be a sequence of positive numbers convergent to 0.

The **left** g- **AC** of K gives rise to a sequence

$$r_{\varepsilon_n} \in (0, \varepsilon_n), \; n \in \mathbb{N}$$

satisfying:

for any set $\{(t'_\gamma, t''_\gamma)\}_{\gamma\in\Gamma}$ of non-overlapping subintervals of $[0,1]$ with Γ at most countable,

$$\sum_{\gamma\in\Gamma}(g(t''_\gamma) - g(t'_\gamma)) < r_{\varepsilon_n} \Longrightarrow \sum_{\gamma\in\Gamma} e(K(t'_\gamma), K(t''_\gamma)) < \varepsilon_n.$$

For each $n \in \mathbb{N}$, choose an **open set** E_n with

$$E \subset E_n \,, \quad \mu_g(E_n) < r_{\varepsilon_n}.$$

We may suppose that $(E_n)_n$ is a decreasing sequence.

By virtue of Theorem 3.1, there exists a sequence
$v_n : [0,1] \to \mathbb{R}^d$, $n \in \mathbb{N}$ of $g-$ AC functions satisfying the following conditions:

$v_n(0) = x_0$;
$v_n(t) \in K(V^l_{g,\varepsilon_n}(t)) + B_{(2+M)\varepsilon_n}(0)$, $\forall t \in [0,1]$;
$[0,1) = \cup_{i \in I^n} [\alpha_i^n, \beta_i^n)$,
where I^n is at most countable such that for each $i \in I^n$,
• if $\alpha_i^n \in E$, then $[\alpha_i^n, \beta_i^n) \subset E_n$ and either
g is constant on $[\alpha_i^n, \beta_i^n)$ or

$$\int_{[\alpha_i^n, \beta_i^n)} \|(v_n)'_g(t)\| dg(t) \leq e(K(\alpha_i^n), K(\beta_i^n))$$

• if $\alpha_i^n \notin E$,

$$(v_n)'_g(t) \in$$

$$\in F((V_{g,\varepsilon_n}(t) \times (v_n(t) + B_{(2+M)\varepsilon_n}(0))) \cap Graph K) + B_{\varepsilon_n}(0), \quad \mu_g - a.e. \text{ on } [\alpha_i^n, \beta_i^n).$$

Due to **hypothesis 3)**, the last one implies

$$||(v_n)'_g(t)|| \leq M + \varepsilon_n, \quad \mu_g - a.e. \text{ on } [\alpha_i^n, \beta_i^n).$$

Step 2: By using the properties of the sequence $((v_n)'_g)_n$ we prove that

$$\sup_n \int_0^1 ||(v_n)'_g(t)|| d\mu_g(t) < \infty$$

and that the sequence $((v_n)'_g)_n$ is μ_g- **uniformly integrable,** i.e.,

for each $\eta > 0$, $\exists\, \theta_\eta > 0$ s.t. for each $g-$ measurable set $A \subset [0,1]$,

$$\mu_g(A) < \theta_\eta \Longrightarrow \int_A ||(v_n)'_g(t)|| d\mu_g(t) < \eta, \quad \forall\ n \in \mathbb{N}.$$

It follows that the sequence $((v_n)'_g)_n$ is **weakly** $L^1_g([0,1])-$ **relatively compact.**

On a subsequence (not relabelled) it **weakly converges** in $L^1_g([0,1])$ to some function $w \in L^1_g([0,1])$.

By the **fundamental theorem of calculus** we infer that

$$v_n(t) \to v(t) = x_0 + \int_{[0,t)} w(t) d\mu_g(t), \quad \forall\ t \in [0,1].$$

Clearly, v is $g-$ AC.

Step 3: By using the properties of the sequence $(v_n)_n$ combined with the **left** $g-$ **AC** of K, we prove that

$$v(t) \in K(t), \quad \text{for every } t \in [0,1].$$

Step 4: By using the properties of the sequence $((v_n)'_g)_n$, we prove that $\exists\, \tilde{C} > 0$ such that for each $N \in \mathbb{N}$,

$$||(v_n)'_g(t)|| \leq \tilde{C}, \quad \mu_g - \text{a.e. in} \quad [0,1) \setminus E_N, \quad \text{for all } n \geq N.$$

Step 5: Fix $N \in \mathbb{N}$.

