DaPrato and Grisvard on free interface

based by the joint results by

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Needs of FBPs — *classical approach*

– a good linearization

– maximal regularity estimate/property

– control/stability in time

[a good linearization]

– *quasi-linear* problem

like the Navier-Stokes equations + free boundary

nonlinearity can not be too high

non-Newtonian fluids (power law models)
can not be considered that way

[maximal regularity]

we look for a good estimate for the linear system

one can think about the abstract problem

$$u_t + Au = f \quad \text{with} \quad u|_{t=0} = 0$$

operator A codes the boundary conditions (homogeneous)

for a spacetime $\mathbf{Z}$ we would have

$$\|u_t\|_{\mathbf{Z}} + \|Au\|_{\mathbf{Z}} \lesssim \|f\|_{\mathbf{Z}}$$

[control in time]

– why it is important?

it is not only the question of regularity of interface
we need to avoid self-crossing

Approaches:

– decay in time like ($v \sim e^{-t}$)

– or $L^1(\mathbb{R}_+)$ integrability in time

thanks that we can control

$$\frac{dX}{dt} = v(t, X(t,y)), \qquad X(0,y) = y.$$

[Our approach]

$$\mathbf{Z} \sim L^1(0, T; \mathbf{X})$$

it is not a UMD space, it is not reflexive, so it is impossible to use a general theory of $L^p(0, T; L^q)$-spaces

So we need to give an examples of $\mathbf{X}$, we consider

$$u_t - \Delta u = f \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^d, \qquad u|_{t=0} = 0$$

as $f \in L^1(\mathbb{R}_+; \mathbf{X})$ we have

$$\|u_t\|_{L^1(\mathbb{R}_+;\mathbf{X})} + \|\Delta u\|_{L^1(\mathbb{R}_+;\mathbf{X})} \lesssim \|f\|_{L^1(\mathbb{R}_+;\mathbf{X})}$$

the analysis of this simple example leads to

$$\mathbf{X} \sim \dot{B}^s_{p,1}(\mathbb{R}^d)$$

Welcome to Besov Spaces

To avoid the definition in language of Fourier analysis we give one by the real interpolation

$$B^s_{p,q} == \left(L^p; W^1_p\right)_{s,q}$$

for $s \in (0,1)$ and $q \in [1,\infty]$, and its homogeneous version

$$\dot{B}^s_{p,q} == \left(L^p; \dot{W}^1_p\right)_{s,q}$$

where $\|u\|_{W^1_p} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$ and $\|u\|_{\dot{W}^1_p} = \|\nabla u\|_{L^p}$

or by the integral representation ($q = 1$ the particular case)

$$\|u\|_{\dot{B}^s_{p,1}(\mathbb{R}^d)} \sim \int_{\mathbb{R}^d} \frac{dh}{|h|^{d+s}} \|u(\cdot + h) - u(\cdot)\|_{L^p}$$

and definition by the Fourier transform ...

Help of abstract theory

in order to apply the framework of Besov spaces we are required to modify it to bounded sets setting.

However it is very technical, somehow too technical...

so we found the very classical result of DaPrato and Grisvard [J.Math.Pures Appl. 1975] — 45 years ago ....

DaPrato–Grisvard theorem

Let $-A$ generates a bounded analytic semigroup (e^{-tA}) for $t \geqslant 0$

$$\dot{u} + Au = f(t), \qquad u(0) = x \qquad (ACP)$$

with solution given by

$$u(t) = e^{-tA}x + \int_0^t e^{(t-s)A} f(s) ds$$

We distinguish between $\mathcal{D}(A)$ and $\mathcal{D}(\dot{A})$ introduce

$$\dot{\mathcal{D}}_A(\theta, q) \sim (X, \mathcal{D}(\dot{A}))_{\theta,q}$$

as a real interpolation

Thm. [DaPrato and Grisvard J.Math.Pures Appl. 1975]

Let $\theta \in (0,1)$, $1 \leqslant q < \infty$ and $0 < T \leqslant \infty$.

