Penalty method in the study of elliptic variational inequalities*

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Outline

I. A class of elliptic variational-hemivariational inequalities

II. A penalty method
  - general case
  - a new penalty method

III. Application: a frictional contact problem

Elliptic variational-hemivariational inequality

X a reflexive Banach space

$K \subset X$ a nonempty convex closed set

$A \colon X \to X^*$ an operator

$\varphi \colon K \to \mathbb{R}$ a convex function

$j \colon X \to \mathbb{R}$ a locally Lipschitz function

$f \in X^*$

Problem $\mathcal{P}$

Find an element $u \in K$ such that

$$\langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K.$$

$j^0(u; v)$ denotes the generalized (Clarke) directional derivative

Hypotheses

<u>$H(A)$</u>: $A\colon X \to X^*$ is such that

(a) it is pseudomonotone.

(b) there exist $\alpha_A > 0, \beta, \gamma \in \mathbb{R}$ and $u_0 \in K$ such that

$$\langle Av, v - u_0 \rangle \geq \alpha_A \|v\|_X^2 - \beta \|v\|_X - \gamma \quad \text{for all } v \in X.$$

(c) strongly monotone, i.e., there exists $m_A > 0$ such that

$$\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \text{for all } v_1, v_2 \in X.$$

<u>$H(\varphi)$</u> : $\varphi\colon K \to \mathbb{R}$ is convex and lower semicontinuous on K.

$A\colon X \to X^*$ is **pseudomonotone** if and only if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u \rangle \leq 0$ imply $\lim \langle Au_n, u_n - u \rangle = 0$ and $Au_n \to Au$ weakly in X^*.

Hypotheses

<u>$H(j)$</u> : $j \colon X \to \mathbb{R}$ is such that

(a) j is locally Lipschitz.

(b) $\|\partial j(v)\|_{X^*} \le c_0 + c_1 \|v\|_X$ for all $v \in X$ with $c_0, c_1 \ge 0$.

(c) there exists $\alpha_j > 0$ such that

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2$$

for all $v_1, v_2 \in X$.

The Clarke directional derivative of j at x in direction v

$$j^0(x; v) := \limsup_{y \to x, \lambda \searrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}$$

The generalized subdifferential of j at x

$$\partial j(x) := \{\xi \in X^* \mid \langle \xi, v \rangle \le j^0(x; v) \ \text{ for all } v \in X\}$$

Remark

The condition H(j)(c)

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2$$

is equivalent to the so-called **relaxed monotonicity** condition

$$\langle \partial j(v_1) - \partial j(v_2), v_1 - v_2 \rangle \geq -\alpha_j \|v_1 - v_2\|_X^2 \quad \text{for all} \quad v_1, v_2 \in X.$$

If $j \colon X \to \mathbb{R}$ is a convex function, then this condition reduces to monotonicity of the (convex) subdifferential, i.e., $\alpha_j = 0$.

Particular cases of Problem $\mathcal{P}$

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K$$

1. For $j \equiv 0$, Problem $\mathcal{P}$ reduces to the elliptic variational inequality of the first kind of the form

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K$$

J.L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519

M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012

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2. For $j \equiv 0$ and $K = X$ Problem $\mathcal{P}$ reduces to the elliptic variational inequality of the second kind of the form

$$u \in X, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v-u\rangle \quad \text{for all } v \in X$$

H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* **18** (1968), 115–175

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Particular cases of Problem $\mathcal{P}$

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K$$

3. For $j \equiv 0$ and $\varphi \equiv 0$, Problem $\mathcal{P}$ reduces to the elliptic variational inequality of the form

$$u \in K, \quad \langle Au, v-u\rangle \geq \langle f, v-u\rangle \quad \text{for all } v \in K$$

H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* **18** (1968), 115–175

F.E. Browder, Nonlinear monotone operators and convex sets in Banach spaces, *Bull. Amer. Math. Soc.* **71** (1965), 780–785

D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Classics in Applied Mathematics **31**, SIAM, Philadelphia, 2000

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Particular cases of Problem $\mathcal{P}$

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K$$

4. For $\varphi \equiv 0$, $K = X$, Problem $\mathcal{P}$ reduces to the elliptic hemivariational inequality of the form

$$u \in X, \quad \langle Au, v\rangle + j^0(u; v) \geq \langle f, v\rangle \quad \text{for all } v \in X$$

Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995

5. For $j \equiv 0$, $\varphi \equiv 0$ and $K = X$, Problem $\mathcal{P}$ reduces to the elliptic equation

$$u \in X, \ Au = f.$$

Existence result

Theorem 1

Under the hypotheses $H(A)$, $H(\varphi)$, $H(j)$, $H(K, f)$ and the smallness condition

$$\alpha_j < m_A$$

Problem $\mathcal{P}$ has a unique solution.

