Global Bifurcation of Forced Oscillations of ODE's involving the Φ-Laplacian

Alessandro Calamai
Università Politecnica delle Marche, Ancona, Italy

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In this talk I will present some results recently obtained in collaboration with Maria Patrizia Pera and Marco Spadini (Firenze)

- Calamai, A.; Pera, M.P.; Spadini, M., *Branches of forced oscillations for a class of implicit equations involving the Φ-Laplacian*, in: Topological Methods for Delay and Ordinary Differential Equations, P. Amster – P. Benevieri Editors, Advances in Mechanics and Mathematics 51, 2024, pp. 151–166.
- Calamai, A.; Spadini, M., *Carathéodory periodic perturbations of degenerate systems*, Electronic Journal of Differential Equations, art. no. 39 (2024).
- Calamai, A.; Pera, M.P.; Spadini, M., *Forced oscillations for generalized Φ-Laplacian equations with Carathéodory perturbations*, Communications in Contemporary Mathematics, to appear.

Outline

- Setting of the problem
- Continuation results for equations on manifolds
- The degree of a tangent vector field
- Equations with generalized Φ-Laplacian type term

Setting of the problem

Consider the following two equations

$$[\phi(\lambda, x(t), x'(t))]' = \lambda f\big(t, x(t), x'(t), \lambda\big), \quad \lambda \geq 0, \qquad (1)$$

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where

- $g\colon U \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, $U \subseteq \mathbb{R}^n$ open;
- $f\colon \mathbb{R} \times U \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ Carathéodory (or continuous) and T-periodic in t;
- $\phi\colon [0, \infty) \times U \times \mathbb{R}^n \to \mathbb{R}^n$ continuous (generalized Φ-Laplacian): recall the p-Laplacian operator $\Phi(y) := y|y|^{p-2}$.

Carathéodory assumption

Recall that $f\colon \mathbb{R} \times U \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ as in (1) and (2) is said to be T-periodic Carathéodory if:

(F1) $f(t+T,p,q,\lambda) = f(t,p,q,\lambda)$, $\forall (p,q,\lambda) \in U \times \mathbb{R}^n \times [0,\infty)$ and for a.e. $t \in \mathbb{R}$;

(F2) $t \mapsto f(t,p,q,\lambda)$ is measurable, $\forall (p,q,\lambda) \in U \times \mathbb{R}^n \times [0,\infty)$;

(F3) $(p,q,\lambda) \mapsto f(t,p,q,\lambda)$ is continuous, for a.e. $t \in \mathbb{R}$;

(F4) for any compact set $K \subseteq U \times \mathbb{R}^n \times [0,\infty)$, there exists a nonnegative function $\sigma_K \in L^1_T(\mathbb{R})$ such that $|f(t,p,q,\lambda)| \leq \sigma_K(t)$, $\forall (p,q,\lambda) \in K$ and for a.e. $t \in \mathbb{R}$.

$L^1_T(\mathbb{R})$ denote the Banach space, isomorphic to $L^1((0,T),\mathbb{R})$, of the L^1_{loc} maps $\xi\colon \mathbb{R} \to \mathbb{R}$ that are T-periodic, in the sense that $\xi(t) = \xi(t+T)$ for a.e. $t \in \mathbb{R}$.

Goal:

to study the structure of the set of T-periodic solutions of equations (1) and (2) and to prove global continuation results.
Atypical bifurcation results in the sense of Prodi-Ambrosetti.

Usually called *forced oscillations* in the case of second order equations.

Tools: topological methods

- We write equations (1) and (2) as equivalent systems in $\mathbb{R}^{2n}$.
- We use earlier results, obtained by the authors with M. Furi, about periodically perturbed ODEs on differentiable manifolds in the Carathéodory setting.
- We are able to state our results in terms of the well-known Brouwer degree.

Why the Φ-Laplacian? Applications in:

- non-Newtonian fluid theory
- diffusion of flows in porous media
- nonlinear elasticity
- theory of capillary surfaces
- models in glaciology
- models in fluid dynamics

Part I: Continuation results for equations on manifolds and the degree of a tangent vector field

In a series of papers, Furi, Pera and collaborators investigated the set of harmonic solutions of parametrized periodic ODEs on a smooth constraining manifold.
See in particular:

- Furi, M.; Pera, M.P., *Carathéodory periodic perturbations of the zero vector field on manifolds*, Topological methods in nonlinear analysis 10 (1997), n. 1, 79–92.
- Spadini, M., *Harmonic solutions of periodic Carathéodory perturbations of autonomous ODE's on manifolds*, Nonlinear Analysis 41A (2000), 477–487.