Step 4 combined with **Lemma 3.2** imply that

$$v'_g(t) \in \bigcap_{p \in \mathbb{N}} \overline{co} \bigcup_{n=p}^{\infty} \{(v_n)'_g(t)\},\ \mu_g - \text{a.e. on } [0,1) \setminus E_N.$$

Moreover, by passing to subsequences, we may assume that

$$v_n \to v, \quad \text{uniformly on} \quad [0,1] \setminus E_N\ .$$

Step 6: We prove that for each $N \in \mathbb{N}$,

$$v_g'(t) \in F(t, v(t)), \quad \mu_g - \text{a.e. on } [0,1) \setminus E_N \ .$$

The proof is based on Step 5,

- the last assertion satisfied by $((v_n)_g')_n$.

- the fact that $F : [0,1] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ is $g-$ **upper semicontinuous with compact convex values (hypothesis 2).**

Step 7: We prove that

$$v_g'(t) \in F(t, v(t)), \quad \mu_g - \text{a.e. on } [0,1).$$

This follows easily from Step 6 combined with the fact that

$$\mu_g \left(\bigcap_n E_n \right) = \lim_n \mu_g(E_n) = 0.$$

$\square$

4. A Filippov-type result.

First, we recall two classical results from the theory of ODEs.

Let $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous satisfying

$$||f(t,x) - f(t,y)|| \leq L(t)||x-y||, \quad \forall\ (t,x),\ (t,y) \in [0,1] \times \mathbb{R}^d,$$

where $L(\cdot) \in L^1_+([0,1])$.

Proposition 4.1. *If u, v are AC solutions of the ODE*

$$x'(t) = f(t,x(t)), \quad \textit{a.e. in} \quad [0,1],$$

then

$$||u(t) - v(t)|| \leq ||u(0) - v(0)||e^{\int_0^t L(s)ds}, \quad \forall\, t \in [0,1].$$

Next proposition extends the above result to the case where v is not necessarily a solution but "differs from being a solution" by some integrable function.

Proposition 4.2. *Let u, $v : [0,1] \to \mathbb{R}^d$ be AC functions s.t.*

$$u'(t) = f(t,u(t)), \quad ||v'(t) - f(t,v(t))|| \leq \gamma(t), \quad \textit{a.e. in} \quad [0,1],$$

where $\gamma(\cdot) \in L^1_+([0,1])$.

Then $\forall\, t \in [0,1]$,

$$||u(t) - v(t)|| \leq e^{\int_0^t L(s)ds} \left[||x_0 - v(0)|| \, + \, \int_0^t \gamma(s) e^{-\int_0^s L(\tau)d\tau} ds \right].$$

An extension of Prop. 4.2 to the **multi-valued case** is the celebrated **Filippov lemma:**

Theorem 4.1 (A. Filippov (1967)). *Let* $F : [0,1] \times \mathbb{R}^d \to \mathcal{P}_c(\mathbb{R}^d)$ *be of Caratheodory type s.t.*

$$D(\ F(t,x),\ F(t,y)\ )\ \leq\ L(t)||x-y||,\ \ \forall\ \ (t,x),\ (t,y) \in [0,1] \times \mathbb{R}^d,$$

where $L(\cdot) \in L^1_+([0,1])$*. Moreover, let* $v : [0,1] \to \mathbb{R}^d$ *be an AC function s.t.*

$$d(v'(t),\ F(t,v(t))|| \leq \gamma(t), \quad \text{a.e. in} \quad [0,1],$$

where $\gamma(\cdot) \in L^1_+([0,1])$*. Then* $\forall\ x_0 \in \mathbb{R}^d$*, there exists an AC solution* u *of the IVP*

$$u'(t) \in F(t,u(t)), \quad \text{a.e. in} \quad [0,1], \quad u(0) = x_0$$

s.t.

$$||u(t) - v(t)|| \leq r(t), \quad \forall\, t \in [0,1],$$

where

$$r(t) = e^{\int_0^t L(s)ds}\left[||x_0 - v(0)||\ +\ \int_0^t \gamma(s)e^{-\int_0^s L(\tau)d\tau}ds\right], \quad t \in [0,1].$$

Remark: The function $r(\cdot)$ mentioned above is the unique AC solution of the IVP

$$r'(t) = L(t)r(t) + \gamma(t), \quad \text{a.e. in} \quad [0,1], \quad r(0) = ||x_0 - v(0)||.$$

A characteristic **Filippov -type result for Stieltjes differential inclusions** is recalled below.

Theorem 4.2 (A. Fryszkowski, J. Sadowski (2021)).

