Invariance and strict invariance for nonlinear evolution problems with applications

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We study evolution problems of the form

$$\begin{cases} \dot{u} \in -Au + f(u), \ t \geqslant 0, \\ u(0) = x_0, \end{cases} \tag{1}$$

where:

- $A : D(A) \multimap X$ is a quasi-m-accretive operator in a Banach space $(X, \|\cdot\|)$;
- $f : \Omega \to X$, with open $\Omega \subset X$, is a continuous map and
- $x_0 \in \Omega \cap \overline{D(A)}$.

Questions

• the behavior of the so-called *integral* solutions u to (1) related to a closed set $K \subset X$.
• the *invariance* of K w.r.t. (1): *any* integral solution u to (1), $u(0) \in K$, remains in K, i.e. $u(t) \in K$ for every t from the maximal interval of existence $[0, \tau_u)$, $0 < \tau_u \leqslant \infty$.
• the *strict invariance*: *all* solutions to (1) stay for $t \in (0, \tau_u)$ in the interior $\mathrm{int}K$ of K.

Concepts of invariance and strict invariance differ from the so-called *viability* of K w.r.t. (1): *there exists* a solution u starting at $x_0 \in K$ and staying there.

Road safety - invariance illustration

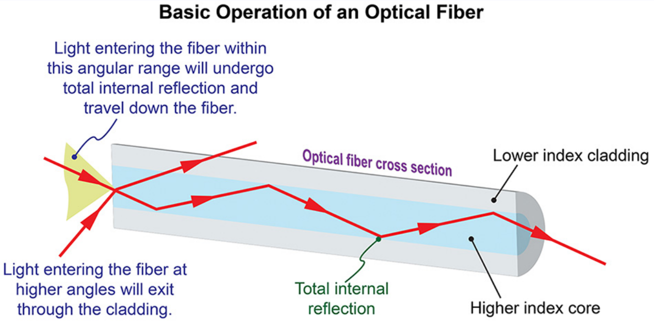
Let us imagine the following situation: we are driving along a highway in a car. Along the edges of the road, safety barriers are installed whose purpose is to prevent a vehicle from leaving the roadway.

Rysunek: Ivariance problem

Rysunek: Strict invariance

The Watt governor is a mechanism used in steam engines to regulate steam pressure and thus speed speed automatically. The pressure should not enter the critical valued, as it may cause the explosion

Real motivation - optical fiber control



We consider the propagation of an optical field envelope

$$A = A(z, x_\perp, t)$$

, where $z \in \mathbb{R}$ is the longitudinal (propagation) variable, $x_\perp \in \mathbb{R}^d$ with $d = 1, 2$ denotes the transverse spatial variables, and $t \in \mathbb{R}$ is the retarded time.

The (generalized) spatio-temporal nonlinear Schroedinger equation reads:

$$i\partial_z A + \frac{1}{2k_0}\Delta_\perp \varrho_\perp(A) + \frac{\beta_2}{2}\partial_t^2 \varrho(A) + \gamma|A|^2 A - i\frac{\alpha}{2}A = 0. \tag{2}$$

Particular results

Assume that $A \equiv 0$:

$$\begin{cases} \dot{u} = f(u), \ t \geqslant 0, \\ u(0) = x_0 \in \Omega, \end{cases} \tag{3}$$

(integral and C^1-solutions to (3) coincide); $K \subset X$ be closed, $K \cap \Omega \neq \emptyset$.

- [Vrabie *et al.*] K is invariant w.r.t. (3) if it is locally viable and f satisfies a one-sided estimate

$$\left[x - y, f(x) - f(y)\right]_+ \leqslant \omega\left(\|x - y\|\right) \text{ for } x \in U \setminus K \text{ and } y \in K \cap \Omega, \tag{4}$$

where $U \subset \Omega$ is a nghbd of $K \cap \Omega$, ω is a uniqueness function and $[\cdot, \cdot]_+$ is the right semi-inner product.

- The viability of K implies that f is tangent to K, i.e.

$$f(x) \in T_K(x) \ \text{ for all } \ x \in \partial K \cap \Omega, \tag{5}$$

where $T_K(x)$ is the so-called contingent cone to K at x. Condition (5) along with (4) imply the invariance, too. In this case a solution starting at $x_0 \in K$ is compared with an approximate solution living in K.

- [Volkmann, Brezis] If K is proximal (i.e. there is a nghbd V of K s.t. $\forall\, x \in V$ $\{y \in K \mid \|x - y\| = d_K(x)\} \neq \emptyset$, then conditions of tangency and (4) (in a slightly relaxed form) imply the so-called exterior tangency condition saying that a lower right Dini derivative of the distance function $d_K = d(\cdot, K) := \inf_{k \in K} \|\cdot - k\|$ at x in the direction $f(x)$ satisfies

$$D_+ d_K(x; f(x)) \leqslant \omega\left(d_K(x)\right) \ \text{ for } \ x \in U. \tag{6}$$

The restrictive assumption concerning proximality of K is actually superfluous if f is Lipschitz in the sense $\|f(x) - f(y)\| \leqslant \omega\left(\|x - y\|\right)$ for $x \in U \setminus K$, $y \in \partial K \cap \Omega$. Condition (6) actually entails the invariance of K.

(1) with a nontrivial m-accretive operator A

- [Pavel, Shi, Carja, Donchev, Postolache] for a linear A and [Bothe] for a nonlinear A. Generally speaking, a closed set K is invariant with respect to (1) if f satisfies condition (4) above and either K is viable or f is A-tangent to K.

- [Cannarsa, Da Prato, Frankowska] for linear A, conditions outside K expressed in terms of Dini derivatives in directions of $-A + f$ were considered.

Two possible approaches to invariance:

1. Controlled 'monotonicity' (or a one-sided Lipschitz) (4) helps to compare a solution of (3) with a one surviving in K.

2. The second approach does not use monotonicity, but instead the Lagrange type stability assumption (6): d_K plays a role of a Lyapunov function of sorts that does not allow solutions to escape from K.

We deal with the 'stability' approach, and get the analogue of the exterior tangency condition (6) in the general situation of (1). The approach directly inspired by papers of Cannarsa *et al.*

In applications the constraint set K is a sublevel

$$K = \{x \in \overline{D(A)} \mid V(x) \leqslant 0\} \tag{7}$$

where $V : X \to \mathbb{R}$ is a locally Lipschitz potential.

Example 1

Any closed set $K \subset \overline{D(A)}$ is represented by $V = d_K$ or $V = \Delta_K := d_K - d_{X \setminus \mathrm{int} K}$ if $\mathrm{int} K \neq \emptyset$.

It is natural to look for invariance conditions in terms of the constraining functional V.

The key ingredient of our approach relies on the analysis of behavior of V along solutions of (1). The concept of the Dini A-directional derivative $D_A V(x; v)$ of V at point $x \in \Omega \cap \overline{D(A)}$ in the direction $v = f(x)$ is useful.

- Assume $A : D(A) \to X$ is quasi m-accretive operator;
- $V : X \to \mathbb{R}$ locally Lipschitz function representing K,
- $x \in \overline{D(A)}$ and $v \in X$. Suppose that $u := u_A(\cdot; x, v)$ is the integral solution to

$$\begin{cases} \dot{u} \in -Au + v, \ t \geqslant 0, \\ u(0) = x_0, \end{cases} \tag{8}$$

Definition 2

By the *A-derivative of V at x in the direction v* we mean the Dini type derivative

$$D_A V(x; v) := \liminf_{h \to 0^+} \frac{(V \circ u)(h) - V(x)}{h} = D_+ \big(V \circ u_A(\cdot; x, v) \big)(0). \tag{9}$$

- $D_A V(x; v)$ measures the rate of growth of V along the integral curve $u = u_A(\cdot; x, v)$.

The following results **without the reflexivity** of X:

Theorem 3

If $\forall\ z \in \partial K \cap \Omega\ \exists$ *nghbd* $U(z) \subset \Omega$ *of* z *and a uniqueness function* ω *such that*

$$D_A V\big(x; f(x)\big) \leqslant \omega\big(V(x)\big) \quad \text{for } x \in \big(U(z) \setminus K\big) \cap \overline{D(A)}, \tag{10}$$

then K *is invariant w.r.t.* (1)*. In particular this holds if*

$$D_A V\big(x; f(x)\big) \leqslant CV(x).$$

A version convenient for applications to PDE of parabolic type:

Theorem 4

Assume: $\forall\ z \in \partial K \cap \Omega\ \exists$ *nghbd* $U(z) \subset \Omega$ *of* z *and a uniqueness function* ω *such that, for any integral solution* $u : [0, \tau) \to X$ *of* (1) *with* $u(0) = z$ *one has*

$$D_+(V \circ u)(t) \leqslant \omega\Big(V\big(u(t)\big)\Big), \quad \text{for a.e. } t \in (0, \tau), \quad \text{whenever } u(t) \in U(z) \setminus K. \tag{11}$$

Then K *is invariant w.r.t.* (1).

Sometimes (e.g. for first-order partial differential equations) the verification of (37) or (39) for integral solutions is not obvious (if possible). Then the following result is suitable.