In (C.-Spadini, EJDE 2024) we unified the previous results, and extended them to the Carathéodory setting.

Let $\mathcal{M} \subseteq \mathbb{R}^k$ be a smooth manifold.
We consider

$$\dot{\boldsymbol{x}} = G(\boldsymbol{x}) + \lambda F(t, \boldsymbol{x}), \quad \lambda \geq 0,$$

where $G\colon \mathcal{M} \to \mathbb{R}^k$ and $F\colon \mathbb{R} \times \mathcal{M} \to \mathbb{R}^k$ are tangent vector fields on $\mathcal{M}$. That is, $G(p) \in T_p\mathcal{M}$ and $F(t,p) \in T_p\mathcal{M}$ for all $(t,p) \in \mathbb{R} \times \mathcal{M}$. Here $T_p\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at p.

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Moreover, G is continuous and the perturbing term F is Carathéodory and T-periodic in t for some given $T > 0$.

Crucial Remark: $G^{-1}(0)$ need not be compact.

We aim to study the structure of the set of the pairs $(\lambda, \boldsymbol{x})$, where $\boldsymbol{x} : \mathbb{R} \to \mathcal{M}$ is an absolutely continuous T-periodic function such that

$$\dot{\boldsymbol{x}}(t) = G(\boldsymbol{x}(t)) + \lambda F(t, \boldsymbol{x}(t)) \text{ for a.e. } t \in \mathbb{R}.$$

For this purpose we follow a topological approach, based on the concept of *topological degree of a tangent vector field* (also called Euler characteristic).

The degree of a tangent vector field

Let $\mathcal{M} \subseteq \mathbb{R}^k$ be a manifold, and w a tangent vector field on $\mathcal{M}$, that is a map $w : \mathcal{M} \to \mathbb{R}^k$ s.t. $w(p) \in T_p\mathcal{M}$, $\forall p \in \mathcal{M}$.
Let $V \subseteq \mathcal{M}$ be open. If $\{p \in V : w(p) = 0\}$ is compact, then the pair (w, V) is called *admissible* for the degree:
$\to$ *degree (or characteristic) of the vector field w in V*

$$\boxed{\deg(w, V) \in \mathbb{Z}}$$

Sometimes this degree is called *index*, or *Euler characteristic*, or *rotation number*.
For the construction see, e.g., Milnor, Hirsch, Guillemin-Pollack, Furi-Pera.

The regular case

If w is of class C^1, a zero $p \in \mathcal{M}$ of w is said to be *nondegenerate* if the differential $\mathrm{d}w(p) : T_p\mathcal{M} \to \mathbb{R}^k$ is one-to-one.
In this case the *index* of w at p is

$$\mathrm{i}\,(w,p) = \operatorname{sign}\det \mathrm{d}w(p)$$

If (w, V) is *regular* (i.e. w is C^1 with only nondegenerate zeros):

$$\deg(w, V) = \sum_{p \in w^{-1}(0) \cap V} \mathrm{i}\,(w,p).$$

Properties

- When $\mathcal{M} = \mathbb{R}^k$, i.e. $w : \mathbb{R}^k \to \mathbb{R}^k$ and $V \subseteq \mathbb{R}^k$ open,

$\rightarrow$ $\deg(w, V)$ is just the classical Brouwer degree

$$\boxed{\deg_B(w, V, 0)}$$

of the map w on V with respect to zero.

All the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces still hold in the more general context of differentiable manifolds
(*Homotopy invariance, Excision, Additivity, Existence, ...*)

Remark

By the Poincaré-Hopf Theorem, when $\mathcal{M}$ is a compact manifold, $\deg(w, \mathcal{M})$ coincides with the Euler-Poincaré characteristic of $\mathcal{M}$, so it is independent of w.

Observe, in particular, that when $\mathcal{M} = \{p\}$ is a singleton, one has

$$\deg(\mathbf{0}, \mathcal{M}) = 1$$

where $\mathbf{0}$ denotes the zero vector field.

