Coupled reaction-diffusion equations with degenerate diffusivity: wavefront analysis

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Joint work with

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Model: Bacterial colonies grown on the surface of thin agar plates

$n(t,x)$: nutrient concentration $\quad 0 \leq n \leq 1$
$b(t,x)$: bacterial concentration $\quad 0 \leq b \leq 1$

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Assumptions

$f\colon [0,1]^2 \to [0,+\infty)$ of class C^1

$f(s,r) = 0 \Leftrightarrow s = 0$ or $r = 0$

$\exists\, 0 < L_1 \leq L_2 : \quad L_1 sr \leq f(s,r) \leq L_2 sr \quad \forall\,(s,r) \in (0,1)^2$

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$f(n,b)$: reaction term
$g(n)h(b)$: diffusivity

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$g, h\colon [0,1] \to [0,+\infty)$ of class C^1
$g(s) = 0 \Leftrightarrow s = 0 \qquad h(r) = 0 \Leftrightarrow r = 0$
$\exists\, M_g > 0 : \quad g(s) \geq M_g g(s_1) \quad \forall\, s \geq s_1 \geq 0$
$\dot{g}(0), \dot{h}(0) > 0$

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Remark Null diffusivity of the nutrient: modelling the case of the so called hard agar

Non degenerate systems with positive nutrient diffusion

Example 1 **Berestycki-Nicolaenko-Scheurer 1985, Marion 1985**

$$\begin{cases} -\eta'' + c\eta' = f(\eta)\beta \\ -\lambda\beta'' + c\beta' = -f(\eta)\beta \end{cases}$$

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- **Existence for c sufficiently big**
- **For $\lambda \in (0,1)$: Existence of a threeshold speed and uniqueness**

Non degenerate systems with null nutrient diffusion

Example 2 **Logak-Loubeau 1996, Logak 1997**

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+ additional technical condition

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- **For $d \in (0,1)$: Existence of a threeshold speed**

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$$\begin{cases} n_t = D_n n_{xx} - n^q b^l \\ b_t = [D_b(1-b)n^p b b_x]_x + n^q b^l \end{cases} \qquad t, x \in \mathbb{R}$$

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$$\begin{cases} n_t = -knb \\ b_t = [(1-n)b_x]_x + (1-b)b \end{cases}$$

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▶ **Existence and uniqueness for every $c > 0$**

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$$\begin{cases} n_t = n(1 - n - db) \\ b_t = [(1 - n)b_x]_x + rb(1 - b) \end{cases}$$

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Assumptions

$d, r > 0$

- **For $d \in (0, 1)$: Existence and uniqueness for every $c > 0$**
- **For $d > 1$: Non-existence**

$$c \in \mathbb{R} : \text{wave speed}$$

$$\eta(\xi) = \eta(x - ct) = n(x,t) \qquad \beta(\xi) = \beta(x - ct) = b(x,t) : \quad \text{wave profiles}$$

$$\begin{cases} c\eta' - f(\eta,\beta) = 0 \\ (g(\eta)h(\beta)\beta')' + c\beta' + f(\eta,\beta) = 0 \end{cases} \qquad \xi \in \mathbb{R}$$

$c \in \mathbb{R}$: wave speed

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η of class C^1 classical solution

Traveling waves solutions

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β continuous and differentiable a.e., weak solution
$h(\beta)\beta' \in L^1_{loc}(\mathbb{R})$

$$\int_{\mathbb{R}} \Big(\left(g(\eta(\xi))h(\beta(\xi))\beta'(\xi) + c\beta(\xi)\right) \psi'(\xi) - f(\eta(\xi),\beta(\xi))\psi(\xi) \Big) \, \mathrm{d}\xi = 0$$

$$\forall \, \psi \in C_0^\infty(\mathbb{R})$$

We look for travelling waves connecting natural steady states, i.e. satisfying the boundary conditions

$$\begin{cases} \eta(-\infty) = 0 & \beta(-\infty) = 1 \\ \eta(+\infty) = 1 & \beta(+\infty) = 0 \end{cases}$$

Proposition If (η, β) is a traveling wave satisfying the boundary conditions and $J = \{\xi \in \mathbb{R} \colon \beta(\xi) = 0\}$ then

$$g(\eta)h(\beta)\beta' + c\beta + c\eta - c = 0 \quad \text{in } \mathbb{R}\backslash J$$

Only two cases can occur:

$J = \emptyset \Rightarrow (\eta, \beta)$ classical

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$J \neq \emptyset \Rightarrow (\eta, \beta)$ sharp at 0 : $\exists \xi_\ell \in \mathbb{R} : \beta(\xi_\ell) = 0, \beta$ is classical on $\mathbb{R} \setminus \{\xi_\ell\}, \beta$ is not differentiable at ξ_ℓ

Proposition Denoted

$$\tau := \sup\{\xi \in \mathbb{R} \colon \beta(\xi) > 0\} \in \mathbb{R} \cup \{+\infty\}$$

$\eta'(\xi) > 0, \beta'(\xi) < 0 \quad \forall \xi < \tau$

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$\beta(\xi) = 0, \eta(\xi) = 1, \forall \xi \geq \tau$

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if $\tau = +\infty \Rightarrow \beta'(+\infty) = 0$