Theorem 5

Assume that X is reflexive and:

(i) $\forall\, z \in \partial K \cap \Omega$, $\exists\, \delta > 0$ *and a* **slow function** β *such that* $D(z,\delta) \subset \Omega$ *and*

$$\big[x-y, f(x)-f(y)\big]_+ \leqslant \beta\big(\|x-y\|\big) \quad \text{for } x,y \in D(z,\delta); \tag{12}$$

(ii) f *maps bounded sets into bounded ones;*

(iii) $\forall\, z \in \partial K \cap \Omega$ $\exists$ *a nghbd* $U(z) \subset \Omega$ *of* z *such that*

$$D_A V\big(x; f(x)\big) \leqslant \omega\big(V(x)\big) \quad \text{for } x \in \big(U(z) \setminus K\big) \cap D(A), \tag{13}$$

where ω *is a nondecreasing uniqueness function.*

Then K *is invariant w.r.t.* (1).

Condition (12) implies, that integral solutions to (1) starting in a neighborhood of ∂K are locally unique. This is the price for relaxing condition (37).

If X is not reflexive, then the applicable to PDE result holds true:.

Theorem 6

Assume that conditions (i) *and* (ii) *from Theorem 5 are satisfied and that*

(iii)' $\forall\, z \in \partial K \cap \Omega\ \exists$ *nghbd* $U(z) \subset \Omega$ *of* z *and a nondecreasing uniqueness function* ω *such that, for any integral solution* $u : [0,\tau) \to X$ *of* (1) *with* $u(0) \in D(A)$ *and* $u([0,\tau)) \subset U(z)$ *one has*

$$D_+(V \circ u)(t) \leqslant \omega\Big(V\big(u(t)\big)\Big) \ \text{for a.e. } \ t \in (0,\tau) \ \text{ with } \ u(t) \in U(z) \setminus K. \qquad (14)$$

Then K *is invariant with respect to* (1).

Theorems 3 – 6 generalize results from [Cannarsa *et al.*] where a linear operator A was considered, f was assumed to be globally quasi-dissipative. Here A may be nonlinear and f only continuous or locally quasi-dissipative. Moreover, we have come up with the invariance criteria that do not require the reflexivity of X.

The necessity of exterior tangency conditions.

Theorem 7

Assume assumptions (i) and (ii) of Theorem 5 are satisfied, $\overline{D(A)} = X$.

• *If a closed set K given by* (7) *is invariant w.r.t.* (1), *then* (iii) *of Theorem* 6 *holds for some nondecreasing uniqueness function ω.*

• *More precisely, every $z \in \partial K \cap \Omega$ has a neighborhood $U(z) \subset \Omega$ such that for any integral solution $u : [0, \tau_0) \to X$ of* (1) *if $u(0) \in U(z)$, then*

$$D_+(d_K \circ u)(t) \leqslant \omega\big(d_K(u(t))\big)$$

for all sufficiently small $t \in [0, \tau_0)$.

• *Moreover conditions* (37) *and* (iii) *from Theorem 5 are satisfied, too; i.e. each point $z \in \partial K \cap \Omega$ has a neighborhood $U(z)$ and some nondecreasing uniqueness function ω such that*

$$D_A d_K\big(x; f(x)\big) \leqslant \omega\big(d_K(x)\big) \ \text{ for } \ x \in U(z) \setminus K.$$

Exterior tangency conditions like (37), (13) or (14) appear to be necessary when V is a distance function d_K, even without the reflexivity assumption on X. This result corresponds to results of Cannarsa, Da Prato and Frankowska, where A is linear.

Recall (7) and assume that

$$\mathrm{int}K = \{x \in X \mid V(x) < 0\} \neq \emptyset. \tag{15}$$

Theorem 8

If $\forall\ z \in \partial K \cap \Omega\ \exists$ nghbd $U(z) \subset \Omega$ of z and a uniqueness function ω such that

$$D_A V(x; f(x)) \leqslant \omega\big(-V(x)\big) \ \text{ for } \ x \in \big(U(z) \cap \mathrm{int}K\big) \cap \overline{D(A)}, \tag{16}$$

then $\mathrm{int}K$ is invariant with respect to (1).

Theorem 9

If X is reflexive, assumptions (i) and (ii) from Theorem 5 hold and

(iv) $\forall\ z \in \partial K \cap \Omega\ \exists$ nghb $U(z) \subset \Omega$ of z such that

$$D_A V(x; f(x)) \leqslant \omega\big(-V(x)\big) \ \text{ for } \ x \in \big(U(z) \cap \mathrm{int}K\big) \cap D(A), \tag{17}$$

where ω is a nondecreasing uniqueness function.

Then $\mathrm{int}K$ is invariant with respect to (1).

Results: Strict invariance

Introducing additionally the inwardness condition (19) we obtain the following strict invariance results.

Theorem 10

If for every $z \in \partial K \cap \Omega$ there are a neighborhood $U(z) \subset \Omega$ of z and a uniqueness function ω such that

$$D_A V(x; f(x)) \leqslant \omega(|V(x)|) \quad \text{for } x \in U(z), \tag{18}$$

$$D_A V(x; f(x)) < 0 \quad \text{for } x \in U(z) \cap \partial K, \tag{19}$$

then K is strictly invariant with respect to (1).

Theorem 11

Assume that X is reflexive, assumptions (i) and (ii) of Theorem 5 hold,

(v) for every $z \in \partial K \cap \Omega$ there is a neighborhood $U(z) \subset \Omega$ of z such that

$$D_A V(x; f(x)) \leqslant \omega(|V(x)|) \quad \text{for } x \in U(z) \cap D(A), \tag{20}$$

where ω is a nondecreasing uniqueness function,

and the strong inwardness condition (19) holds true.

Then K is strictly invariant with respect to (1).

Notation

Given a metric space (Y,d), $K \subset Y$ and $x \in Y$,

- $d_K(x) := \inf_{y\in K} d(x,y)$; by $\overline{K}$, $\mathrm{int}K$ and ∂K we denote the closure, the interior and the boundary of K;

- $B(x,r)$ (resp. $D(x,r)$) is the open (resp. closed) ball around $x \in Y$ of radius $r > 0$; $B(K,r)$ denotes the r-neighborhood of K, i.e. $B(K,r) = \{y \in Y \mid d_K(x) < r\}$.

- In what follows $(X, \|\cdot\|)$ is a real Banach space, X^* stands for the dual of X; $\langle\cdot,\cdot\rangle$ is the conjugation duality in X, i.e. if $x \in X$, $p \in X^*$, then $\langle p,x\rangle := p(x)$; by default X^* is normed.

- The use of function spaces (L^p, Sobolev $W^{k,p}$, etc.), linear (unbounded in general) operators in Banach spaces, C_0 semigroups is standard.

- In particular, given real functions u, v, we put $u \vee v := \max\{u,v\}$, $u \wedge v := \min\{u,v\}$, $u_\pm = (\pm u) \vee 0$.

Definition 12 (Dini derivatives)

For a function $u : (a, b) \to \mathbb{R}$ one defines the *Dini derivatives*

$$D_{\pm} u(t) := \liminf_{h \to 0^{\pm}} \frac{u(t+h) - u(t)}{h}, \quad D^{\pm} u(t) := \limsup_{h \to 0^{\pm}} \frac{u(t+h) - u(t)}{h}, \qquad t \in (a, b).$$

If $f : \Omega \to \mathbb{R}$, where $\Omega \subset X$ is open, $x \in \Omega$ and $v \in X$, then the *Dini directional derivatives at x in the direction v* are given by

$$D_{\pm} f(x; v) := \liminf_{h \to 0^{\pm}} \frac{f(x+hv) - f(x)}{h}, \qquad D^{\pm} f(x; v) := \limsup_{h \to 0^{\pm}} \frac{f(x+hv) - f(x)}{h}.$$

If the function f is convex, then $D_{-} f(x; v) = D^{-} f(x; v)$ and $D_{+} f(x; v) = D^{+} f(x; v)$.

Definition 13 (semi inner products)

If $x, y \in X$, then we put

$$[x, y]_{\pm} := \lim_{h \to 0^{\pm}} \frac{\|x + hy\| - \|x\|}{h},$$

i.e. $[x, y]_{\pm}$ is the lower right (resp. left) Dini directional derivative of $\| \cdot \|$ at x in the direction of y

Uwagi do definicji

Let

$$\langle x, y\rangle_{\pm} := \lim_{h\to 0^{\pm}} \frac{\|x+hy\|^2 - \|x\|^2}{2h}.$$

Then

$$\langle x, y\rangle_{\pm} = \|x\|[x, y]_{\pm}.$$

Let

$$J(x) := \{p \in X^* \mid \langle p, x\rangle = \|x\|^2 = \|p\|^2\}, \;\; x \in X,$$

be the duality map. Then

$$\langle x, y\rangle_{+} = \sup_{p\in J(x)} \langle p, y\rangle, \;\; \langle x, y\rangle_{-} = \inf_{p\in J(x)} \langle p, y\rangle.$$

Lemma 14

If a function $u : [a, b] \to X$ is left (right) differentiable at $t_0 \in (a, b]$ ($t_0 \in [a, b)$), then $\|u(\cdot)\|$ and $\|u(\cdot)\|^2$ is left(right) differentiable at t_0 and

$$\frac{d^{\pm}}{dt}\|u(t_0)\| = \left[u(t_0), \frac{d^{\pm}}{dt}u(t_0)\right]_{\pm},$$

$$\frac{1}{2}\frac{d^{\pm}}{dt}\|u(t_0)\|^2 = \|u(t_0)\|\frac{d^{\pm}}{dt}\|u(t_0)\| = \left\langle u(t_0), \frac{d^{\pm}}{dt}u(t_0)\right\rangle_{\pm}.$$

Definition 15 (Uniqueness function)

A continuous function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$ is a *uniqueness* (or a *Perron*) function if the only nonnegative solution to the problem $\dot{u} = \omega(u)$ on an interval $[0, \tau)$, $0 < \tau \leqslant \infty$, such that $u(0) = 0$ is the null function

By a maximality of a solution x we mean that $u(t) \leqslant x(t)$ for every other solution u and every t in a common interval of existence.