A unified approach

Let $\mathcal{M} = M \times N$, $\boldsymbol{x} = (x, y)$, $G(\boldsymbol{x}) = (0, g(x, y))$ and $F(t, \boldsymbol{x}, \lambda) = (f(t, x, y, \lambda), h(t, x, y, \lambda))$.
Equation

$$\dot{\boldsymbol{x}} = G(\boldsymbol{x}) + \lambda F(t, \boldsymbol{x}, \lambda)$$

becomes the following system of coupled equations:

$$\begin{cases} \dot{x} = \lambda f(t, x, y, \lambda), \\ \dot{y} = g(x, y) + \lambda h(t, x, y, \lambda), \end{cases} \tag{3}$$

depending on the parameter $\lambda \geq 0$, on the product manifold $M \times N$, where $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^s$ are (smooth, boundaryless) differentiable manifolds.

System

$$
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includes, as particular cases, the problems studies in [FuPe97] and [Sp00].

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includes, as particular cases, the problems studies in [FuPe97] and [Sp00].
Assumptions:

• the map $f\colon \mathbb{R} \times M \times N \times [0, \infty) \to \mathbb{R}^k$ is a Carathéodory, T-periodic vector field tangent to M;

• the map $h\colon \mathbb{R} \times M \times N \times [0, \infty) \to \mathbb{R}^s$ is a Carathéodory, T-periodic vector field tangent to N;

• the map $g\colon M \times N \to \mathbb{R}^s$ is a continuous, autonomous vector field tangent to N; that is, $g(p, q) \in T_q N$, $\forall (p, q) \in M \times N$.

Carathéodory assumption

In this context, $f\colon \mathbb{R} \times M \times N \times [0,\infty) \to \mathbb{R}^k$ is said to be a Carathéodory, T-periodic vector field tangent to M if:

(F1) $f(t+T,p,q,\lambda) = f(t,p,q,\lambda) \in T_pM$, $\forall (p,q,\lambda) \in M \times N \times [0,\infty)$ and for a.e. $t \in \mathbb{R}$;

(F2) $t \mapsto f(t,p,q,\lambda)$ is measurable, $\forall (p,q,\lambda) \in M \times N \times [0,\infty)$;

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(F4) for any compact set $K \subseteq M \times N \times [0,\infty)$, there exists a nonnegative function $\sigma_K \in L^1_T(\mathbb{R})$ such that $|f(t,p,q,\lambda)| \leq \sigma_K(t)$, $\forall (p,q,\lambda) \in K$ and for a.e. $t \in \mathbb{R}$.

By a *solution* of system (3) we mean a pair (x, y) of absolutely continuous functions such that the equalities

$$\begin{cases} \dot{x}(t) = \lambda f\big(t, x(t), y(t), \lambda\big), \\ \dot{y}(t) = g(x(t), y(t)) + \lambda h\big(t, x(t), y(t), \lambda\big) \end{cases}$$

hold for a.e. $t \in \mathbb{R}$.

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hold for a.e. $t \in \mathbb{R}$.

• It is convenient to investigate the properties of the T-periodic solutions of (3) in the metric space of the continuous functions.

Notation

We denote by $C_T(M \times N)$ the set of the $M \times N$-valued, T-periodic, continuous functions with the topology induced by the Banach space $C_T(\mathbb{R}^{k+s})$.

Definition

A triple $(\lambda, x, y) \in [0, \infty) \times C_T(M \times N)$ is called a T-triple for (3) if the pair (x, y) is a T-periodic solution of (3). A T-triple (λ, x, y) is called trivial if (x, y) is constant and $\lambda = 0$.

Notation: Given $(p, q) \in M \times N$, by $\overline{p}$ and $\overline{q}$ we denote the functions constantly equal to p and q, respectively.
Note that a T-triple is trivial if and only if it is of the form $(0, \overline{p}, \overline{q})$ with $(p, q) \in g^{-1}(0)$.

Let $w : M \times N \to \mathbb{R}^k$ be the mean value vector field defined by

$$w(p, q) = \frac{1}{T} \int_0^T f(t, p, q, 0)\,\mathrm{d}t$$

and observe that this is a continuous, autonomous vector field tangent to M; that is, $w(p, q) \in T_p M$, $\forall (p, q) \in M \times N$.

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Let now $\nu : M \times N \to \mathbb{R}^{k+s}$ be defined as

$$\nu(p,q) = \left(w(p,q)\ ,\ g(p,q)\right). \tag{4}$$

Note that ν is a vector field tangent to the product manifold $M \times N \subseteq \mathbb{R}^{k+s}$. In fact, w and g are tangent, respectively, to M and N.