Let $F : [0,1] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ *be a compact valued map of Carathéodory-type s.t.*

$$D(\ F(t,x),\ F(t,y)\ )\ \leq\ L(t)||x-y||,\ \ \forall\ \ (t,x),\ (t,y) \in [0,1] \times \mathbb{R}^d,$$

where $L(\cdot) \in L^1_+([0,1], \mu_g)$.

Moreover, let $v : [0,1] \to \mathbb{R}^d$ *be a* $g-$ *AC function s.t.*

$$d(v_g'(t),\ F(t, v(t)) \leq \gamma(t), \quad \mu_g - \text{ a.e. in } \quad [0,1),$$

where $\gamma(\cdot) \in L^1_+([0,1], \mu_g)$.

Then $\forall\, x_0 \in \mathbb{R}^d$, *there exists a* $g-$ *AC solution* u *of the IVP*

$$u_g'(t) \in F(t, u(t)), \quad \mu_g - \text{ a.e. in } \quad [0,1), \quad u(0) = x_0$$

s.t. $\forall\, t \in [0,1]$,

$$||u(t) - v(t)|| \leq e^{\int_{[0,t)} L(s) d\mu_g(s)} \cdot \left(\|x_0 - v(0)\| + \int_{[0,t)} \gamma(s) d\mu_g(s) \right).$$

We derive a **new Filippov-type lemma**, a direct extension of the original one (Th. 4.1.).

Theorem 4.3. (I. Márquez Albés (2021)). Let $L, \gamma \in L^1([0,1], \mu_g)$ such that $\forall t \in [0,1]$,

$$1 + L(t)\Delta g(t) > 0, \quad \text{where } \Delta g(t) = g(t+) - g(t).$$

Then the Stieltjes linear Cauchy problem

$$r_g'(t) = L(t)r(t) + \gamma(t), \;\; \mu_g- \text{ a.e. in } [0,1), \;\; r(0) = r_0$$

has a unique $g-$ AC solution

$$r(t) = e_L(t)\left[r_0 + \int_{[0,t)} \frac{\gamma(s)}{e_L(s)(1 + L(s)\Delta g(s))} d\mu_g(s)\right], \forall t \in [0,1]$$

where

$$e_L(t) = e^{\int_{[0,t)} \tilde{L}(s)dg(s)}$$

with

$$\tilde{L}(t) = \begin{cases} L(t), & \text{if } g \text{ is continuous at } t \\ \dfrac{\log[\,1 + L(t)\Delta g(t)\,]}{\Delta g(t)}, & \text{otherwise.} \end{cases}$$

We consider the following problem

$$\begin{cases} u'_g(t) \in F(t, u(t)), \ \mu_g - a.e.\ t \in [0, 1), \\ \\ u(0) = x_0, \ \ x_0 \in \mathbb{R}^d. \end{cases} \tag{4}$$

Theorem 4.4. *Let $F : [0, 1] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ be compact-valued s.t. for all $s, t \in [0, 1],\ x, y \in \mathbb{R}^d$,*

$$D(F(t, x), F(s, y)) \leq L(\|x - y\| + |g(t) - g(s)|),$$

where L is a positive constant.

Then given a $g-$ AC function $v : [0, 1] \to \mathbb{R}^d$, denoting by

$$d(v'_g(t), F(t, v(t)) = \gamma(t),$$

there exists a solution u of (4) such that

$$\|u(t) - v(t)\| \leq r(t), \quad \forall \ t \in [0, 1]$$

where $r : [0,1] \to \mathbb{R}$ *is the unique* $g-$ *AC solution of the linear* g*-differential problem*

$$
\begin{aligned}
&r'_g(t) = Lr(t) + \gamma(t),\ \mu_g - a.e. \text{ on } [0,1) \\
&r(0) = \|x_0 - v(0)\|.
\end{aligned}
$$

Now we can combine Theorems 4.3 and 4.4 to obtain

Corollary 4.1. *Under the hypotheses of Th.4.4, we have that* $\forall t \in [0,1]$,

$$||u(t) - v(t)|| \ \le \ r(t) =$$

$$= \ \ e_L(t)\left[\|x_0 - v(0)\| + \int_{[0,t)} \frac{\gamma(s)}{e_L(s)(1 + L\Delta g(s))} d\mu_g(s)\right],$$

where

$$e_L(t) = e^{\int_{[0,t)} \tilde{L}(s) d\mu_g(s)}$$

with

$$\tilde{L}(t) = \begin{cases} L, & \text{if } g \text{ is continuous at } t \\ \dfrac{\log[\ 1 + L\Delta g(t)\ ]}{\Delta g(t)}, & \text{otherwise.} \end{cases}$$

Sketch of the proof of Th.4.4:

Let E be a g-null set such that on $[0,1) \setminus E$ the functions v and r are $g-$ differentiable.