Lemma 16

Let $\omega : [0, \infty) \to [0, \infty)$ be continuous and $\tau > 0$.

(1) *If x is the maximal solution $\dot{x} = \omega(x)$ on $[0, \tau]$, $u : [0, \tau] \to [0, \infty)$ is continuous, $u(0) \leqslant x(0)$ and $Du \leqslant \omega(u)$ on $(0, \tau)$, where D stands for any Dini derivative, then $u(t) \leqslant x(t)$ for $t \in [0, \tau]$. In particular if ω is a uniqueness function and $u(0) = 0$, then $u \equiv 0$ on $[0, \tau]$.*

(2) *If ω is a uniqueness function, $u : [-\tau, 0] \to (-\infty, 0]$ is continuous, $u(0) = 0$ and $Du \leqslant \omega(-u)$ on $(-\tau, 0)$, where D stands for any Dini derivative, then $u \equiv 0$ on $[-\tau, 0]$.*

(3) *If x is the maximal solution to $\dot{x} = \omega(x)$ on $[0, \tau]$, $u : [0, \tau] \to [0, \infty)$ is continuous and $u \in W^{1,1}_{loc}((0, \tau])$, $u(0) \leqslant x(0)$ and $\dot{u}(t) \leqslant \omega(u(t))$ for a.a $t \in (0, \tau]$, then $u \leqslant x$ on $[0, \tau]$. If ω is a uniqueness function, $u : [0, \tau] \to [0, \infty)$ is continuous and $u \in W^{1,1}_{loc}((0, \tau])$, $u(0) = 0$ and $\dot{u} \leqslant \omega(u)$ a.e. on $(0, \tau]$, then $u \equiv 0$ on $[0, \tau]$.*

Definition 17 (Slow function)

A continuous nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ is a slow function if there are $\varepsilon > 0$, $M > 0$ and $\tau > 0$ such that if $u : [0, \tau] \to [0, \infty)$ is continuous and

$$u(t) \leqslant a + \int_0^t \beta(u(s))\, ds \quad \text{for} \quad t \in [0, \tau], \tag{21}$$

where $a \in [0, \varepsilon]$, then one has $u(t) \leqslant aM$ for $t \in [0, \tau]$.

Fact 18

If $\beta : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that

$$\liminf_{x \to 0^+} \frac{x}{\beta(x)} > 0, \tag{22}$$

then β is slow.

We collect and recall some general concepts and relevant facts concerning evolution problems involving accretive operators.

Definition 19

Let $A : D(A) \multimap X$, $D(A) \subset X$, be a (possibly) set-valued operator, i.e. $\emptyset \neq Ax \subset X$ for $x \in D(A)$. Let $\mathrm{Gr(A)} := \{(\mathrm{x}, \mathrm{u}) \in \mathrm{X} \times \mathrm{X} \mid \mathrm{x} \in \mathrm{D(A)},\ \mathrm{u} \in \mathrm{Ax}\}$ be the graph of A.

(a) A is accretive if

$$[x - y, u - v]_{+} \geqslant 0 \ \text{ for all } \ (x, u), (y, v) \in \mathrm{Gr(A)}.$$

A is *m*-accretive if it is accretive and

$$\mathrm{Range}(I + \lambda A) := \{y \in X \mid y \in x + \lambda Ax \text{ for some } x \in D(A)\} = X$$

for some (equivalently for all) $\lambda > 0$.

(b) A is *α-accretive* (resp. α-m-accretive), $\alpha \in \mathbb{R}$, if $\alpha I + A$ is accretive (resp. m-accretive). Hence A is α-accretive if and only if

$$[x - y, u - v]_{+} \geqslant -\alpha \|x - y\| \ \text{ for all } \ (x, u), (y, v) \in \mathrm{Gr(A)}.$$

A is quasi *m*-accretive if it is α-m-accretive for some $\alpha \in \mathbb{R}$.

Uwagi do definicji

$A : D(A) \multimap X$ is accretive if and only if for all $x, y \in D(A)$, $u \in Ax, v \in Ay$ and $\lambda > 0$,

$$\|x - y\| \leqslant \|x - y + \lambda(u - v)\|.$$

Dissipative operators

$A : D(A) \multimap X$ is dissipative if

$$[x - y, u - v]_- \leqslant 0 \text{ for all } (x, u), (y, v) \in \mathrm{Gr(A)}$$

i.e. $-A$ is accretive. A is α-dissipative if $A - \alpha I$ is dissipative. A is m-dissipative if $\mathrm{Range}(I - \lambda A) = X$ for some (all) $\lambda > 0$.

Hence A is α-dissipative if

$$[x - y, u - v]_- \leqslant \alpha\|x - y\| \text{ for all } (x, u), (y, v) \in \mathrm{Gr(A)}$$

Remark 20

(1) If A is quasi m-accretive, then Gr(A) is closed and Ax, $x \in D(A)$, is closed.

If X^* is uniformly convex, then Ax is convex.

If X and X^* are uniformly convex, then $\overline{D(A)}$ is convex and for each $x \in D(A)$ and $w \in X$ there is a unique element $(Ax - w)^0 \in Ax - w$ of minimal norm, i.e. $\|(Ax - w)^0\| = |Ax - w| := \inf_{u \in Ax} \|u - w\|$.

(2) By the Lumer theorem a linear operator $A : D(A) \to X$ is α-m-accretive $\Leftrightarrow$ $-A$ is a closed, densely defined generator of a strongly continuous semigroup of linear operators $\{e^{-tA}\}_{t \geqslant 0}$ such that $\|e^{-tA}\| \leqslant e^{t\alpha}$ for $t \geqslant 0$.

(3) If A is α-m-accretive, $\lambda > 0$ with $\lambda\alpha < 1$, then the *resolvent*

$$J_\lambda = J_\lambda^A := (I + \lambda A)^{-1} : X \to D(A)$$

and the *Yosida approximation*

$$A_\lambda = \lambda^{-1}(I - J_\lambda) : X \to X$$

are well-defined, single-valued, and

$$\|J_\lambda x - J_\lambda y\| \leqslant (1 - \lambda\alpha)^{-1}\|x - y\|, \quad A_\lambda x \in AJ_\lambda x \ \text{ for all } \ x, y \in X, \tag{23}$$

$$\lim_{\lambda \to 0^+} J_\lambda x = x \ \text{ for } \ x \in \overline{D(A)}. \tag{24}$$

A is an α-m-accretive operator; $T > 0$, $w \in L^1\big([0,T], X\big)$ and consider the problem

$$\begin{cases} \dot u(t) \in -Au(t) + w(t), \ t \in [0,T], \\ u(0) = x \in \overline{D(A)}. \end{cases} \qquad (25)$$

Definition 21 (Solutions)

• A continuous function $u : [0,T] \to X$ is a strong solution to (25) if $u \in W^{1,1}_{loc}\big((0,T], X\big)$, $u(t) \in D(A)$, $u(0) = x$ and

$$\dot u(t) - w(t) \in -Au(t)$$

for a.a. $t \in (0,T]$ ([a]).

• A continuous $u : [0,T] \to X$ is an integral solution to (25) if $u(0) = x$ and for any $0 \leqslant s \leqslant t \leqslant T$ and $(y,v) \in \mathrm{Gr(A)}$,

$$e^{-t\alpha}\big\|u(t) - y\big\| \leqslant e^{-s\alpha}\big\|u(s) - y\big\| + \int_s^t e^{-z\alpha}\big[u(z) - y, w(z) - v\big]_+ \, dz. \qquad (26)$$

[a] Here $\dot u(t)$ stands for the ordinary strong derivative; the formula makes sense since u is differentiable a.a.

There are examples [Deville] showing that in general Banach spaces strong solutions don't exist in general.

