Rate-dependent Prandtl-Ishlinskii model of hysteresis and its application in engineering

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Overview

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2. Prandtl-Ishlinskii model and rate dependency
3. Inversion formula for rate-dependent Prandtl-Ishlinskii operator

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4. Application in hysteresis compensation

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Play operator: $u \mapsto \Phi_r[u]$

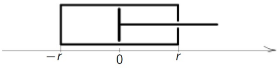

Fig.1: mechanical play

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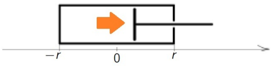

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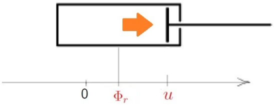

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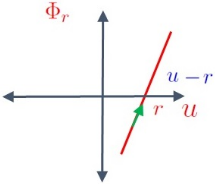

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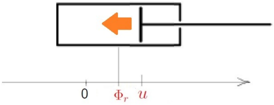

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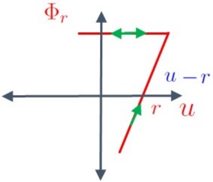

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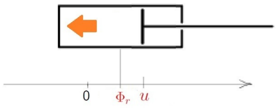

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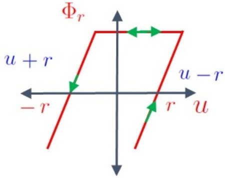

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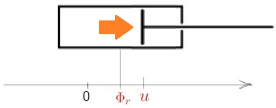

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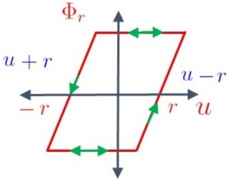

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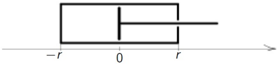
Play operator: $u \mapsto \Phi_r[u] := \xi$



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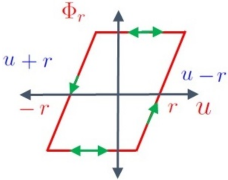

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Variational inequality: (absolutely continuous inputs)

$$\begin{cases} \xi(0) = u(0) - x_0, , \\ |u(t) - \xi(t)| \leq r, & t \in [0, T], \\ \dot{\xi}(t)(u(t) - \xi(t) - z) \geq 0 & \text{a.e. in } [0, T], \ \forall z \in [-r, r] \end{cases}$$

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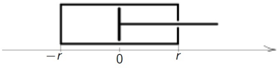

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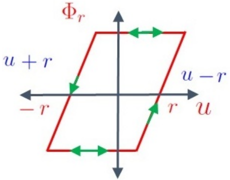

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Variational integral inequality: (inputs regulated and left continuous)

$$\begin{cases} \xi(0) = u(0) - x_0, , & \\ |u(t) - \xi(t)| \le r, & t \in [0, T], \\ \displaystyle\int_0^T (u(t+) - \xi(t+) - z(t))\,\mathrm{d}\xi(t) \ge 0 & \forall z \in G(0, T; [-r, r]) \end{cases}$$

where the integral is understood as the Kurzweil-Stieltjes integral

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Original construction by **Prandtl** (1928) and **Ishlinskii** (1944):

$$\mathcal{P}[v](t) = \sum_{j=1}^{m} a_j \, \mathfrak{p}_{r_j}[v](t), \quad t \in [0, T],$$

where $0 < r_1 < r_2 < \cdots < r_m$ and $a_j \in \mathbb{R}$

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$$\mathcal{P}[v](t) = a_0 v(t) + \int_0^{\infty} h(r) \, \mathfrak{p}_r[v](t) \, \mathrm{d}r$$

with density function $h : \mathbb{R}_+ \to \mathbb{R}$

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$$\mathcal{P}[v](t) = a_0 v(t) + \int_0^\infty h(r)\, \mathfrak{p}_r[v](t)\, \mathrm{d}r = -\int_0^\infty \psi'(r) \frac{\partial}{\partial r} \mathfrak{p}_r[v](t)\, \mathrm{d}r,$$

where $\psi(\rho) = a_0 \rho + \int_0^\rho h(s)(\rho - s)\, \mathrm{d}s$ is the initial loading curve