S. Migórski, A. Ochal and M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elasticity* **127** (2017), 151–178.

Sketch of the proof

1. Problem $\mathcal{P}$ is equivalent to: find $u \in X$ such that

$$\langle Au, v-u\rangle + \widetilde{\varphi}(v) - \widetilde{\varphi}(u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all } v \in X,$$

where $\widetilde{\varphi}\colon X \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$\widetilde{\varphi}(v) = \begin{cases} \varphi(v) & \text{if } v \in K, \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for} \quad v \in X.$$

$\widetilde{\varphi}$ is proper, convex, lower semicontinuous.

2. Consider the following problem: find $u \in X$ such that

$$Au + \partial j(u) + \partial \widetilde{\varphi}(u) \ni f.$$

3. Introduce $T_1,\ T_2 \colon X \to 2^{X^*}$ such that

$$T_1 v = Av + \partial j(v), \quad T_2 v = \partial \widetilde{\varphi}(v) \ \text{ for } v \in X.$$

4. Prove that

T_1 is bounded, u_0-coercive and pseudomonotone,

$T_2 = \partial \widetilde{\varphi} \colon X \to 2^{X^*}$ is maximal monotone with $D(\partial \widetilde{\varphi}) = K$.

5. Apply the surjectivity result ([Naniewicz, Panagiotopoulos]) to obtain an existence of $u \in X$ a solution to the inclusion.

6. Observe that every solution to the inclusion is a solution to the inequality.
Indeed, let $u \in X$ be such that

$$Au + \xi + \theta = f$$

with $\xi \in \partial\widetilde{\varphi}(u)$ and $\theta \in \partial j(u)$. We have

$$\langle \xi, v - u \rangle \le \widetilde{\varphi}(v) - \widetilde{\varphi}(u) \ \text{ for all } v \in X,$$
$$\langle \theta, v \rangle \le j^0(u; v) \ \text{ for all } v \in X.$$

Hence, we obtain

$$\langle Au, v - u \rangle + \widetilde{\varphi}(v) - \widetilde{\varphi}(u) + j^0(u; v - u) \ge \langle f, v - u \rangle \quad \text{for all } v \in X.$$

7. Uniqueness part:
Let u_1, $u_2 \in K$ be solutions to Problem $\mathcal{P}$, i.e.,

$$\langle Au_1, v - u_1\rangle + \varphi(v) - \varphi(u_1) + j^0(u_1; v - u_1) \geq \langle f, v - u_1\rangle \quad \text{for all } v \in K,$$

$$\langle Au_2, v - u_2\rangle + \varphi(v) - \varphi(u_2) + j^0(u_2; v - u_2) \geq \langle f, v - u_2\rangle \quad \text{for all } v \in K.$$

Taking $v = u_2$ in the first inequality and $v = u_1$ in the second one, adding them, we obtain

$$\langle Au_1 - Au_2, u_2 - u_1\rangle + j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) \geq 0.$$

From the strong monotonicity of A and hypothesis H(j)(c), we have

$$(m_A - \alpha_j)\,\|u_1 - u_2\|_X^2 \leq 0$$

which, due to the similarity condition, implies $u_1 = u_2$. □

Penalty operator

Definition

A single-valued operator $P\colon X \to X^*$ is said to be **a penalty operator** of K if P is bounded, demicontinuous, monotone and
$K = \{x \in X \mid Px = 0_X\}$.

D. Pascali, S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff and Noordhoff International Publishers, Alpen aan den Rijn, 1978

Example

Let $J\colon X \to X^*$ be the duality mapping, $P_K\colon X \to X$ be the projection operator on K and I denote the identity map on X. Then the mapping $P = J(I - P_K)$ is a penalty operator of K.

We remark that the duality map $J\colon X \to 2^{X^*}$, defined by

$$Jx = \{\, x^* \in X^* \mid \langle x^*, x\rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2 \,\} \qquad \text{for all } x \in X$$

Penalized problem

For every $\lambda > 0$, we consider the following penalized problem:

Problem $\mathcal{P}_\lambda$

Find an element $u_\lambda \in X$ such that

$$\langle Au_\lambda, v-u_\lambda\rangle + \frac{1}{\lambda}\langle Pu_\lambda, v-u_\lambda\rangle + \varphi(v) - \varphi(u_\lambda) + j^0(u_\lambda; v-u_\lambda) \geq \langle f, v-u_\lambda\rangle$$

for all $v \in X$, where $P\colon X \to X^*$ is the penalty operator of K.

We need the following additional hypotheses.

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that } \limsup j^0(u_n; v-u_n) \leq j^0(u; v-u) \\ \text{for all } v \in X \text{ and } u_n \to u \text{ weakly in } X. \end{cases}$$

Convergence result

Theorem 2

Under our assumptions and $\alpha_j < \min\{\alpha_A, m_A\}$ the following hold

(i) for each $\lambda > 0$, there exists a unique solution $u_\lambda \in X$ to Problem $\mathcal{P}_\lambda$;

(ii) $u_\lambda \to u$ in X, as $\lambda \to 0$, where $u \in K$ is a unique solution to Problem $\mathcal{P}$.