Then there exists a constant C s.t. for all $f \in L^q(0, T; \mathcal{D}_A(\theta, q))$,

the solution to (ACP) with $x = 0$ satisfies

$u(t) \in \mathcal{D}(A)$ for almost all $0 < t < T$

with homogeneous bound

$$\|Au\|_{L^q(0,T;\dot{\mathcal{D}}_A(\theta,q))} \leqslant C\|f\|_{L^q(0,T;\dot{\mathcal{D}}_A(\theta,q))}$$

What's important $q = 1$, and we have an extension on $x \neq 0$

An application to FBP

We consider

$$v_t + v \cdot \nabla v - \operatorname{div} \mathbf{T}(v,p) = 0 \qquad \Omega_t,$$

$$\operatorname{div} v = 0 \qquad \Omega_t,$$

$$\mathbf{T}(v,p) \cdot n = 0 \qquad \partial\Omega_t,$$

$$v|_{t=0} = v_0, \quad \Omega_t|_{t=0} = \Omega_0$$

$$V = v \cdot n \quad \text{(the speed of free surface)}$$

$$\mathbf{T}(v,p) == \frac{1}{2}(\nabla v + (\nabla v)^T) - p\, Id$$

$$\partial\Omega_t \sim \partial\Omega_0 + \text{"} \int_0^t v(x,t)dt \text{"}$$

What we can prove:

Thm [2020, RD, MH, PBM, PT]
Let $p \in (1,2)$ and Ω_0 be close to $\mathbb{R}^2_+$ in $B^{1+1/p}_{p,1}$. If

$$\|v_0\|_{\dot{B}^{2/p-1}_{p,1}} < c \ \ (small)$$

Then the FBP has a unique global in time solution such that

$$v \in L^\infty(0,\infty; \dot{B}^{2/p-1}_{p,1}(\Omega_t)), \qquad \nabla v \in L^1(0,T; \dot{B}^{2/p}_{p,1}(\Omega_t))$$

$$\partial\Omega_t \in L^\infty(0,\infty; \dot{B}^{1+1/p}_{p,1})$$

since $\dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ we have

$$\int_0^\infty \|\nabla v(t)\|_{L^\infty} dt < \infty, \ \ \text{and} \ \ n \ \ \mathrm{close} \ \ (0,-1)$$

In the talk we remove of dark side of the problem and after some numbers of nontrivial obstacles

the nonlinear system is transformed by
the Lagrangian coordinates defined by the ODEs

$$\frac{d}{dt}X(t,y) = v(t, X(t,y)), \qquad X(0,y) = y$$

in order to define it we need $\nabla v \in L^1(0,T;L^\infty)$

we obtain very technical form of the new system, but ...

we find there nice linearization property we reduce the problem to the kernel

The Stokes system with Neumann boundary conditions

$$\begin{aligned}
&v_t - \operatorname{div} \mathbf{T}(v,p) = f && (0,T)\times\mathbb{R}^2_+,\\
&\operatorname{div} v = 0 \ [\neq 0] && (0,T)\times\mathbb{R}^2_+,\\
&\mathbf{T}(v,p)\cdot n = 0 \ [\neq 0] && (0,T)\times\partial\mathbb{R}^2,\\
&v|_{t=0} = v_0,
\end{aligned}$$

To apply the theory of DaPrato-Grisvard we need just to have the estimate for the resolvent

$$\begin{aligned} &\lambda v - \operatorname{div} \mathbf{T}(v,p) = f && \mathbb{R}^2_+, \\ &\operatorname{div} v = 0 && \mathbb{R}^2_+, \\ &\mathbf{T}(v,p)\cdot n = 0 && \partial\mathbb{R}^2, \end{aligned}$$

with the bound

$$|\lambda|\|u\|_{L^p(\mathbb{R}^2_+)} + \|\nabla^2 u\|_{L^p(\mathbb{R}^2_+)} \leqslant C\|f\|_{L^p(\mathbb{R}^2_+)}$$

and the same for $\dot{W}^1_p$ instead of L^p.

Since

$$\dot{B}^s_{p,1} \sim (L^p, \dot{W}^1_p)_{s,1}$$

So we need to show the bound just for the resolvent in the L^p framework, what is easier... and in the case of the Neumann problem doable..
and we have

$$\|v\|_{L^\infty(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^2_+))} + \|\nabla^2 v\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^2_+))}$$

$$\leqslant C\big(\|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^2_+))} + \|v_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^2_+)}\big)$$

the dark side is located at the boundary terms and in $\operatorname{div} v \neq 0$

there is a need of tedious construction, but in this framework is possible

Dziękuję!!!

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