PI operator with initial loading curve ψ:

$$\mathcal{P}_\psi[v](t) = -\int_0^\infty \psi'(r)\,\frac{\partial}{\partial r}\mathfrak{p}_r[v](t)\,\mathrm{d}r,$$

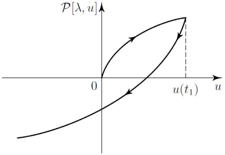

Fig.3: Hysteresis loop of the PI model

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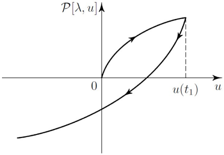

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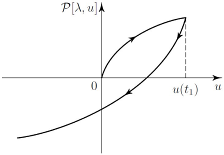

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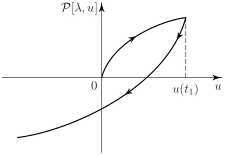

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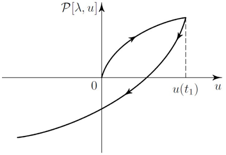

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For $r > 0$ and $x \in [-r, r]$, the play operator

$$\mathfrak{p}_r : \quad v \in W^{1,1}(0,T) \quad \longmapsto \quad \xi \in W^{1,1}(0,T)$$

whose output $\xi(t) = \mathfrak{p}_r[x, v](t)$ is the solution to the variational inequality

$$\begin{cases} \xi(0) = v(0) - x, , & \\ |v(t) - \xi(t)| \leq r, & t \in [0,T], \\ \dot{\xi}(t)(v(t) - \xi(t) - z) \geq 0 & \text{a.e. in } [0,T],\ \forall z \in [-r, r]. \end{cases}$$

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Time dependent play operator

For $\rho \in W^{1,1}(0,T)$ and $x \in [-\rho(0), \rho(0)]$, define

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with $\xi(t) = \mathfrak{p}_{\rho(t)}[x, v](t)$ satisfying

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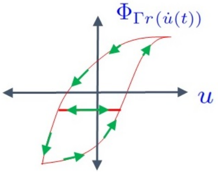
dynamic threshold: $\rho(t) = r + \beta|\dot{u}(t)|$, r constant



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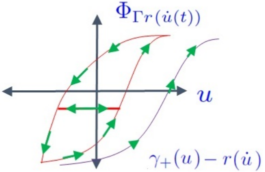

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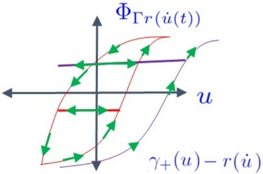

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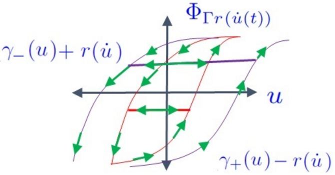

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For $m \in \mathbb{N}$ and $v \in AC[0, T]$ define

$$\mathcal{P}[v](t) = \sum_{j=1}^{m} a_j \, \mathfrak{p}_{r_j(t)}[x_j, v](t), \quad t \in [0, T] \tag{1}$$

where $a_j > 0$, $r_j \in W^{1,1}(0, T)$, $j = 1, \ldots, m$, are such that

$$0 \leq r_1(t) < r_2 < \cdots < r_m(t) \quad \forall t \in [0, T],$$

and $\mathfrak{p}_{r_j}[x_j, v]$ denotes the time-dependent play operator with threshold function r_j, input function v and the initial conditions $x_j \in \mathbb{R}$ satisfying

$$|x_1| \leq r_1(0), \quad |x_{j+1} - x_j| \leq r_{j+1}(0) - r_j(0), \quad j = 1, \ldots, m-1.$$

M. Al Janaideh, P. Krejčí, *An inversion formula for a Prandtl–Ishlinskii operator with time dependent thresholds*, Physica B 406 (2011)

$$\mathcal{P}[x,v](t) = a_0 v(t) + \sum_{i=1}^{m+1} a_i \mathfrak{p}_{\tilde{r}_i(t)}[\tilde{x}_i, v](t) \qquad (1)$$

Inversion formula [M. Al Janaideh, P. Krejčí (2011)]

Assume $0 \leq \tilde{r}_1(t) < \cdots < \tilde{r}_m(t)$ and $\tilde{r}'_{i+1}(t) - \tilde{r}'_i(t) \geq 0$.