S. Migórski, A. Ochal and M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elasticity* **127** (2017), 151–178.

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- (i) for each $\lambda > 0$, there exists a unique solution $u_\lambda \in X$ to Problem $\mathcal{P}_\lambda$;
- (ii) $u_\lambda \to u$ in X, as $\lambda \to 0$, where $u \in K$ is a unique solution to Problem $\mathcal{P}$.

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Idea of the proof of **(i)**

1. Consider $A_\lambda \colon X \to X^*$ defined by

$$A_\lambda = A + \frac{1}{\lambda} P \text{ for } \lambda > 0.$$

2. The operator A_λ is pseudomonotone, u_0-coercive and strongly monotone.
3. Apply Theorem 1.

Idea of the proof of **(ii)**

1. the sequence $\{u_\lambda\}$ is bounded in X
2. the subsequence $u_\lambda \to \widetilde{u}$ weakly in X, as $\lambda \to 0$ (with $\widetilde{u} \in X$)
3. $\widetilde{u} \in K$ is a solution to Problem $\mathcal{P}$
4. Since Problem $\mathcal{P}$ has a unique solution $u \in K$, we deduce that $\widetilde{u} = u$. This implies that every subsequence of $\{u_\lambda\}$ which converges weakly has the same limit and, therefore, it follows that the whole sequence $\{u_\lambda\}$ converges to u.
5. $u_\lambda \to u$ in X, as $\lambda \to 0$

□

General Questions

a) Given Problem $\mathcal{P}$ with some constraints which has a unique solution u, how to choose the sequence of perturbed Problems $\mathcal{P}_n$ governed by a sequence of parameters $\{\lambda_n\}$?

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b) How to guarantee that the solution u_n of Problem $\mathcal{P}_n$ converges to the solution u of Problem $\mathcal{P}$?

General Questions

a) Given Problem $\mathcal{P}$ with some constraints which has a unique solution u, how to choose the sequence of perturbed Problems $\mathcal{P}_n$ governed by a sequence of parameters $\{\lambda_n\}$?

b) How to guarantee that the solution u_n of Problem $\mathcal{P}_n$ converges to the solution u of Problem $\mathcal{P}$?

We provide necessary and sufficient conditions for the convergence

$$u_n \to u \ \text{ in } X,$$

i.e., introduce a *convergence criterium*.

Elliptic variational inequality

X is a reflexive Banach space endowed with the norm $\|\cdot\|_X$
X^* is its dual space and $\langle\cdot,\cdot\rangle$ denotes the duality pairing mapping

$K \subset X$, $A\colon X \to X^*$, $\varphi\colon X \to \mathbb{R}$ and $f \in X^*$

Problem $\mathcal{P}$

Find u such that $u \in K$ and

$$\langle Au, v-u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1}$$

Hypotheses on the data

(2) K is a nonempty closed convex subset of X.

$$(3)\quad \begin{cases} A\colon X \to X^* \text{ is pseudomonotone and strongly monotone, i.e.:} \\ \text{(a) } A \text{ is bounded and } u_n \rightharpoonup u \text{ in } X,\quad \limsup \langle Au_n, u_n - u\rangle \le 0 \\ \qquad \text{imply} \quad \liminf \langle Au_n, u_n - v\rangle \ge \langle Au, u - v\rangle \quad \forall\, v \in X. \\ \text{(b) there exists } m_A > 0 \text{ such that} \\ \qquad \langle Au - Av, u - v\rangle \ge m_A \|u - v\|_X^2 \quad \forall\, u,\ v \in X. \end{cases}$$

(4) $\varphi\colon X \to \mathbb{R}$ is convex and lower semicontinuous.

(5) $f \in X^*$.

Existence and uniqueness result

Theorem 3

Assume (2)–(5). Then, the inequality

$$(1) \qquad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v-u\rangle \quad \forall\, v \in K$$

has a unique solution $u \in K$.

It is the well known Lions-Stampacchia theorem.

Proof (for instance) in the book: M. Sofonea and S. Migórski, Variational-Hemivariational Inequalities with Applications, 2018 (2024)

Tool 1: variational-hemivariational inequality

Problem $\mathcal{Q}$

Find u such that $u \in X$ and

(6) $\langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(\gamma u; \gamma v - \gamma u) \geq \langle f, v-u\rangle \quad \forall\, v \in X.$

(7) $\gamma\colon X \to Y$ is a linear compact operator, Y is a reflexive Banach space.

$$
(8)\quad \begin{cases} j\colon Y \to \mathbb{R} \text{ is such that:} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) } \|\xi\|_{Y^*} \leq c_0 + c_1 \|v\|_Y \text{ for all } v \in Y,\ \xi \in \partial j(v) \\ \qquad \text{with } c_0, c_1 \geq 0. \\ \text{(c) there exists } \alpha_j \geq 0 \text{ such that for all } v_1, v_2 \in Y \\ \qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_Y^2. \end{cases}
$$

(9) $\alpha_j < m_A$.