Define $K : [0,1] \to \mathcal{P}_{cc}(\mathbb{R}^d)$ by

$$K(t) = v(t) + r(t) \cdot \overline{B(0,1)}.$$

The idea is to prove that F, K satisfy the hypotheses of the viability result in Theorem 3.2.

Hypotheses 1), 2), 3) are easily checked.

Fix $t \in [0,1] \setminus E$ and $x \in K(t)$.

We have to prove that

$$F(t,x) \cap D_g K(t,x) \neq \emptyset.$$

Choose $u \in \overline{B(0,1)}$ such that $x = y(t) + r(t) \cdot u$.

We will present the proof only in the case where

$$t \text{ is a discontinuity point of } g, \quad ||u|| < 1.$$

Based on the definition of the contingent $g-$ derivative we may show that $z \in D_g K(t,x)$ iff

$$\|z - v_g'(t)\| \leq r_g'(t) \cdot \|u\| + \frac{r(t+)}{g(t+) - g(t)} \cdot (1 + \|u\|).$$

Since

$$\gamma(t) = d(v_g'(t),\ F(t, v(t)),$$

we may find $w \in F(t, v(t))$ with

$$\|w - v_g'(t)\| = \gamma(t).$$

Moreover,

$$d(w,\ F(t,x)) \ \leq\ D(\ F(t, v(t)),\ F(t,x)\ )\ \leq\ L||x - v(t)||$$

so, we can also choose $z \in F(t, x)$ such that

$$\|w - z\| \leq L\|x - y(t)\| \leq Lr(t)\|u\|.$$

Hence,

$$\|z - v_g'(t)\| \leq Lr(t)\|u\| + \gamma(t). \tag{5}$$

We may see that

$$\begin{aligned}
\gamma(t) \quad &\leq \quad Lr(t) + \gamma(t) + \frac{r(t)}{g(t+) - g(t)} \\
&= \quad r_g'(t) + \frac{r(t)}{g(t+) - g(t)} \\
&= \quad \frac{r(t+)}{g(t+) - g(t)} \leq \frac{r(t+)}{g(t+) - g(t)} \frac{1 + \|u\|}{1 - \|u\|}
\end{aligned}$$

which implies that

$$\gamma(t) \ \leq \ \gamma(t)||u|| \ + \ \frac{r(t+)}{g(t+)-g(t)}(1+||u||). \tag{6}$$

From (5) and (6) we infer that

$$\begin{aligned} \|z - v_g'(t)\| \quad &\leq \quad (Lr(t)+\gamma(t))\cdot\|u\| + \frac{r(t+)}{g(t+)-g(t)}\cdot(1+\|u\|) \\ &= \quad r_g'(t)\cdot\|u\| + \frac{r(t+)}{g(t+)-g(t)}\cdot(1+\|u\|) \end{aligned}$$

i.e.,

$$z \in F(t,x) \cap D_g K(t,x).$$

$\square$

The estimation of the distance $\|u(t)-v(t)\|$ obtained in Corollary 4.1 is **better** than the one obtained in Th.4.2.

Proposition 4.3. *Let* $L(\cdot),\ \gamma(\cdot) \in L^1_+([0,1],\mu_g)$ *and* $v : [0,1] \to \mathbb{R}^d$ *be* $g-$ *AC. Set*

$$q(t) = e^{\int_{[0,t)} L(s)d\mu_g(s)} \cdot \left[\|x_0 - v(0)\| + \int_{[0,t)} \gamma(s)d\mu_g(s)\right]$$

and

$$r(t) = e_L(t)\left[\|x_0 - v(0)\| + \int_{[0,t)} \frac{\gamma(s)}{e_L(s)(1 + L(t)\Delta g(s))}d\mu_g(s)\right],$$

where

$$x_0 \in \mathbb{R}^d, \quad \Delta g(t) = g(t^+) - g(t), \quad e_L(t) = e^{\int_{[0,t)} \tilde{L}(s)d\mu_g(s)}$$

with

$$\tilde{L}(t) = \begin{cases} L(t), & \text{if } g \text{ is continuous at } t \\ \dfrac{\log[\,1 + L(t)\Delta g(t)\,]}{\Delta g(t)}, & \text{otherwise.} \end{cases}$$

Then $r(t) \le q(t), \quad \forall\ t \in [0,1].$