Suppose $u : [0, T] \to X$ is a strong solution ($\alpha = 0$ for simplicity). Then

$$w(z) - \dot{u}(z) \in Au(z) \text{ for a.a. } z \in (0.T].$$

Since A is accretive, this implies that for $(y, v) \in \mathrm{Gr(A)}$ and a.a. $z \in (0, T]$

$$0 \leqslant [u(z) - y, -\dot{u}(z) + w(z) - v]_+ \leqslant [u(z) - y, -\dot{u}(z)]_+ + [u(z) - y, w(z) - v]_+.$$

$$[u(z) - y, \dot{u}(z)]_- = -[u(z) - y, -\dot{u}(z)]_+ \leqslant [u(z) - y.w(z) - v]_+.$$

Clearly $z \mapsto \|u(z) - y\|$ is absolutely continuous and a.e. differentiable. So for a.a. z

$$[u(z) - y, \dot{u}(z)]_- = \frac{d^-}{dt}\|u(z) - y\| = \frac{d}{dt}\|u(z) - y\|.$$

Integrating over $[s, t] \subset [0, T]$

$$\|u(t) - y\| \leqslant \|u(s) - y\| + \int_s^t [u(z) - y, w(z) - v]_+ \, dz,$$

i.e. u is an integral solution.

Remark 22

1. [Brezis] If X is reflexive, $x \in D(A)$ and $w \in W^{1,1}\big([0, T], X\big)$, then (25) has a unique strong solution $u \in W^{1,\infty}\big([0, T], X\big)$; if X and X^* are uniformly convex and $x \in D(A)$, then u is right differentiable, $\dot{u}_+$ is right continuous and $\dot{u}_+(t) + \big(Au(t) - w(t)\big)^0 = 0$ for a.a. $t \in (0, T)$; if in addition w is continuous, then $\dot{u}_+(0) = \big(-Ax + w(0)\big)^0$.

2. A strong solution is an integral one (shown above).

3. [Benilan, Brezis] Equation (25) admits a *unique* integral solution denoted by $u = u_A(\cdot; x, w) : [0, T] \to X$ (or $u(\cdot; x, w)$ if A is default) and $u(t) \in \overline{D(A)}$ for all $t \in [0, T]$.

4. [Benilan, Crandall] Given $x_1, x_2 \in \overline{D(A)}$, $w_1, w_2 \in L^1\big([0, T], X\big)$ the *Benilán inequality* holds: for $0 \leqslant s \leqslant t \leqslant T$

$$e^{-t\alpha}\big\|u_1(t) - u_2(t)\big\| \leqslant e^{-s\alpha}\big\|u_1(s) - u_2(s)\big\| + \int_s^t e^{-z\alpha}\big[u_1(z) - u_2(z), w_1(z) - w_2(z)\big]_+ \, dz$$

$$\leqslant e^{-s\alpha}\big\|u_1(s) - u_2(s)\big\| + \int_s^t e^{-z\alpha}\big\|w_1(z) - w_2(z)\big\| \, dz,$$

where $u_i := u_A(\cdot; x_i, w_i)$, $i = 1, 2$. Analogously

$$e^{-2t\alpha}\left\|u_1(t)-u_2(t)\right\|^2 \leqslant e^{-2s\alpha}\left\|u_1(s)-u_2(s)\right\|^2 + 2\int_s^t e^{-2z\alpha}\left[u_1(z)-u_2(z), w_1(z)-w_2(z)\right]_+ dz$$

$$\leqslant e^{-2s\alpha}\left\|u_1(s)-u_2(s)\right\|^2 + 2\int_s^t e^{-2z\alpha}\left\|u_1(z)-u_2(z)\right\|\left\|w_1(z)-w_2(z)\right\| dz,$$

5. If $w \equiv 0$, then (25) has a unique integral solution $u_A(\cdot; x, 0)$ defined on $[0,\infty)$ and the *Crandall-Liggett formula* holds: for any $x \in X$ and $t \geqslant 0$,

$$u_A(t;x,0) = \lim_{n\to\infty} J^n_{t/n}x.$$

Let us put

$$S_A(t)x := u_A(t;x,0), \quad t \geqslant 0.$$

Then $\forall\ t \geqslant 0$, $S_A(t) : \overline{D(A)} \to \overline{D(A)}$,

$$\forall x, y \in \overline{D(A)} \quad \left\|S_A(t)x - S_A(t)y\right\| \leqslant e^{t\alpha}\|x-y\|.$$

The family $\{S_A(t)\}_{t\geqslant 0}$ is a (strongly continuous) semigroup of continuous maps, i.e. for any $x \in \overline{D(A)}$ the map $[0,\infty) \ni t \mapsto S_A(t)x$ is continuous, $S_A(0) = I$ on $\overline{D(A)}$ and $S_A(t+s) = S(t) \circ S_A(s)$ for any $t, s \geqslant 0$.

6. If $x \in D(A)$, then $S_A(\cdot)\, x : [0, \infty) \to X$ is Lipschitz continuous on every compact interval $[0, \tau]$, $T > 0$. The same is true for integral solutions $u_A(\cdot, x, w)$ with *constant* w.

7. If X is reflexive and $x \in D(A)$, then $u = S_A(\cdot)x \in W^{1,\infty}_{loc}([0,\infty), X)$, $u(t) \in D(A)$ and $\dot{u}(t) \in -Au(t)$ for a.a. $t \geqslant 0$.

8. Suppose that A is linear. Then $S_A(t) = e^{-tA}$ for any $t \geqslant 0$. Moreover $u = u_A(\cdot; x, w)$ is a mild solution of (25) in the sense of the Duhamel formula, i.e.

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-s)A} w(s)\, ds, \;\; t \geqslant 0. \tag{27}$$

9. Let $u = u_A(\cdot; x, w)$ and fix a small $h > 0$. We have the formula

$$u_A(\cdot + h; x, w) = u_A\big(\cdot; u_A(h; x, w), w(\cdot + h)\big), \;\; t \in [0, T - h] \tag{28}$$

and, in view of (27), for any $0 \leqslant s \leqslant t \leqslant T - h$

$$\|u(t+h) - u(t)\| \leqslant e^{(t-s)\alpha}\big\|u(s+h) - u(s)\big\| + \int_s^t e^{(t-z)\alpha}\big[u(z+h) - u(z), w(z+h) - w(z)\big]_+ \, dz \tag{29}$$

$$\leqslant e^{(t-s)\alpha}\big\|u(s+h) - u(s)\big\| + \int_s^t e^{(t-z)\alpha}\big\|w(z+h) - w(z)\big\|\, dz. \tag{30}$$

Energy dissipation

An equivalent definition of an integral solution ($\alpha = 0$) says that for all $(y, v) \in \mathrm{Gr}(A)$ and $0 \leqslant s \leqslant t \leqslant T$

$$\|u(t) - y\|^2 \leqslant \|u(s) - y\|^2 + 2\int_s^t \langle u(z) - y, w(z) - v\rangle_+ \, dz.$$

Assume that $0 \in A(u)$ and $w \equiv 0$

$$\|u(t)\|^2 \leqslant \|u(s)\|^2.$$

Usually $E(t) := \|u(t)\|^2$ is interpreted as being proportional to the energy of the system. We see the dissipation (rozpraszanie) energii.

Let A be an α-m-accretive operator and $f : \Omega \to X$, where $\Omega \subset X$ is open, be continuous.

Definition 23

A continuous $u : [0, T] \to \Omega$, where $T > 0$, is an *integral* (resp. *strong*) solution to (1), i.e.

$$\begin{cases} \dot{u} \in -Au + f(u) \\ u(0) = x \in \Omega \cap \overline{D(A)}, \end{cases} \tag{31}$$

if u is an *integral* (resp. *strong*) *solution* to (25) with $w := f \circ u$.

More generally (if f is not assumed to be continuous), u is an integral solution of (31) if $f \circ u \in L^1([0, T], X)$ and u is an integral solution to (25) with $w = f \circ u$.

A continuous function $u : [0, \tau) \to X$, $0 < \tau \leqslant \infty$, is an integral solution to (31) if for any $0 < T < \tau$, u restricted to $[0, T]$ is an integral solution to (31) on $[0, T]$.

An integral solution $u : [0, \tau) \to X$ is *noncontinuable* if it has no extension to a solution defined on the interval $[0, \tau')$ with $\tau' > \tau$.

Corollary 24

(1) Along with (31) consider the problem

$$\begin{cases} \dot{u} \in -Bu + g(u) \\ u(0) = x \in \Omega \cap \overline{D(A)}, \end{cases} \tag{32}$$

where $B = A + \alpha I$ and $g(u) = \alpha u + f(u)$, $u \in \Omega$. B is m-accretive. Then integral solutions to (31) and (32) coincide, i.e. one can shift the part αI from the perturbation term to the accretive operator and vice-versa. In particular we may, w.l.o.g, consider *only* m-accretive operators A.

(2) Let $u : [0, \tau) \to \Omega$, $\tau > 0$, an integral solution of eq. (31) and $\exists\ M > 0$ such that $\left\|f(u(t))\right\| \leqslant M$ for any $0 \leqslant t < \tau$. Let $w(t) := f\big(u(t)\big)$, $t \in [0, \tau)$. Then $w \in L^1\big([0, \tau], X\big)$.
Hence there is an integral solution $\bar{u} : [0, \tau] \to X$ on $[0, \tau]$ of the problem $\dot{u} \in -Au + w$, $u(0) = x$.
Evidently $u = \bar{u}$ on $[0, \tau)$. Hence

$$\lim_{t \to \tau^-} u(t) = \bar{u}(\tau) \in \overline{\Omega} \ \text{ exists}$$

. (3) A 'semigroup' property of sorts of integral solutions to (31). Namely, putting

$$S_A(t; f)(x) := \{u(t) \mid u \text{ is an integral solution to (31)}\}, \quad t \in [0, T)$$

we have $S_A(t + h; f)(x) = S_A(t; f)\big(S_A(h; f)(x)\big)$ for $t, h \in [0, T]$ such that $t + h \leqslant T$.