VHVIs

Theorem 4

Assume (2)–(5), (7)–(9). Then, the inequality

$$(6)\ \langle Au, v-u\rangle + \varphi(v) - \varphi(u) + j^0(\gamma u; \gamma v - \gamma u) \geq \langle f, v-u\rangle \quad \forall\, v \in X.$$

has a solution $u \in X$.

Proof in the paper: S. Migórski, A. Ochal and M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elasticity* **127** (2017), 151–178

Tool 2: a convergence criterium

Additional assumptions:

$$(10) \qquad \begin{cases} A \text{ is a Lipschitz continuous operator, i.e.,} \\ \exists M_A > 0 \ \|Au - Av\|_{X^*} \leq M_A \|u - v\|_X \qquad \forall\, u,\, v \in X. \end{cases}$$

$$(11) \qquad \begin{cases} \varphi \text{ is a Lipschitz continuous function, i.e.,} \\ \exists L_\varphi > 0 \ |\varphi(v) - \varphi(w)| \leq L_\varphi \|v - w\|_X \qquad \forall\, v,\ w \in X. \end{cases}$$

Remark. If A is strongly monotone and Lipschitz continuous operator, then it is pseudomonotone.

A convergence criterium

Theorem 5

Assume (2)–(5) and (10)–(11), denote by u the solution of the variational inequality

$$(1) \qquad u \in K, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) \ge \langle f, v-u\rangle \quad \forall\, v \in K$$

and let $\{u_n\} \subset X$. Then the following statements are equivalent:

$$(12) \qquad u_n \to u \qquad \text{in } X.$$

$$(13) \quad \begin{cases} \text{(a) } d(u_n, K) \to 0\ ; \\ \text{(b) there exists } 0 \le \varepsilon_n \to 0 \text{ such that} \\ \qquad \langle Au_n, v-u_n\rangle + \varphi(v) - \varphi(u_n) \\ \qquad\qquad +\varepsilon_n(1 + \|v-u_n\|_X) \ge \langle f, v-u_n\rangle \quad \forall\, v \in K,\ n \in \mathbb{N}. \end{cases}$$

C. Gariboldi, A. Ochal, M. Sofonea and D. A. Tarzia, Convergence criterion for elliptic variational inequalities, *Applicable Analysis* **103(10)** (2024), 1810–1830, arXiv:2309.04805

Sketch of the proof of a convergence criterium

i) Any $\{u_n\} \subset X$ which satisfies condition (13) (b) is bounded.
ii) (12) $\Longrightarrow$ (13)
For $n \in \mathbb{N}$ fixed, $d(u_n, K) \leq \|u_n - u\|_X \Longrightarrow$ (13)(a) holds and

$$(14) \quad \langle Au_n, v - u_n \rangle + \varphi(v) - \varphi(u_n) + \varepsilon_n(1 + \|v - u_n\|_X) \geq \langle f, v - u_n \rangle$$

where $\varepsilon_n = \max\{\, M_A, \|Au\|_{X^*} + \|f\|_{X^*} + L_\varphi \,\} \, \|u - u_n\|_X.$

Hence, $\varepsilon_n \to 0$ and condition (13)(b) is satisfied.
iii) (13) $\Longrightarrow$ (12)
Let $u_n = v_n + w_n,\ v_n \in K,\ w_n \in X,\ \|w_n\|_X \to 0$. We deduce that

$$m_A \|u - v_n\|_X^2 \leq (M_A \|w_n\|_X + \varepsilon_n) \|u - v_n\|_X$$
$$+ (\|Au\|_{X^*} + L_\varphi + \varepsilon_n + \|f\|_{X^*} \|w_n\|_X) \|w_n\|_X + \varepsilon_n.$$

Since $\{u_n\}$ is bounded, $\|w_n\|_X \to 0$, $\varepsilon_n \to 0$ we find that $\|u - v_n\|_X \to 0$.
This implies that $v_n \to u$ in X and (14) holds, which concludes the proof.$\square$

A perturbed problem (a penalty method)

Problem $\mathcal{Q}_n$

Find u_n such that $u_n \in X$ and

$$\langle Au_n, v - u_n\rangle + \varphi(v) - \varphi(u_n) + \frac{1}{\lambda_n} j^0(\gamma u_n; \gamma v - \gamma u_n) \geq \langle f, v - u_n\rangle \quad \forall\, v \in X.$$

Additional assumptions:

(15) $$j^0(\gamma u; \gamma v - \gamma u) \leq 0 \qquad \forall\, u \in X,\ v \in K.$$

(16) $$u \in X, \quad j^0(\gamma u; \gamma v - \gamma u) \geq 0 \quad \forall\, v \in K \implies u \in K.$$

(17) $$\lambda_n \to 0.$$

Main theorem

Theorem 6

Assume (2)–(5), (10)–(11), (15)–(17), denote by u the solution of Problem $\mathcal{P}$ and let $\{u_n\}$ be a sequence of elements in X such that, for each $n \in \mathbb{N}$, u_n is a solution of Problem $\mathcal{Q}_n$. Then, $u_n \to u$ in X.