Let Ω be an open subset of $[0, \infty) \times C_T(M \times N)$.
Theorem 1 establishes a topological condition in terms of the degree of ν in Ω for the existence of a *connected set of nontrivial T-triples* that in a sense "emanates" from the set of zeros of ν in Ω and is not contained in any compact subset of Ω.

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Theorem 1 establishes a topological condition in terms of the degree of ν in Ω for the existence of a *connected set of nontrivial T-triples* that in a sense "emanates" from the set of zeros of ν in Ω and is not contained in any compact subset of Ω.

Notation

$$\Omega_{M\times N} = \{(p,q)\in M\times N : (0,\overline{p},\overline{q})\in\Omega\}.$$

Considering system (3), we have the following:

Theorem 1 (EJDE 2024)

Let f, g and h be as in system (3), let ν be as in (4), and let Ω be an open subset of $[0,\infty) \times C_T(M \times N)$. Assume that

$$\boxed{\deg\left(\nu, \Omega_{M\times N}\right)}$$

is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-triples in Ω of (3) whose closure in $[0,\infty) \times C_T(M \times N)$ intersects

$$\left\{(0,\overline{p},\overline{q}) \in [0,\infty) \times C_T(M \times N) : (p,q) \in \nu^{-1}(0) \cap \Omega_{M\times N}\right\}$$

and is not contained in any compact subset of Ω. In particular, if $M \times N$ is closed in $\mathbb{R}^{k+s}$ and $\Omega = [0,\infty) \times C_T(M \times N)$, then Γ is unbounded.

Tools in the proof:

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- *topological degree of a tangent vector field*
- *a formula relating the fixed point index of the Poincaré translation operator to the degree of a suitable tangent vector field* (in the spirit of a result due to Krasnosel'skii)
- a point-set topological lemma (Whyburn's type) due to Furi-Pera (deduced from general results by Kuratowski)

Lemma (Furi-Pera, 1993)

Let Y_0 be a compact subset of a locally compact metric space Y. Assume that every compact subset of Y containing Y_0 has nonempty boundary.
Then $Y \setminus Y_0$ contains a connected set whose closure in Y is noncompact and intersects Y_0.

By taking $M = \{p\}$ and $N = \{q\}$ and comparing the notions of T-triple with that of T-pairs in [Sp00] and [FuPe97], respectively, by our Theorem we recover the main results of these papers. That is, for equations equivalent to system (3) when N, resp. M, is a singleton.

In [FuPe97]

$$\dot{x} = \lambda f(t, x) \to \text{ degree of } w(p) = \frac{1}{T}\int_0^T f(t, p)\,\mathrm{d}t$$

In [Sp00]

$$\dot{x} = g(x) + \lambda f(t, x) \to \text{ degree of } g(q)$$

In particular, the vector field $(p, q) \mapsto \big(\mathbf{0}, g(q)\big)$ is not admissible for the degree, unless M is compact.

Part II: Application to equations with generalized Φ-Laplacian type term

Back to equations (1) and (2):

$$[\phi(\lambda, x(t), x'(t))]' = \lambda f(t, x(t), x'(t), \lambda), \ \lambda \geq 0, \tag{1}$$

$$[\phi(\lambda, x(t), x'(t))]' = g(x(t), x'(t)) + \lambda f(t, x(t), x'(t), \lambda), \ \lambda \geq 0, \tag{2}$$

Similar equations but to be handled separately.

Equations with generalized Φ-Laplacian type term

Assumptions on the map $\phi\colon [0,\infty) \times U \times \mathbb{R}^n \to \mathbb{R}^n$

• $\phi(\lambda, p, \cdot)$ is one-to-one and onto, $\forall \lambda \in [0,\infty), p \in U$.

$$u = \phi(\lambda, p, q) \leftrightarrow q = \psi(\lambda, p, u)$$

• The "partial inverse" $\psi(\lambda, p, u)$ is continuous and $\lambda \mapsto \partial_1 \psi(\lambda, p, u)$ is continuous too.

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Moreover:

- $\phi(0, \cdot, \cdot)$ depends only on the third variable, i.e.

$$\phi_0(q) := \phi(0, p, q) \text{ and thus } \psi(0, p, u) = \phi_0^{-1}(u)$$

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In other words,

$$\phi(\lambda, p, q) = \phi_0(q) + \lambda \delta(\lambda, p, q),$$

with ϕ_0 a homeomorphism and δ a perturbation satisfying the above assumptions.