Example 25

Let $\varrho : \mathbb{R} \to \mathbb{R}$ continuous and differentiable on $\mathbb{R} \setminus \{0\}$, $\varrho'(t) > 0$. Let

$$D(A) = \{u \in L^1(\Omega) \mid \varrho \circ u \in W_0^{1,1}(\Omega),\ \Delta(\varrho \circ u) \in L^1(\Omega)\},$$

where $\Delta(\varrho \circ u)$ is the Laplacian (in the distribution sense). We put

$$Au := -\Delta(\varrho \circ u), \quad u \in D(A).$$

The A is m-accretive. If $V \in L^\infty(\Omega)$ and $\operatorname{ess\,sup}_{x\in\Omega} |V(x)| \leqslant \alpha$, then the nonlinear Schrödinger operator $u \mapsto -\Delta(\varrho \circ u) + V(\cdot)u$ is quasi-m-accretive.

We take $u, v \in D(A)$ and show that for any $\lambda > 0$

$$\|u - v\| \leqslant \|u - v + \lambda(Au - Av)\|.$$

Take $f_n \in C^1(\mathbb{R})$ such that $f_n(0) = 0$, $|f_n| \leqslant 1$, $f' \geqslant 0$, $n \in \mathbb{N}$. Then (weak gradient)

$$\nabla f_n \circ (\varrho \circ u - \varrho \circ v)(x) = f_n'(\varrho \circ u(x) - \varrho \circ v(x))\nabla(\varrho \circ u - \varrho \circ v)(x)$$

and, by definition

$$\int_\Omega (Au - Av) f_n(\varrho(u) - \varrho(v))\, dx = \int_\Omega \|\nabla(\varrho(u) - \varrho(v)\|^2 f_n'(\varrho(u) - \varrho(v))\, dx \geqslant 0.$$

Consequently for any $\lambda > 0$

$$\int_\Omega (u-v) f_n\big(\varrho(u)-\varrho(v)\big) \leqslant$$

$$\int_\Omega \big(u-v+\lambda(Au-Av)\big) f_n\big(\varrho(u)-\varrho(v)\big)\, dx \leqslant$$

$$\int_\Omega \big|u-v+\lambda(Au-Av)\big| \Big| f_n\big(\varrho(u)-\varrho(v)\big)\Big|\, dx \leqslant \big\|u-v+\lambda(Au-Av)\big\|_{L^1}.$$

We may choose $f_n \to \operatorname{sgn}(\cdot)$ as $n \to \infty$. e.g,

$$f_n(t) := \frac{nt}{n|t|+1},\ t \in \mathbb{R}.$$

Passing to limit in the LHS and taking into account that ϱ is increasing

$$\operatorname{sgn}\big(\varrho \circ u(x) - \varrho \circ v(x)\big) = \operatorname{sgn}\big(u(x)-v(x)\big).$$

We get the accretivity.

Take $f \in L^1(\Omega)$. We are to show that there is $u \in D(A)$ such that

$$(*) \qquad u - \Delta)\varrho \circ u) = f.$$

Niech

Theorem 26 (Brezis-Strauss)

For any $f \in L^1(\Omega)$ there is a unique $u \in D(B) = \{v \in W_0^{1,1}(\Omega) \mid \Delta u \in L^1(\Omega)\}$ such that (even a.e.)

$$-\Delta u(x) + b\big(u(x)\big) = f,$$

where $b : \mathbb{R} \to \mathbb{R}$ is increasing and continuous.

Hence there is $\varrho^{-1}(v) - \Delta v = f$. Then $u := \varrho^{-1} \circ v$ solves $(*)$.

The following result seems to be a well-known folklore. It seems, however, to be more convenient than the corresponding result of Barbu.

Theorem 27

Assume that $f : \overline{\Omega} \to X$ *is locally Lipschitz continuous,* $A : D(A) \multimap X$ *is m-accretive and* $x \in \Omega \cap \overline{D(A)}$. *Then:*

(a) *there is* $T > 0$ *and a unique integral solution* $u : [0, T] \to \Omega$ *of eq.* (31).

(b) *If* $u : [0, T] \to \Omega$ *is an integral solution of* (31) *with* $x \in D(A) \cap \Omega$, *then* u *is Lipschitz continuous.*

(c) *If* X *is reflexive,* $u : [0, T] \to \Omega$ *is an integral solution of* (31) *with* $x \in \Omega \cap D(A)$, *then* u *is a strong solution and* $u \in W^{1,\infty}([0, T], X)$.

Proof of (a)

(a) There is $R > 0$ such that $D := D(x, R) \subset \Omega$ and f is Lipschitz with the Lipschitz constant $\ell > 0$ on D. Take $y \in D(A) \cap D(x, R/3)$ and $p \in Ay$. Let $M := \sup_{u \in D} \big\| f(u) - p \big\|$ and

$$Y := \big\{ u \in C\big([0, T], X\big) \mid u(t) \in D,\ t \in [0, T] \big\},$$

where $T = \frac{R}{3M}$. Let Y be endowed with the complete metric

$$d(u, v) := \sup_{t \in [0, T]} e^{-\ell t} \big\| u(t) - v(t) \big\|,\ \ u, v \in Y.$$

Consider a map $N : Y \to C\big([0, T], X\big)$ given by

$$[Nu](t) = u_A(t; x, w_u),\ u \in Y,$$

where $w_u := f \circ u$, i.e. Nu is the integral solution to (31) with $w = w_u$. This map is well-defined since $w_u \in C\big([0, T], X\big) \subset L^1\big([0, T], X\big)$. Actually $N : Y \to Y$ and it is a (Banach) contraction. Indeed: for $u \in Y$ and $t \in [0, T]$

$$\big\| w_u(\tau) - p \big\| = \big\| f(u(\tau)) - p \big\| \leqslant M,\ \ \tau \in [0, t],$$

and, in view of (26)

$$\big\|[Nu](t) - x\big\| \leqslant \big\|[Nu](t) - y\big\| + \|x - y\| \leqslant$$
$$2\|x - y\| + \int_0^t \big\|w_u(z) - p\big\| \, dz \leqslant \frac{2}{3}R + MT \leqslant R,$$

i.e. $Nu \in Y$. For $u, v \in Y$ in view of (27) we have

$$\begin{aligned} \big\|[Nu](t) - [Nv](t)\big\| &\leqslant \textstyle\int_0^t \big\|w_u(z) - w_v(z)\big\| \, dz \leqslant \ell \int_0^t \big\|u(z) - v(z)\big\| \, dz \\ &\leqslant \ell d(u, v) \textstyle\int_0^t e^{\ell z} \, dz = (e^{\ell t} - 1) d(u, v) \end{aligned}$$

and thus $d(Nu, Nv) \leqslant cd(u, v)$ with $c = 1 - e^{-\ell T} < 1$. Hence there is $u \in Y$ such that $Nu = u$, i.e. u is an integral solution to (31). Its uniqueness is straightforward.

Theorem 28

Let $A : D(A) \multimap X$ be m-accretive and suppose that $f : \Omega \to X$ is continuous and

$$\big[u - v, f(u) - f(v)\big]_+ \leqslant \beta\big(\|u - v\|\big) \text{ for } u, v \in \Omega,$$

where β is a slow function.

Assume that f maps bounded sets into bounded ones.

For any $x_0 \in \overline{D(A)} \cap \Omega$ and $0 < r < R$ such that $D(x_0, R) \subset \Omega$, there is $T > 0$ such that for any $x \in D(x_0, r) \cap \overline{D(A)}$ the problem (31) has a unique integral solution on $[0, T]$.

If $x \in D(x_0, r) \cap D(A)$, then this solution is Lipschitz continuous.

Proof

1. The Lasota-Yorke theorem implies that for any $n \in \mathbb{N}$ there is a locally Lipschitz $f_n : \Omega \to X$ such that

$$\big\|f(x) - f_n(x)\big\| \leqslant \frac{1}{2n} \ textfor \ x \in \Omega.$$

2. Then for any $n \in \mathbb{N}$ and $u, v \in \Omega$

$$\big[u - v, f_n(u) - f_n(v)\big]_+ \leqslant \beta\big(\|u - v\|\big) + \frac{1}{n} \leqslant \beta\big(\|u - v\|\big) + 1.$$

3. For any $n \in \mathbb{N}$ and $x \in D(x_0, r) \cap \overline{D(A)}$ there is noncontinuable integral solution u_n to the problem

$$\begin{cases} \dot{u} \in -Au + f_n(u) \\ u(0) = x, \end{cases}$$

defined on $[0, \tau_n)$.
We claim that there is a $0 < T \leqslant \tau_n$ for any $n \in \mathbb{N}$.

4. The functional sequence (u_n) converges uniformly to an integral solution.

Differentiation along trajectories

Assume $K \subset X$ be closed, $x \in X$.

Let $u : [0, \tau) \to X$, $\tau > 0$ be continuous, $u(t) = x$ for some $t \in [0, \tau)$.