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Theorem 6

Assume (2)–(5), (10)–(11), (15)–(17), denote by u the solution of Problem $\mathcal{P}$ and let $\{u_n\}$ be a sequence of elements in X such that, for each $n \in \mathbb{N}$, u_n is a solution of Problem $\mathcal{Q}_n$. Then, $u_n \to u$ in X.

Sketch of the proof.

i) The sequence $\{u_n\}$ satisfies condition (13)(b).

Main theorem

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Sketch of the proof.

i) The sequence $\{u_n\}$ satisfies condition (13)(b).

ii) Any weakly convergent subsequence of the sequence $\{u_n\}$ satisfies condition (13)(a).

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Sketch of the proof.

i) The sequence $\{u_n\}$ satisfies condition (13)(b).

ii) Any weakly convergent subsequence of the sequence $\{u_n\}$ satisfies condition (13)(a).
iii) Any weakly convergent subsequence of the sequence $\{u_n\}$ converges to the solution u of inequality (1).
(This is a direct consequence of steps i), ii) and Theorem 5 on convergence criterium.)

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Sketch of the proof.

i) The sequence $\{u_n\}$ satisfies condition (13)(b).

ii) Any weakly convergent subsequence of the sequence $\{u_n\}$ satisfies condition (13)(a).
iii) Any weakly convergent subsequence of the sequence $\{u_n\}$ converges to the solution u of inequality (1).
(This is a direct consequence of steps i), ii) and Theorem 5 on convergence criterium.)
iv) The whole sequence $\{u_n\}$ converges to the solution u of inequality (1). □

Motivation

- Nonlinear problems governed by unilateral constraints
  - the Signorini contact problem which describes the equilibrium of an elastic body with a rigid foundation
  - the heat transfer problem across a semipermeable membrane
- The penalty method in the study of constainted problems
  - solving an unconstrainted problem is more convenient from numerical point of view
  - from theoretic point of view the penalty methods establish the link between problems with a different feature (to approach the solution of a contact problem with a rigid foundation by the solution of a contact problem with a deformable foundation, for a small deformability coefficient)

Example: a frictional contact problem

Problem $\mathcal{P}_{\mathcal{M}}$

Find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ such that

$$
\begin{aligned}
&\boldsymbol{\sigma} = \mathcal{F}\varepsilon(\boldsymbol{u}) && \text{in } \Omega,\\
&\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 && \text{in } \Omega,\\
&\boldsymbol{u} = 0 && \text{on } \Gamma_1,\\
&\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 && \text{on } \Gamma_2,\\
&u_\nu \le 0, \quad \sigma_\nu \le 0, \quad u_\nu\,\sigma_\nu = 0 && \text{on } \Gamma_3,\\
&\begin{cases} \|\boldsymbol{\sigma}_\tau\| \le F_b \\ -\boldsymbol{\sigma}_\tau = F_b \frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if } \boldsymbol{u}_\tau \neq 0 \end{cases} && \text{on } \Gamma_3.
\end{aligned}
$$

Here $\boldsymbol{u}$ represents the displacement field, $\mathcal{F}$ is the elasticity operator, $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ denote the density of applied body forces and tractions which act on the body and the surface Γ_2, respectively and F_b is the friction bound.

$$u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}, \quad \sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$$

A weak formulation of a frictional contact problem

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d \;:\; \boldsymbol{v} = 0 \;\text{ on }\; \Gamma_1 \,\}$$

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_Q, \quad Q = L^2(\Omega; \mathbb{R}^{d\times d}_{sym})$$

$$K = \{\, \boldsymbol{v} \in V \;:\; v_\nu \le 0 \;\text{ a.e. on } \Gamma_3 \,\}$$

Problem $\mathcal{P}^c$

Find $\boldsymbol{u}$ such that $\boldsymbol{u} \in K$ and

$$\int_\Omega \mathcal{F}\varepsilon(\boldsymbol{u}) \cdot (\varepsilon(\boldsymbol{v}) - \varepsilon(\boldsymbol{u}))\, dx + \int_{\Gamma_3} F_b(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, da$$
$$\ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \gamma(\boldsymbol{v} - \boldsymbol{u})\, da \quad \forall\, \boldsymbol{v} \in K.$$