Then the inverse of P is given by

$$\mathcal{P}^{-1}[x,w](t) = b_0 w(t) + \sum_{i=1}^{m+1} b_i \mathfrak{p}_{\tilde{s}_i(t)}[\tilde{y}_i, w](t) \qquad (2)$$

where $b_0 = \frac{1}{a_0}$, $b_i = \frac{1}{A_i} - \frac{1}{A_{i-1}}$, with $A_i = \sum_{j=0}^{i} a_j$, and

$$\tilde{s}_1(t) = a_0 \tilde{r}_1(t), \qquad \tilde{s}_{i+1}(t) - \tilde{s}_i(t) = A_i(\tilde{r}_{i+1}(t) - \tilde{r}_i(t)),$$

$$\tilde{y}_1 = a_0 \tilde{x}_1, \qquad \tilde{y}_{i+1} - \tilde{y}_i = A_i(\tilde{x}_{i+1} - \tilde{x}_i)$$

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where $b_0 = \frac{1}{a_0}$, $b_i = \frac{1}{A_i} - \frac{1}{A_{i-1}}$, with $A_i = \sum_{j=0}^{i} a_j$, and

$$\tilde{s}_1(t) = a_0 \tilde{r}_1(t), \qquad \tilde{s}_{i+1}(t) - \tilde{s}_i(t) = A_i(\tilde{r}_{i+1}(t) - \tilde{r}_i(t)),$$

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Inversion formula [M. Al Janaideh, P. Krejčí (2011)]

Assume $0 \le \tilde{r}_1(t) < \cdots < \tilde{r}_m(t)$ and $\tilde{r}'_{i+1}(t) - \tilde{r}'_i(t) \ge 0$.

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Discrete case

$$\mathcal{P}_m[v](t) = \sum_{j=1}^{m} a_j \, \mathfrak{p}_{r_j(t)}[x_j, v](t), \quad t \in [0, T] \tag{1}$$

The classical PI operator (with initial loading curve ψ):

$$\mathcal{P}[v](t) = -\int_0^\infty \psi'(r) \, \frac{\partial}{\partial r} \mathfrak{p}_r[v](t) \, \mathrm{d}r$$

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Rate dependent PI operator with shape function $a_0 I + \varphi$

$$\mathcal{P}_\varphi[x, v](t) = a_0 v(t) - \int_0^\infty \varphi'(r) \frac{\partial}{\partial r} \mathfrak{p}_{r+z(t)}[x(r), v](t) \, \mathrm{d}r \tag{3}$$

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Given a division $0 < \tilde{\rho}_1 < \cdots < \tilde{\rho}_m$, let φ be so that

$$\varphi'(r) = \sum_{i=1}^{m} \hat{\varphi}^*_{i-1} \chi_{[r_{i-1}, r_i)}(r), \qquad \hat{\varphi}^*_{i-1} \in \mathbb{R}$$

The corresponding PI operator can be written as

$$P_{\varphi^*}[x, v](t) = (a_0 + \hat{\varphi}^*_0) v(t) + \sum_{i=1}^{m} a^*_i \mathfrak{p}_{r_i + z(t)}[x(r_i), v](t)$$

with $a^*_i = \hat{\varphi}^*_i - \hat{\varphi}^*_{i-1}$

Given $R > 0$, for an input $v \in W^{1,1}(0,T)$, $|v(t)| \leq R$, let

$$\mathcal{P}_\varphi[x,v](t) = a_0 v(t) - \int_0^\infty \varphi'(r) \frac{\partial}{\partial r} \mathfrak{p}_{r+z(t)}[x(r), v](t)\, \mathrm{d}r \qquad (3)$$

Basic hypothesis:

1. $z \in C(0,T)$ with $z(t) \geq 0$
2. initial value function $x \in W^{1,\infty}(0,\infty)$ such that
$$|x(0)| \leq z(0), \ |x'(r)| \leq 1 \ \text{ a.e.}, \ x(r) = v(0) \ \text{ for } \ r \geq R,$$
3. $\varphi \in W^{1,\infty}(0,\infty)$ such that $\varphi' \in BV_{loc}(0,\infty)$,
$$\varphi(0) = \varphi'(0) = \varphi'(0+) = 0, \quad \sup_{r>0} |\varphi'(r)| =: \bar{\varphi} < a_0$$