The Dini derivative $D_+(d_K \circ u)(t)$ measures the rate of changes of the distance of $u(s)$ from K for s in a neighborhood of t.

For instance that $D_+(d_K \circ u)(t) = 0 \Leftrightarrow \exists\ h_n \to 0^+$, $v_n \to 0$ such that

$$d_K\big(u(t + h_n) + h_n v_n\big) \leqslant d_K(x) \;\text{ for all }\; n \geqslant 1,$$

i.e. u is tangential to

$$K^\alpha := \big\{y \in X \mid d_K(y) \leqslant \alpha = d_K(x)\big\}.$$

- If u is (right) differentiable at t, i.e. $u'_+(t)$ exists, then u is tangential to $K^\alpha \Leftrightarrow$

$$D_+ d_K(x; v) := \liminf_{h \to 0^+} \frac{d_K(x + hv) - \alpha}{h} = 0;$$

in other words if and only if $v \in T_{K^\alpha}(x)$.

- Assume $A : D(A) \to X$ is quasi m-accretive operator;
- $V : X \to \mathbb{R}$ locally Lipschitz function representing K,
- $x \in \overline{D(A)}$ and $v \in X$. Suppose that $u := u(\cdot; x, v)$ is the integral solution to (25)

with $w(\cdot) \equiv v \in X$.

Definition 29

By the *A-derivative of V at x in the direction v* we mean the Dini type derivative

$$D_A V(x; v) := \liminf_{h \to 0^+} \frac{(V \circ u)(h) - V(x)}{h} = D_+ \big(V \circ u_A(\cdot; x, v)\big)(0). \qquad (33)$$

Note that if $x \in D(A)$, then the derivative $D_A V(x; v)$ is finite sincethe function $V \circ u$ is Lipschitz around 0.

$D_A V(x; v)$ measures the rate of growth of V along the integral curve $u = u_A(\cdot; x, v)$.

• If $D_A V(x; v) > \alpha$, then there is $\eta > 0$ such that $V\big(u(t)\big) > \alpha t + V(x)$ for $0 < t < \eta$. If $\dot{u}_+(0)$ exists, then

$$D_A V(x; v) = D_+ V\big(x; \dot{u}_+(0)\big). \tag{34}$$

Indeed $u(h) = x + hu'_+(0) + o(h)$ when $h \to 0$; hence

$$h^{-1}\Big|\big(V\big(u(h)\big) - V(x)\big) - \big(V\big(x + h\dot{u}_+(0)\big) - V(x)\big)\Big| \leqslant h^{-1}\ell\big|o(h)\big| \to 0 \;\text{ as }\; h \to 0,$$

where ℓ is the Lipschitz constant of V at x.

Fact 30

(i) If X and X^* are uniformly convex, then

$$D_A V(x; v) = D_+ V(x; v - y),$$

where $y \in Ax$ s. t. $y - v = (Ax - v)^0$ is the element of $Ax - v$ with minimal norm. This holds in any Banach space if A is a linear operator and then $D_A V(x; v) = D_+ V(x; v - Ax)$.

(ii) *Assume A is single-valued and X, X^* are uniformly convex or X is an arbitrary Banach space but A is linear. If for $x \in \overline{D(A)}$*

$$V(S_A(t)x) \leqslant V(x) \text{ for all } t \geqslant 0, \tag{35}$$

then

$$D_A V(x; v) \leqslant V^\circ(x; v) \text{ for any } x \in D(A),\ v \in X, \tag{36}$$

where $V^\circ(x; v)$ is the generalized Clarke derivative at x in the direction of v.

Theorem 31

Let $x_0 \in \overline{D(A)}$ *and* $u : [0,\tau) \to X$ *be an integral solution to* (1)*. Then*

$$D_+(V \circ u)(t) = D_A V\Big(u(t); f\big(u(t)\big)\Big), \quad t \in [0,\tau).$$

Proof

Fix $t \in [0,\tau)$, let $x := u(t)$ and $v := f\big(u(t)\big)$. Recall

$$D_A V(x; v) = D_+\big(V \circ u_A(\cdot; x, v)\big)(0).$$

By the semigroup property

$$u(t+h) = u(h; x, w), \quad \text{where} \;\; w(h) := f\big(u(t+h)\big),$$

for $h \in [0, T-t)$. By (27) we have

$$\big|V\big(u(t+h)\big) - V\big(u_A(h; x, v)\big)\big| \leqslant \ell \big\|u(t+h) - u_A(h; x, v)\big\| =$$

$$\ell\big\|u_A(h; x, w) - u_A(h; x, v)\big\| \leqslant \ell e^{\alpha h} \int_0^h \big\|w(s) - v\big\|\, ds,$$

where ℓ is the Lipschitz constant of V around x.

Since

$$h^{-1}\ell e^{\alpha h}\int_0^h \left\|w(s)-v\right\| ds \to 0 \text{ as } t \to 0^+$$

and

$$V\big(u(t+h)\big)-V\big(u(t)\big)=V\big(u(t+h)\big)-V\big(u_A(h;x,v)\big)+V\big(u_A(h;x,v)\big)-V(x)$$

this yields that $D_+(V\circ u)(t)=D_AV(x;v)=D_AV\Big(u(t);f\big(u(t)\big)\Big)$ as required.

Theorem 32 (Theorem 4)

If $\forall\ z \in \partial K \cap \Omega\ \exists$ *nghbd* $U(z) \subset \Omega$ *of* z *and a uniqueness function* ω *such that*

$$D_A V\big(x; f(x)\big) \leqslant \omega\big(V(x)\big) \quad \text{for } x \in \big(U(z) \setminus K\big) \cap \overline{D(A)}, \tag{37}$$

then K *is invariant w.r.t.* (1)*. In particular this holds if*

$$D_A V\big(x; f(x)\big) \leqslant CV(x).$$

Proof

Suppose to the contrary that there is an integral solution $u : [0, \tau) \to X$ that leaves K, i.e. there is $T \in (0, \tau)$ such that $V\big(u(T)\big) > 0$. Let

$$\bar{t} := \sup\big\{t \in [0, T] \mid V\big(u(t)\big) \leqslant 0\big\}.$$

Clearly $\bar{t} < T$, $V\big(u(\bar{t})\big) = 0$ and

$$V\big(u(t)\big) > 0 \quad \text{for all} \quad t \in (\bar{t}, T].$$

W.l.o.g. we may assume that $\bar{t} = 0$ and $u(t) \in U\big(u(0)\big)$ for $t \in [0, T]$. In view of the above Theorem, for any $t \in (0, T]$

$$D_+(V \circ u)(t) = D_A V\Big(u(t); f\big(u(t)\big)\Big) \leqslant \omega\Big(V\big(u(t)\big)\Big). \tag{38}$$

This however, by the Perron Lemma we get that $V\big(u(t)\big) = 0$ for all $t \in [0, T]$ since ω is a uniqueness function.

A version convenient for applications to PDE of parabolic type:

Theorem 4

Assume: $\forall\ z \in \partial K \cap \Omega\ \exists$ nghbd $U(z) \subset \Omega$ of z and a uniqueness function ω such that, for any integral solution $u : [0, \tau) \to X$ of (1) with $u(0) = z$ one has

$$D_+(V \circ u)(t) \leqslant \omega\big(V\big(u(t)\big)\big), \quad \text{for a.e. } t \in (0, \tau), \quad \text{whenever } u(t) \in U(z) \setminus K. \tag{39}$$

Then K is invariant w.r.t. (1).

Remark 33

• Assume that $X \hookrightarrow Y$, Y is a Banach space, and $\exists$ is a quasi m-accretive operator $A_Y : D(A_Y) \to Y$ such that the part of $A_{Y|}$ in X is equal to A and V can be extended to a differentiable $V_Y : Y \to \mathbb{R}$.

• Suppose that any integral solution $u : [0, T] \to X$ of (1) is a strong solution to

$$\dot{u}(t) \in -A_Y u(t) + f(u(t)) \quad \text{for a.e. } t \in [0, T], \tag{40}$$

i.e. $u \in W^{\mathbf{1},\mathbf{1}}_{loc}((0, T], Y)$ (i.e. $u \in W^{\mathbf{1},\mathbf{1}}([\delta, T], Y)$, for any $\delta \in (0, T)$) and $u(t) \in D(A_Y)$. Then, for a.e. $t \in [0, T]$,

$$D_+(V \circ u)(t) = D_+(V_Y \circ u)(t) = V'_Y(u(t))\dot{u}(t) = V'_Y(u(t))\big(- v + f(u(t))\big)$$

for some $v \in A_Y u(t)$. Therefore in order to verify (39) it is enough to show that, for all $x \in (U(z) \setminus K) \cap D(A_Y)$,

$$V'_Y(x)\big(- v + f(x)\big) \leqslant \omega\big(V(x)\big) \quad \text{for all } v \in A_Y(x). \tag{41}$$

Theorem 34

Suppose Y, A_Y and V_Y are as in Remark 33. If $u : [0, T] \to X$ is an integral solution of (1) with $u(0) \in D(A)$, then $u \in W^{1,1}([0, T], Y)$, $u(t) \in D(A_Y)$ and

$$\dot{u}(t) \in -A_Y u(t) + f(u(t)),$$

for a.e. $t \in [0, T]$.