Perturbation of Problem $\mathcal{P}^c$

Problem $\mathcal{P}_n^c$

Find $\boldsymbol{u}_n$ such that $\boldsymbol{u}_n \in V$ and

$$\int_\Omega \mathcal{F}\varepsilon(\boldsymbol{u}_n)\cdot(\varepsilon(\boldsymbol{v})-\varepsilon(\boldsymbol{u}_n))\,dx$$

$$+\int_{\Gamma_3} F_b(\|\boldsymbol{v}_\tau\|-\|\boldsymbol{u}_{n\tau}\|)\,da+\frac{1}{\lambda_n}\int_{\Gamma_3} j_\nu^0(\gamma u_{n\nu};\gamma v_\nu-\gamma u_{n\nu})\,da$$

$$\geq \int_\Omega \boldsymbol{f}_0\cdot(\boldsymbol{v}-\boldsymbol{u}_n)\,dx+\int_{\Gamma_2}\boldsymbol{f}_2\cdot\gamma(\boldsymbol{v}-\boldsymbol{u}_n)\,da \quad \forall\, \boldsymbol{v}\in V.$$

Notation

We consider $A\colon V \to V^*$, $\varphi\colon V \to \mathbb{R}$, $j\colon L^2(\Gamma_3) \to \mathbb{R}$, and $\boldsymbol{f} \in V^*$ defined for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\xi \in L^2(\Gamma_3)$ as

$$\begin{aligned}
\langle A\boldsymbol{u}, \boldsymbol{v}\rangle &= \int_\Omega \mathcal{F}\varepsilon(\boldsymbol{u}) \cdot \varepsilon(\boldsymbol{v})\, dx,\\
\varphi(\boldsymbol{v}) &= \int_{\Gamma_3} F_b \|\boldsymbol{v}_\tau\|\, da,\\
j(\xi) &= \int_{\Gamma_3} j_\nu(\xi)\, da,\\
\langle \boldsymbol{f}, \boldsymbol{v}\rangle &= \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \gamma\boldsymbol{v}\, da.
\end{aligned}$$

Reference to abstract problems

Then it is easy to see that

$$\begin{cases} \boldsymbol{u} \text{ is a solution of Problem } \mathcal{P}^c \text{ if and only if} \\ \boldsymbol{u} \in K, \quad \langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u} \rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle \quad \forall\, \boldsymbol{v} \in K. \end{cases}$$

Moreover, for each $n \in \mathbb{N}$, the following equivalence holds:

$$\begin{cases} \boldsymbol{u}_n \text{ is a solution of Problem } \mathcal{P}_n^c \text{ if and only if} \\ \boldsymbol{u}_n \in V, \quad \langle A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n \rangle + \varphi(\boldsymbol{v}) - \varphi(\boldsymbol{u}_n) \\ \qquad\qquad + \frac{1}{\lambda_n} j^0(\gamma \boldsymbol{u}_n; \gamma \boldsymbol{v} - \gamma \boldsymbol{u}_n) \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n \rangle \quad \forall\, \boldsymbol{v} \in V. \end{cases}$$

Additional assumptions

$$(18)\qquad \lambda_n > 0 \quad \forall\, n \in \mathbb{N},\ \ \lambda_n \to 0 \ \text{ as } n \to \infty.$$

$$(19)\quad \left\{\begin{array}{l} j_\nu \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there exists} \\ \qquad\quad e_1 \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, e_1(\cdot)) \in L^1(\Gamma_3). \\ \quad \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \\ \quad \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \le c_{0\nu} + c_{1\nu}|r| \text{ for all } r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \qquad\quad \text{with } c_{0\nu}, c_{1\nu} \ge 0. \\ \quad \text{(d) } j_\nu^0(\boldsymbol{x}, r; -r) \le d_\nu\,(1 + |r|) \ \text{ for all } r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \qquad\quad \text{with } d_\nu \ge 0. \\ \quad \text{(e) } \ j_\nu(\boldsymbol{x}, \cdot) \text{ is regular for a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right.$$

Additional assumptions

$$\lambda_n > 0 \quad \forall\, n \in \mathbb{N},\ \lambda_n \to 0 \ \text{ as } n \to \infty. \tag{18}$$

$$\begin{cases} j_\nu\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there exists} \\ \qquad\quad e_1 \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, e_1(\cdot)) \in L^1(\Gamma_3). \\ \quad \text{(b) } j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3. \\ \quad \text{(c) } |\partial j_\nu(\boldsymbol{x}, r)| \le c_{0\nu} + c_{1\nu}|r| \text{ for all } r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \qquad\quad \text{with } c_{0\nu}, c_{1\nu} \ge 0. \\ \quad \text{(d) } j_\nu^0(\boldsymbol{x}, r; -r) \le d_\nu\,(1 + |r|) \ \text{ for all } r \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \qquad\quad \text{with } d_\nu \ge 0. \\ \quad \text{(e) } \ j_\nu(\boldsymbol{x}, \cdot) \text{ is regular for a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{19}$$

Consider now the following additional conditions.