M. Al Janaideh, P. Krejčí, G. A. Monteiro, Approximation error bounds for rate-dependent Prandtl-Ishlinskii compensators, Appl. Math. (2023)

Consider the PI operator with shape function $a_0 I + \varphi$ and initial value function $x \in W^{1,\infty}(0,\infty)$ satisfying the basic hypothesis

$$\mathcal{P}_\varphi[x,v](t) = a_0 v(t) - \int_0^\infty \varphi'(r) \frac{\partial}{\partial r} \mathfrak{p}_{r+z(t)}[x(r),v](t)\,\mathrm{d}r \tag{3}$$

Inversion formula [M. Al Janaideh, P. Krejčí, G. A. Monteiro]

For an input $w \in W^{1,1}(0,T)$, let

$$\mathcal{P}_\psi[y,w](t) = \frac{1}{a_0} w(t) - \int_0^\infty \psi'(s) \frac{\partial}{\partial s} \mathfrak{p}_{s+a_0 z(t)}[y(s),w](t)\,\mathrm{d}s \tag{4}$$

where

$$\psi(s) = (a_0 I + \varphi)^{-1}(s) - \frac{s}{a_0} \quad \text{for} \quad s \geq 0.$$

$$y(s) = a_0 x(0) + \int_0^{\frac{1}{a_0}s + \psi(s)} x'(r)(a_0 + \varphi'(r))\,\mathrm{d}r$$

Then the operators P_φ and P_ψ are mutually inverse.

Consider the PI operator with shape function $a_0 I + \varphi$ and initial value function $x \in W^{1,\infty}(0,\infty)$ satisfying the basic hypothesis

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Wafer Scanner illustration

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micropositioning issues $\rightsquigarrow$ smart actuators

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challenge: hysteresis effects

micropositioning issues $\rightsquigarrow$ smart actuators

challenge: hysteresis effects

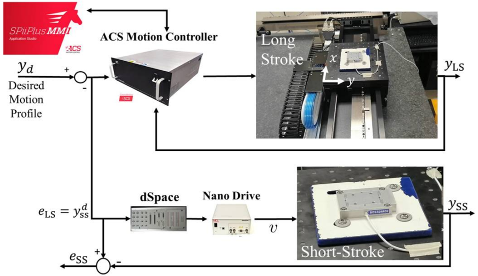

Fig.5: The experimental setup of the dual-stage positioning system

M. Al Janaideh, P. Krejčí, G. A. Monteiro, Memory reduction of rate-dependent Prandtl-Ishlinskii compensators in applications on high-precision motion systems, Physica B (2024) - HMM proceedings micropositioning issues ⇝ smart actuators

challenge: hysteresis effects

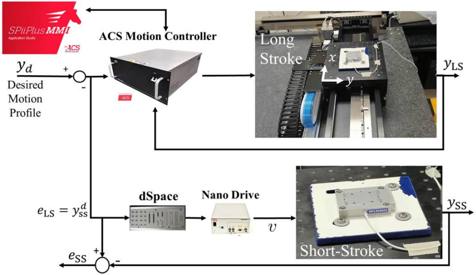

Fig.5: The experimental setup of the dual-stage positioning system

⇝ Short-stroke: piezoelectric actuator & Nano-OP30

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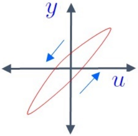
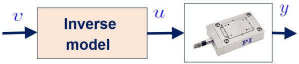
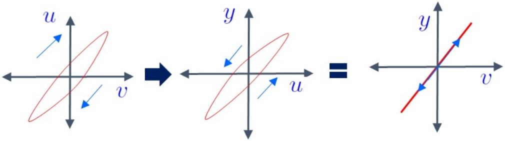
Hysteresis compensation (theory):



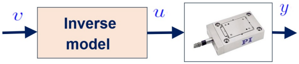
Hysteresis compensation (theory): $P^{-1}_{\varphi_m}[P_{\varphi_m}[u]] = u$

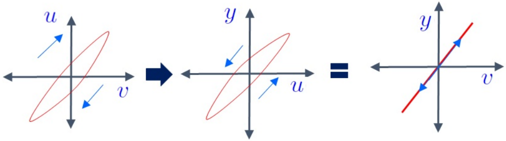




Application in hysteresis compensation

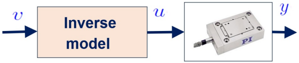
Hysteresis compensation (theory): $P^{-1}_{\varphi_m}[P_{\varphi_m}[u]] = u$