Proof

Observe that $w := f \circ u \in L^{\mathbf{1}}([0, T], Y)$ and that, since the part of A in X is equal to A, we have $(I + \lambda A_Y)^{-\mathbf{1}} v = (I + \lambda A)^{-\mathbf{1}} v \in D(A)$ for any $v \in X$ and $\lambda > 0$. By the construction of integral solutions (see [Barbu]) it is easily seen that u is also an integral solution of

$$\dot{u}(t) \in A_Y u(t) + w(t), \;\; t \in [0, T]. \tag{42}$$

Hence, in view of Proposition 26 (c) and the reflexivity of Y, we infer that $u \in W^{\mathbf{1},\infty}([0, T], Y)$ and u is a strong solution of (42).

Consider a problem with state dependent impulses of the form

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & t \in [0, T],\ t \neq \tau_j(y(t)),\ j = 1, \ldots, k, \\ y(0) = x_0, & \\ y(t^+) = y(t) + I_j(y(t)) & \text{for } t = \tau_j(y(t)),\ j = 1, \ldots, k, \end{cases} \tag{43}$$

where:

- $T > 0$, $F : [0, T] \times X \multimap X$ is a set-valued dynamics,
- for $j = 1, ..., k$, $\tau_j : X \to (0, T)$ is a *barrier function* and $I_j : X \to X$ an *impulse function*.

To characterize the suitable function space, where solutions can be considered, one looks for sufficient conditions implying that every trajectory of (43) meets a barrier $\Gamma_j = \mathrm{Gr}(\tau_j)$ *exactly* once. Note that if the global existence is achieved, then each barrier is hit *at least* once. One however demands that after the j-th jump a solution stays in the epigraph $\mathrm{Epi}\,(\tau_j)$ of τ_j, i.e. it immediately enters its interior and does not return to Γ_j; in other words one needs conditions implying that epigraph $\mathrm{Epi}\,(\tau_j)$, $j = 1, ..., k$, is strictly invariant.

Our results do fit well to this problem if $F(t,y) = -Ay + f(t,y)$, where

$A : D(A) \multimap X$ is an m-accretive operator.

Let us consider the following problem

$$\begin{cases} \dot{u} \in -Au + f(t,u), & t \in [0,T], \\ u(0) = x \in \overline{D(A)}, \end{cases} \tag{44}$$

where $f : \mathbb{R} \times X \to X$ is continuous. Let $\tau : X \to \mathbb{R}$ be a locally Lipschitz barrier function. By a *solution* to (44) we understand an integral solution $u : [0,T] \to X$, $T > 0$, to (25) with $w = f\big(\cdot, u(\cdot)\big)$.

Theorem 35

Assume that for every $(z,\theta) \in \mathrm{Gr}(\tau)$ there are a neighborhood $U = U(z,\theta)$ of (z,θ) and a uniqueness function ω such that

$$D_A\tau\big(x; f(t,x)\big) \leqslant \omega\Big(\big|\tau(x) - t\big|\Big) + 1 \text{ for } (t,x) \in U,$$

$$D_A\tau\big(x; f(t,x)\big) < 1 \text{ for } (t,x) \in U \cap \mathrm{Gr}(\tau).$$

If $u : [0,T] \to X$ is a solution to (44) and $\tau(x_0) \leqslant t$, then $\tau\big(u(h)\big) < t + h$ for any $0 < h \leqslant T$, i.e. $\big(t+h, u(h)\big) \in \mathrm{Epi}\,(\tau)$ for $h \in (0,T]$.

Proof

• Define $\mathrm{A} : D(\mathrm{A}) \to \mathrm{X} := \mathbb{R} \times X$ by $\mathrm{A}(t,u) := (0, Au)$ for $(t,u) \in D(\mathrm{A}) := \mathbb{R} \times D(A)$;

• $\mathrm{F} : \mathrm{X} \to \mathrm{X}$ by $\mathrm{F}(t,x) := \big(1, f(t,x)\big)$ for $(t,x) \in \mathrm{X}$; • X is a Banach space with the norm $\big\|(t,x)\big\| := |t| + \|x\|$ for $(t,x) \in \mathrm{X}$.

It is immediate to see that A is *m*-accretive and F is continuous. A simple calculation shows that a continuous function $\mathrm{u} : [0,T] \to \mathrm{X}$ is an integral solution to the problem

$$\begin{cases} \dot{\mathrm{u}} \in -\mathrm{A}\mathrm{u} + \mathrm{w}, \\ \mathrm{u}(0) = (t,x), \end{cases} \tag{45}$$

$(t,x) \in \mathrm{X}$ and $\mathrm{w} = (1,w)$, where $w \in L^1([0,T],X)$, if and only if

$$\mathrm{u}(h) = \big(t+h, u(h)\big)$$

for $h \in [0,T]$, where $u : [0,T] \to X$ is an integral solution to (25).

In particular, $u : [0,T] \to X$ is a solution of (44) if and only if $\mathrm{u}(h) = \big(t+h, u(h)\big)$ is a solution to (45) with $\mathrm{w} = \mathrm{F} \circ \mathrm{u}$.

• $V : \mathrm{X} \to \mathbb{R}$ be given by $V(t,x) := \tau(x) - t$, $(t,x) \in \mathrm{X}$. Clearly V is locally Lipschitz,

• $\mathrm{K} := \big\{(t,x) \in \mathrm{X} \mid V(t,x) \leqslant 0\big\} = \mathrm{Epi}\,(\tau)$, $\mathrm{int}\mathrm{K} = \big\{(t,x) \in \mathrm{X} \mid V(t,x) < 0\big\}$ and $\partial\mathrm{K} = \mathrm{Gr}(\tau)$.

In view of assumptions we get that,

$$D_{\mathsf{A}} V\big((t,x);\mathsf{F}(t,x)\big) \leqslant \omega\big(|V(t,x)|\big) \text{ for each } (t,x) \in U,$$
$$D_{\mathsf{A}} V\big((t,x);\mathsf{F}(t,x)\big) < 0 \text{ for each } (t,x) \in U \cap \mathrm{Gr}(\tau).$$

In view of Theorem 10, the set K is strictly invariant with respect to (45). This completes the proof.

Remark 36

Assumptions allow to consider nonsmooth barriers and deal with integral (not strong) solutions in contrast to many other papers. Note also that in our approach the operator A may be nonlinear and solutions need not be mild.

Corollary 37

Assume that A and f satisfy the assumptions of Theorem 35, and let $\tau_j : H \to (0, \infty)$ be locally Lipschitz functions such that (45) and (45) hold for τ_j instead of τ, for every $j = 1, \ldots, k$. We also assume standard conditions on barriers:

1. $0 < \tau_j(x) < \tau_{j+1}(x)$ *for each* $x \in X$ *and* $j = 1, \ldots, k$,
2. $\tau_j\big(x + I_j(x)\big) \leqslant \tau_j(x) < \tau_{j+1}\big(x + I_j(x)\big)$ *for each* $x \in X$ *and* $j = 1, \ldots, k$.

Then any solution to (44) meets each barrier Γ_j at most once. □

Consider the following nonlinear problem

$$\begin{cases} u_t = \Delta_p u + f(x,u), & x \in (0,l),\ t \in [0,T], \\ u(0,t) = u(l,t) = 0, & t \in [0,T], \end{cases} \tag{46}$$

where $l > 0$, $T > 0$, $\Delta_p u := \left(|u_x|^{p-2}u_x\right)_x$, $p \geqslant 2$, is the so-called p-Laplacian and $f : [0,l] \times \mathbb{R} \to \mathbb{R}$. Suppose that functions $m, M : [0,l] \to \mathbb{R}$ such that

$$m \leqslant M, \quad m(0) \leqslant 0 \leqslant M(0) \text{ and } m(l) \leqslant 0 \leqslant M(l) \tag{47}$$

represent the obstacles. In the so-called obstacle problem we look for conditions on f, m and M implying that for any continuous $u_0 : [0,l] \to \mathbb{R}$ such that $u_0(0) = u_0(l) = 0$ and $m(x) \leqslant u_0(x) \leqslant M(x)$ for $x \in [0,l]$, all solutions of (46) starting at u_0 satisfy

$$m(x) \leqslant u(x,t) \leqslant M(x) \text{ for all } x \in [0,l],\ t \in [0,T]. \tag{48}$$

- Let $X = C_0[0,l] = \{u \in C[0,l] \mid u(0) = u(l) = 0\}$. X endowed with the sup-norm $\|\cdot\|_\infty$ is a Banach space.