Some examples of a map ϕ as above in the scalar case $n = 1$

Example (1)

$$\phi(\lambda, x, q) = (q + \lambda)|q + \lambda|^{p-2}, \quad p = 2, 3, \dots$$

a perturbation of the p-Laplacian, s.t. the "inverse" ψ is

$$\psi(\lambda, x, u) = \begin{cases} u - \lambda & \text{if} \quad p = 2 \\ \operatorname{sign}(u)|u|^{1/(p-1)} - \lambda, & \text{if} \quad p > 2 \end{cases}$$

Thus, $\partial_1 \psi(\lambda, x, u) = -1$ and the above assumptions are satisfied.

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Thus, $\partial_1 \psi(\lambda, x, u) = -1$ and the above assumptions are satisfied.

Example (2)

$$\phi(\lambda, p, q) = \Phi(q) + \lambda(q - p),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$, a Φ-Laplacian type operator, is a diffeomorphism such that $\Phi'(q) > 0$ for any $q \in \mathbb{R}$. Even in this case one can check that the above assumptions are satisfied.

The set of assumptions on ϕ may seem complicated but in case of more regularity they are quite straightforward

Remark

When the map ϕ is of class C^1, then, by the Implicit Function Theorem, the above conditions are satisfied if, for any $(\lambda, p, q) \in [0, \infty) \times U \times \mathbb{R}^n$, the partial derivative $\partial_3\phi(\lambda, p, q)$ is invertible.

Let us now clarify the meaning of solution of (1) and (2).

Definition

Let $I \subseteq \mathbb{R}$ be an interval, and $\lambda \geq 0$ be given. A function $x \in W^{1,1}_{loc}(I)$ is said to be a solution of (1) or (2) if the function

$$y : I \to \mathbb{R}^n, \quad y(t) = \phi\big(\lambda, x(t), x'(t)\big)$$

belongs to $W^{1,1}_{loc}(I)$ as well and

$y'(t) = \lambda f\big(t, x(t), x'(t), \lambda\big)$ for a.e. $t \in I$, in case (1),
or
$y'(t) = g\big(x(t), x'(t)\big) + \lambda f\big(t, x(t), x'(t), \lambda\big)$ a.e. $t \in I$, in case (2).

T-forced pairs of the equation

Remark

When x is a solution of (1) or (2), the function $y \in W^{1,1}_{loc}(I)$ introduced above is, in particular, continuous. Thus, recalling that we have $x'(t) = \psi\big(\lambda, x(t), y(t)\big)$ for all $t \in I$ and that ψ is continuous, it follows that x is actually C^1.

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The above Remark justifies the following definition.

Definition

A pair $(\lambda, x) \in [0,\infty) \times C^1_T(U)$, such that x is a T-periodic solution of (1) (resp. (2)), is said to be a T-forced pair for (1) (resp. (2)). A T-forced pair (λ, x) is called trivial if x is constant and $\lambda = 0$.

Notation: $\bar{p}$ denotes the constant function $p(t) = p$ for all $t \in \mathbb{R}$.

Global bifurcation branches

Our aim is to prove the existence of bifurcation branches of T-forced pairs of (1) and (2), i.e. connected sets of nontrivial T-forced pairs "emanating" from the set of trivial pairs.

Definition

$p \in U$ is said to be a bifurcation point for (1) or (2) if any neighborhood of $(0, \bar{p})$ in $[0, +\infty) \times C^1_T(U)$ contains nontrivial T-forced pairs (λ, x)

Notation: $\deg(w, V)$ is the Brouwer degree $\boxed{\deg_B(w, V, 0)}$ of the map w on V with respect to zero.