$$j_\nu^0(\boldsymbol{x}, r; s - r) \le 0 \qquad \forall\, r \in \mathbb{R},\ s \le 0,\ \text{ a.e. } \ \boldsymbol{x} \in \Gamma_3. \tag{20}$$

$$r \in \mathbb{R},\ j_\nu^0(\boldsymbol{x}, r; s - r) \ge 0\ \forall\, s \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 \Longrightarrow r \le 0. \tag{21}$$

Theorem 7

Under "suitable" assumptions, Problem $\mathcal{P}^c$ has a unique solution and, for each $n \in \mathbb{N}$, Problem $\mathcal{P}_n^c$ has at least one solution. Moreover, if $\boldsymbol{u}$ is the solution of Problem $\mathcal{P}^c$ and $\{\boldsymbol{u}_n\}$ is a sequence of elements in V such that, for each $n \in \mathbb{N}$, $\boldsymbol{u}_n$ is a solution of Problem $\mathcal{P}_n^c$, then

$$\boldsymbol{u}_n \to \boldsymbol{u} \qquad \text{in} \quad V \quad \text{as} \quad n \to \infty.$$

Example of contact conditions

Consider the normal compliance contact condition

$$-\sigma_\nu = \frac{1}{\lambda_n} p_\nu(u_\nu) \quad \text{on } \Gamma_3$$

where $p_\nu \colon \mathbb{R} \to \mathbb{R}$ is a prescribed nonnegative continuous function.

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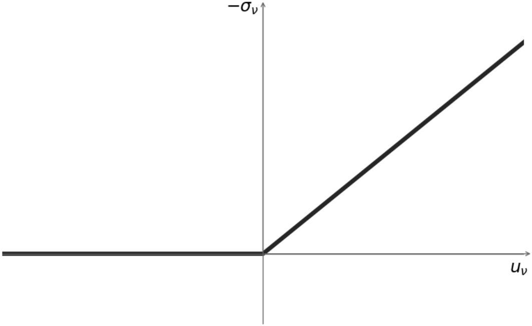

$$p_\nu(u_\nu) = u_\nu^+ = \max\{u_\nu, 0\}$$

Example of contact conditions

Consider the normal compliance contact condition

$$(22) \qquad -\sigma_\nu = \frac{1}{\lambda_n} p_\nu(u_\nu) \quad \text{on } \Gamma_3$$

where $p_\nu \colon \mathbb{R} \to \mathbb{R}$ is a prescribed nonnegative continuous function.

Let $j_\nu \colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$j_\nu(r) = \int_0^r p_\nu(s)\, ds \quad \text{for all} \quad r \in \mathbb{R}.$$

Then, we have

$$\partial j_\nu(r) = \{p_\nu(r)\}$$

for all $r \in \mathbb{R}$ and, therefore, it is easy to see that the contact condition (22) is of the subdifferential form $-\sigma_\nu \in \frac{1}{\lambda_n} \partial j_\nu(u_\nu)$.

Physical interpretation

Remark

The variational inequality (Problem $\mathcal{P}^c$) is obtained by using the Signorini contact conditions in the form without gap, that is

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0 \quad \text{a.e. on} \quad \Gamma_3.$$

Physical interpretation

Remark

The variational inequality (Problem $\mathcal{P}^c$) is obtained by using the Signorini contact conditions in the form without gap, that is

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Problem $\mathcal{P}^c$

Physical interpretation

Remark

The variational inequality (Problem $\mathcal{P}^c$) is obtained by using the Signorini contact conditions in the form without gap, that is

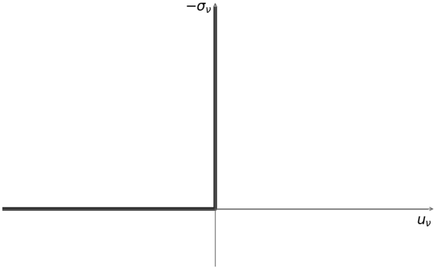
$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu u_\nu = 0 \quad \text{a.e. on} \ \ \Gamma_3.$$

In contrast, the variational-hemivariational inequality (Problem $\mathcal{P}^c_n$) is obtained by using the nonsmooth contact condition

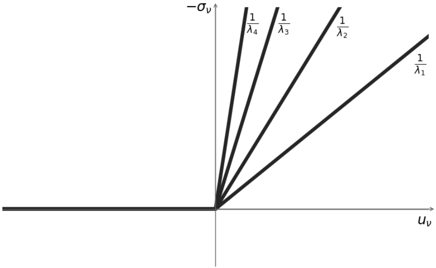
$$-\sigma_\nu \in \frac{1}{\lambda_n}\,\partial j_\nu(u_\nu) \quad \text{a.e. on} \ \ \Gamma_3. \tag{23}$$

This condition represents a contact condition with normal compliance. It describes the contact with a deformable foundation. Here $\frac{1}{\lambda_n}$ can be interpreted as a stiffness coefficient of the foundation. Indeed, in (23) the penetration is allowed but penalized.