- Let $A : D(A) \to X$ be given by $Au := -\left(|u'|^{p-2}u'\right)'$ for $u \in D(A)$, where

$$D(A) := \left\{u \in X \cap C^1(0,l) \mid \left(|u'|^{p-2}u'\right)' \in X\right\}.$$

$D(A)$ is dense and the operator A is m-accretive (see e.g. [**?**, Lemma 6.1]). In addition to the above assumptions suppose that:

1. $f : [0,l] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(\cdot, 0) \equiv 0$ and $f(x,\cdot)$ is locally Lipschitz continuous uniformly with respect to $x \in [0,l]$, i.e. for any $s \in \mathbb{R}$ there are $L > 0$ and $\delta > 0$ such that

$$\left|f(x,s_1) - f(x,s_2)\right| \leqslant L|s_1 - s_2| \text{ for all } x \in [0,l] \text{ and } s_1, s_2 \in (s-\delta, s+\delta); \quad (49)$$

2. $m, M \in C[0,l] \cap C^2(0,l)$, m is a subsolution and M is a supersolution of the stationary problem related to (46), i.e.

$$-\Delta_p m(x) \leqslant f\left(x, m(x)\right), \ x \in (0,l) \quad \text{and} \quad -\Delta_p M(x) \geqslant f\left(x, M(x)\right), \ x \in (0,l).$$

Condition (1) implies that the Nemytskii operator $F : X \to X$ given by

$F(u)(x) := f\left(x, u(x)\right)$, $u \in X$ and $x \in [0, l]$, is well-defined and locally Lipschitz.

By a solution to (46) on $[0, T]$ we understand an integral solution $u : [0, T] \to X$ of the problem

$$\dot{u} = -Au + F(u), \ \ t \in [0, T]. \tag{51}$$

It is clear that a solution u satisfies condition (48) if and only if

$$u(t) \in K_m \cap K_M \ \text{ for all } \ t \in [0, T],$$

where $K_m := \left\{u \in X \mid u \geqslant m \text{ on } [0, l]\right\}$, $K_M := \left\{u \in X \mid u \leqslant M \text{ on } [0, l]\right\}$.

Clearly $K_m = \{u \in X \mid V_m(u) \leqslant 0\}$, $K_M = \{u \in X \mid V_M(u) \leqslant 0\}$, where

$V_m, V_M : X \to \mathbb{R}$ are given by

$$V_m(u) := \frac{1}{2}\int_0^l (u-m)_-^2 \, dx, \quad V_M(u) := \frac{1}{2}\int_0^l (u-M)_+^2 \, dx \quad \text{for} \quad u \in X. \tag{52}$$

It can be easily verified that $d_{K_m}(u) = \|(u-m)_-\|_\infty$ and $d_{K_M}(u) = \|(u-M)_+\|_\infty$. Observe that neither $V_m = d_{K_m}$ nor $V_M = d_{K_M}$.

In order to verify condition (39) of Theorem 4 we shall follow the idea from Remark 32 (2) with $Y = L^2(0, l)$ and the L^2-realization of the p-Laplace operator.

Remark 38

It is known (see e.g. Cwiszewski-Maciejewski) that, for any $u_0 \in X$, any integral solution of (51) has the following properties

$$u \in C([0,T],X) \cap C((0,T],W_0^{1,p}(0,l)) \cap W_{loc}^{1,2}((0,T],L^2[0,l]),$$

$\Delta_p u(t) \in L^2[0,l]$ and

$$\dot{u}(t) = \Delta_p u(t) + F(u(t)) \text{ for a.e. } t \in [0,T], \tag{53}$$

i.e.

$$\dot{u}(t) = -A_{L^2} u(t) + F(u(t)), \quad \text{for a.e. } t \in [0,T],$$

where $A_{L^2} : D(A_{L^2}) \to L^2[0,l]$ is given by

$$A_{L^2} u := -\Delta_p u, \quad D(A_{L^2}) := \{u \in W^{1,p}(0,l) \mid \Delta_p u \in L^2[0,l]\}.$$

It is well known that A_{L^2} is m-accretive (see e.g. Brezis, Showalter) and clearly the part of A_{L^2} in X is equal to A.

Theorem 39

If m and M satisfy (50)*, then any integral solution of* (46) *such that* $u(0,\cdot) = u_0 \in K_m \cap K_M$ *stays there for all times* $0 \leqslant t \leqslant T$*.* □

Let $\Omega \subset \mathbb{R}^N$ be open bounded with Lipschitz boundary $\partial\Omega$ and consider the following parabolic problem

$$\begin{cases} u_t = \Delta u + f(x,u), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{54}$$

where $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and such that $f(x,0) = 0$ for all $x \in \overline{\Omega}$. We shall deal with the related obstacle problem, i.e. we look for solutions u of (54) such that

$$u(x,t) \geqslant m(x) \ \text{ and } \ u(x,t) \leqslant M(x) \ \text{ for all } \ x \in \Omega,\ t > 0, \tag{55}$$

where obstacles $m, M \in C^2(\Omega)$ such that

$$m|_{\partial\Omega} \leqslant 0 \ \text{ and } \ M|_{\partial\Omega} \geqslant 0 \tag{56}$$

are given. As above we look for conditions on f, m and M implying that for any u_0 such that $m \leqslant u_0 \leqslant M$ on Ω all solutions of (54) starting at u_0 satisfy (55).

Let $X = C_0(\Omega) = \{u \in C(\overline{\Omega}) \mid u(x) = 0 \text{ for all } x \in \partial\Omega\}$. Define $A : D(A) \to X$ by $Au := -\Delta u$ (here Δ stands for the L^2-realization of the Laplacian), where $u \in D(A)$ and

$$D(A) := \{u \in X \cap H_0^1(\Omega) \mid \Delta u \in X\}.$$

It is known (see e.g. Cazenave-Haraux) that A is m-accretive and its domain is dense in X. Define $F : X \to X$ by $F(u)(x) := f(x, u(x))$ for $u \in X$ and $x \in \overline{\Omega}$. It is clear that F is well-defined and continuous. By a solution to (54) on $[0, T]$ we mean a function

$$u \in C([0,T], X) \cap C((0,T], H_0^1(\Omega)) \cap C^1((0,T], L^2(\Omega)) \tag{57}$$

such that $\Delta u(t) \in L^2(\Omega)$ and

$$\dot{u}(t) = \Delta u(t) + F(u(t)), \quad \text{for each } \ t \in (0,T], \tag{58}$$

i.e. u is a strong solution to

$$\dot{u}(t) = -A_{L^2} u(t) + F(u(t)), \quad \text{for each } \ t \in (0,T], \tag{59}$$

where $A_{L^2} : D(A_{L^2}) \to L^2(\Omega)$ is given by

$$A_{L^2} u := -\Delta u, \ \ D(A_{L^2}) := \{u \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega)\}.$$

It is well-known that A_{L^2} generates a strongly continuous semigroup of contractions, i.e. it is an m-accretive operator with dense domain and that the part of A_{L^2} in X is equal to A (see e.g. Cazenave-Haraux).

Observe that, since any solution to (54) in the above sense is, in particular, a strong solution to (59), each solution of (54) with $u(0) = u_0$ is also an integral solution of

$$\dot{u}(t) = -A_{L^2} u(t) + w(t) \ \text{ on } [0, T], \ u(0) = u_0 \tag{60}$$

with $w = F \circ u$. On the other hand, if $\tilde{u}$ is an integral solution of

$$\dot{u}(t) = -Au(t) + F\big(u(t)\big), \ t \in [0, T], \tag{61}$$

with $\tilde{u}(0) = u_0$, i.e. $\tilde{u} = u_A(\cdot, u_0, w)$, then $\tilde{u} = u_{A_{L^2}}(\cdot; u_0, w)$, i.e. $\tilde{u}$ is also an integral solution of (60) (see the proof of Theorem 33). This means that $u = \tilde{u}$. Therefore, any solution of (54) is an integral solution of (61). Hence, we are able to use Theorem 4 for (61) and Remark 32 (1) to verify condition (39).

The obstacle conditions (55) can be rewritten as

$$u(t) \in K_m \cap K_M,$$

where

$$K_m := \{u \in X \mid u(x) \geqslant m(x) \;\text{ for }\; x \in \Omega\}, \quad K_M := \{u \in X \mid u(x) \leqslant M(x) \;\text{ for }\; x \in \Omega\}.$$

If we define $V_m, V_M : X \to \mathbb{R}$ by

$$V_m(u) := \frac{1}{2}\int_\Omega (u-m)_-^2 \, dx, \quad V_M(u) := \frac{1}{2}\int_\Omega (u-M)_+^2 \, dx,$$

then clearly $K_m = \{u \in X \mid V_m(u) \leqslant 0\}$ and $K_M = \{u \in X \mid V_M(u) \leqslant 0\}$.

Assumption 40

Let m (resp. M) is a subsolution (resp. supersolution) of the stationary problem related to (54), i.e.

$$-\Delta m(x) \leqslant f\left(x, m(x)\right) \quad (\text{resp.} \ -\Delta M(x) \geqslant f\left(x, M(x)\right)) \quad \text{for} \quad x \in \Omega,$$

and

$$\limsup_{s \to 0^-} \frac{f\left(x, m(x)+s\right) - f\left(x, m(x)\right)}{s} < +\infty \tag{62}$$

$$\left(\text{resp.} \quad \limsup_{s \to 0^+} \frac{f\left(x, M(x)+s\right) - f\left(x, M(x)\right)}{s} < +\infty\right), \tag{63}$$

uniformly with respect to $x \in \overline{\Omega}$.

It is clear that each of the latter conditions is always satisfied whenever f is locally Lipschitz.

Theorem 41

If f satisfies Assumption 39 and u is a solution of (54) *such that* (55) *holds for $t = 0$, then* (55) *holds for all positive times t.* □

THANKS FOR ATTENTION :)