Concerning equation (1), an important role is played by the map $w\colon U \to \mathbb{R}^n$,

$$w(p) := \frac{1}{T}\int_0^T f(t, p, 0, 0)\, dt, \qquad \text{“average wind”}.$$

We have: $p \in U$ bifurcation point $\implies w(p) = 0$

Theorem 2 (global branches of the unperturbed equation)

Let Ω be an open subset of $[0,\infty) \times C^1_T(U)$ and $w\colon U \to \mathbb{R}^n$ the average wind. Assume $\deg(w, \Omega_U) \neq 0$, where Ω_U denotes the open set $\Omega_U = \{p \in U : (0, \overline{p}) \in \Omega\}$. Then, the equation

$$[\phi(\lambda, x(t), x'(t))]' = \lambda f\big(t, x(t), x'(t), \lambda\big), \quad \lambda \geq 0,$$

has a global bifurcation branch Γ, i.e. a connected set of nontrivial T-forced pairs in Ω whose closure in $[0,\infty) \times C^1_T(U)$ intersects the set $\{(0, \overline{p}) \in [0,\infty) \times C^1_T(U) : p \in w^{-1}(0) \cap \Omega_U\}$ and is not contained in any compact subset of Ω.

Theorem 2 (global branches of the unperturbed equation)

Let Ω be an open subset of $[0,\infty) \times C^1_T(U)$ and $w \colon U \to \mathbb{R}^n$ the average wind. Assume $\deg(w, \Omega_U) \neq 0$, where Ω_U denotes the open set $\Omega_U = \{p \in U : (0, \overline{p}) \in \Omega\}$. Then, the equation

$$[\phi(\lambda, x(t), x'(t))]' = \lambda f\left(t, x(t), x'(t), \lambda\right), \quad \lambda \geq 0,$$

has a global bifurcation branch Γ, i.e. a connected set of nontrivial T-forced pairs in Ω whose closure in $[0,\infty) \times C^1_T(U)$ intersects the set $\{(0, \overline{p}) \in [0,\infty) \times C^1_T(U) : p \in w^{-1}(0) \cap \Omega_U\}$ and is not contained in any compact subset of Ω.

Remark

In particular, when $U = \mathbb{R}^n$ and $\Omega = [0,\infty) \times C^1_T(\mathbb{R}^n)$ then Γ is unbounded.

Considering the perturbed equation (2), we have the following:

Theorem 3 (global branches of the perturbed equation)

Let Ω be an open subset of $[0,\infty) \times C^1_T(U)$ and let $\gamma\colon U \to \mathbb{R}^n$ be $\gamma(p) := g(p,0)$. Assume $\deg(\gamma, \Omega_U) \neq 0$. Then, the equation

$$[\phi(\lambda, x(t), x'(t))]' = g(x(t), x'(t)) + \lambda f(t, x(t), x'(t), \lambda), \quad \lambda \geq 0,$$

has a global bifurcation branch Γ, i.e. a connected set of nontrivial T-forced pairs in Ω whose closure in $[0,\infty) \times C^1_T(U)$ intersects the set $\{(0,\overline{p}) \in [0,\infty) \times C^1_T(U) : p \in \gamma^{-1}(0) \cap \Omega_U\}$ and is not contained in any compact subset of Ω. In particular, when $U = \mathbb{R}^n$ and $\Omega = [0,\infty) \times C^1_T(\mathbb{R}^n)$ then Γ is unbounded.

Remark

Observe that the two theorems above have a similar statement and yield similar conclusions.
Yet, it is not possible to consider one as a particular case of the other, even in the case when g vanishes identically so that equation (2) reduces to (1).
In fact the degree of w, that is crucial in Theorem 2, plays no role in Theorem 3 (in principle, it could not be even defined).
Conversely, for equation (1), the degree of γ does not even make sense.

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In fact the degree of w, that is crucial in Theorem 2, plays no role in Theorem 3 (in principle, it could not be even defined).
Conversely, for equation (1), the degree of γ does not even make sense.

Question:

Obtain a unifying result that provides global bifurcation of both equations (1) and (2).

Concluding remarks

Remark

Our results are not directly deducible from Implicit Function Theorem:

• *The Implicit Function Theorem provides information on local properties, while our results are of global nature.*

• *The use of the Implicit Function Theorem requires more regularity than that we assume here (i.e., the involved maps need to be of class C^1).*

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Our results are not directly deducible from Implicit Function Theorem:

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A further line of study

Global bifurcation of T-periodic solutions of equations (1) and (2) when a dependence on delayed arguments is introduced in ϕ and f. Namely if one considers equations of the form

$$[\phi(\lambda, x(t), x(t-r), x'(t))]' = g\big(x(t), x'(t)\big)$$
$$+\lambda f\big(t, x(t), x(t-r), x'(t), x(t-r)\big), \qquad \lambda \geq 0$$

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THANK YOU FOR YOUR ATTENTION!