Physical interpretation

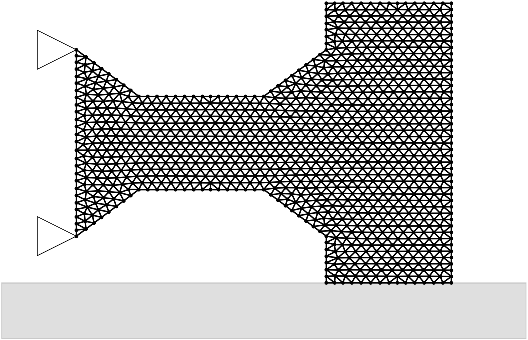


Problem $\mathcal{P}^c$



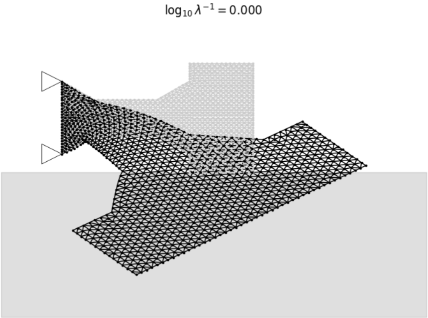
Problem $\mathcal{P}_n^c$: $\lambda_n \to 0$, $\frac{1}{\lambda_n} \to \infty$

Numerical simulations (by Piotr Bartman-Szwarc)



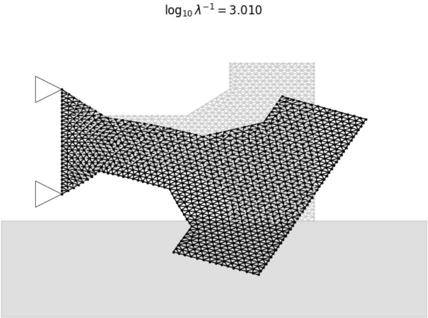
Mesh representation.

Simulation results for selected λ_n values $_{(1)}$



$\log_{10} \lambda^{-1} = 0.000$

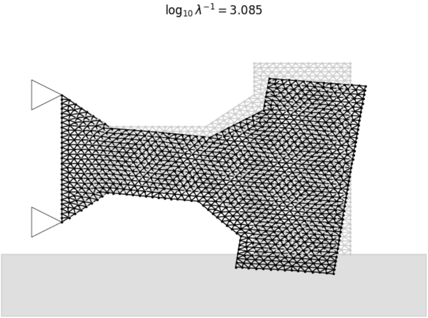
Almost no foundation.



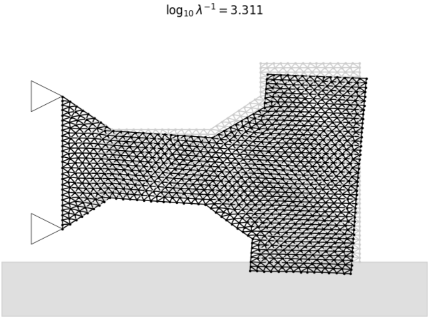
$\log_{10} \lambda^{-1} = 3.010$

Soft foundation.

Simulation results for selected λ_n values $_{(2)}$

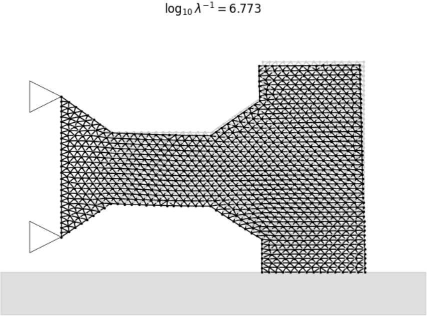


Moderately soft foundation.

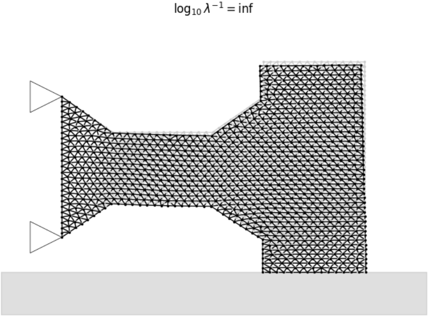


Moderately hard foundation.

Simulation results for selected λ_n values (3)



Hard foundation.



Rigid foundation.

References

P. Bartman-Szwarc, A. Ochal, M. Sofonea and D. A. Tarzia, A convergence criterium for penalty quasivariational inequalities, *Discrete and Continuous Dynamical Systems - Series B* **30**(11) (2025), 4206-4225, doi:10.3934/dcdsb.2025021

C. Gariboldi, A. Ochal, M. Sofonea and D. A. Tarzia, Convergence criterion for elliptic variational inequalities, *Applicable Analysis* **103(10)** (2024), 1810–1830, doi:10.1080/00036811.2023.2268636

S. Migórski, A. Ochal and M. Sofonea, A class of variational - hemivariational inequalities in reflexive Banach spaces, *J. Elasticity* **127** (2017), 151–178, doi:10.1007/s10659-016-9600-7

Thank you for your attention!