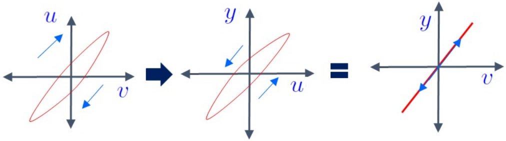




For an initial loading curve φ and a division $0 = r_0 < \cdots < r_m = R$, let φ_i^* be the approximate value of $\varphi(r_i)$ with a measurement error ε: $|\varphi(r_i) - \varphi_i^*| \leq \varepsilon$

Hysteresis compensation (theory): $P_{\varphi_m}^{-1}[P_{\varphi_m}[u]] = u$





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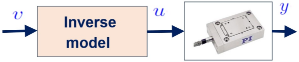
Consider the piecewise linear ('error' approximation) function φ^* and the corresponding PI operator for an input function v

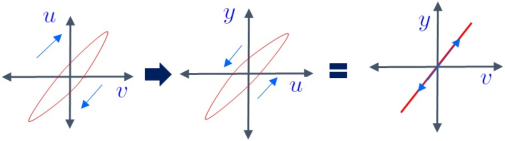
$$P_{\varphi^*}[x, v](t) = a_0 v(t) - \int_0^\infty (\varphi^*)'(r) \frac{\partial}{\partial r} \mathfrak{p}_{r+z(t)}[x(r), v](t)\, \mathrm{d}r$$

where

$$(\varphi^*)'(r) = \sum_{i=1}^{m} \hat{\varphi}_{i-1}^* \chi_{[r_{i-1}, r_i)}(r), \quad \text{with } \hat{\varphi}_{i-1}^* = \frac{\varphi_i^* - \varphi_{i-1}^*}{r_i - r_{i-1}}$$

Hysteresis compensation (theory): $P^{-1}_{\varphi_m}[P_{\varphi_m}[u]] = u$





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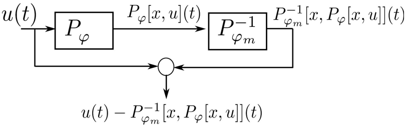
Hysteresis compensation (in practice): $P^{-1}_{\varphi^*}[P_\varphi[u]] \approx u$

Hysteresis compensation diagram:



Approximate compensation error

$$E = \sup \left| u - P^{-1}_{\varphi^*}\left[x, P_\varphi[x,u]\right] \right|$$

where the supremum is taken over $u \in W^{1,1}(0,T)$ with $|u| \leq R$

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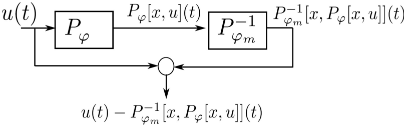

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Error bound [M. Al Janaideh, P. Krejčí, G. A. Monteiro (2023)]

The approximate inversion error, when the operator with continuous thresholds P_φ is replaced with an operator with discrete thresholds P^*_φ, is bounded.

$$E \le \left(\frac{2\varepsilon R}{\min_{i=1,\dots,m} |r_i - r_{i-1}|} + \max_{i=1,\dots,m} |r_i - r_{i-1}| \operatorname*{Var}_{[0,\infty)} \varphi' \right) \sum_{i=0}^{m+1} |b^*_i| \qquad (4)$$

where $b^*_0 = \frac{1}{a_0}$, $\quad b^*_i = \frac{1}{a_0 + \hat{\varphi}^*_{i-1}} - \frac{1}{a_0 + \hat{\varphi}^*_{i-2}}$, and $\hat{\varphi}^*_{i-1} = \frac{\varphi^*_i - \varphi^*_{i-1}}{r_i - r_{i-1}}$

References

M. Al Janaideh, P. Krejčí, *An inversion formula for a Prandtl–Ishlinskii operator with time dependent thresholds*, Physica B 406 (2011)

M. Al Janaideh, P. Krejčí, G. A. Monteiro, *Approximation error bounds for rate-dependent Prandtl-Ishlinskii compensators*, Appl. Math. (2023)

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Thank you!