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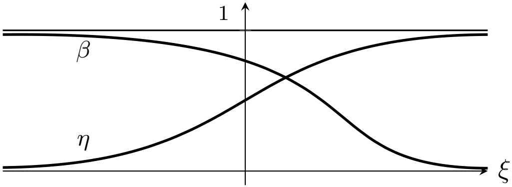
if $\tau = +\infty \Rightarrow \beta'(+\infty) = 0$
if $\tau \in \mathbb{R} \Rightarrow \beta'(\tau) = 0$ or $\beta'(\tau) = -\frac{c}{g(1)\dot{h}(0)}$

Traveling wave (η, β) with non-constant monotone profiles functions

If $\tau = +\infty$, then (η, β) is a classical wavefront



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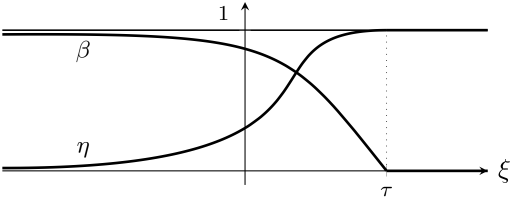
If $\tau \in \mathbb{R}$ and $\beta'(\tau) = 0 \Rightarrow (\eta, \beta)$ is a classical wavefront
If $\tau \in \mathbb{R}$ and $\beta'(\tau) = -c/g(1)\dot{h}(0) \Rightarrow (\eta, \beta)$ is a sharp wavefront at 0



Theorem There exists $c_0 \in \mathbb{R}$:

- if $c < c_0$ there are no wavefronts
- if $c = c_0$ there is a unique (up to shifts) wavefront which is sharp at 0
- if $c > c_0$ there is a unique (up to shifts) wavefront which is classical

Moreover, denoted

$$\underline{c} := \max\left\{\sqrt{L_1 M_g \int_0^1 g(1-r)h(r)r\,\mathrm{d}r},\ \sqrt{2L_1 M_g \int_0^1 (1-r)g(1-r)h(r)r\,\mathrm{d}r}\right\}$$

and

$$\overline{c} := 2\sqrt{L_2 \max_{s\in[0,1]} g(s) \sup_{r\in(0,1]} \frac{h(r)}{r}}$$

it holds

$$\underline{c} \le c_0 \le \overline{c}$$

Existence of a semi-wavefront

Theorem $\forall\, c > 0, \eta_0 \in (0, 1)$, there exists a semi-wavefront (η, β) defined in $(-\infty, 0]$ such that

$$\begin{cases} \eta(-\infty) = 0 \quad \beta(-\infty) = 1 \\ \eta(0) = \eta_0 \end{cases}$$

Proof: Define

$$\mathfrak{N} := \left\{ \eta \in C^1(-\infty, 0] \colon \eta_0 e^{\frac{L_2 \xi}{c}} \leq \eta(\xi) \leq \eta_0 e^{\frac{L_1(1-\eta_0)\xi}{c}},\ \eta'(\xi) \geq 0,\ \forall \xi \in (-\infty, 0] \right\}$$

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$\Rightarrow \forall \eta \in \mathfrak{N}, c > 0\, \exists\, ! \beta_0 \in (0, 1)$:

$$\begin{cases} \beta' = \dfrac{c(1-\beta-\eta)}{g(\eta)h(\beta)} \\ \beta(0) = \beta_0 \end{cases}$$

has a solution β with $\beta(-\infty) = 1$

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Define $\mathcal{T} : \mathfrak{N} \to C^1(-\infty, 0]$ as $\eta \mapsto \beta \mapsto \tilde{\eta}$, where $\tilde{\eta}$ is the solution of

$$\begin{cases} \tilde{\eta}' = \dfrac{f(\tilde{\eta}, \beta)}{c}, \\ \tilde{\eta}(0) = \eta_0. \end{cases}$$

Fixed point theorem

Uniqueness of the semi-wavefront

Theorem For every $c > 0$ there exists at most one (up to shift) semi-wavefront (η, β) defined in $(-\infty, 0]$ with $\eta(-\infty) = 0$ and $\beta(-\infty) = 1$

Proof: Fixed $c > 0, \eta_0 \in (0, 1)$ and a corresponding semi-wavefront (η, β) define

$$\Phi(\xi) = \int_0^\xi \frac{1}{g(\eta(s))h(\beta(s))}\, ds, \quad \xi \in (-\infty, 0]$$

$\Rightarrow$ Φ is a diffeomorphism and $\Phi(-\infty) = -\infty$

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Define

$$p(y) := \eta\left(\Phi^{-1}(y)\right) \quad q(y) := \beta\left(\Phi^{-1}(y)\right) + \eta\left(\Phi^{-1}(y)\right) - 1$$

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$\Rightarrow$ (p, q) solves

$$\begin{cases} \dot{p}(y) = \dfrac{f(p(y), q(y) - p(y) + 1)g(p(y))h(q(y) - p(y) + 1)}{c}, \\ \dot{q}(y) = -cq(y) + \dfrac{f(p(y), q(y) - p(y) + 1)g(p(y))h(q(y) - p(y) + 1)}{c} \\ p(-\infty) = 0, \quad q(-\infty) = 0 \end{cases}$$

Center Manifold theorem

Properties of the extended semi-wavefront

Proposition Denoted

$$\tau := \sup\{\xi \in \mathbb{R} \colon \beta(\xi) > 0\} \in \mathbb{R} \cup \{+\infty\}$$

and

$$\sigma := \inf\{\xi \in \mathbb{R} \colon \eta(\xi) = 1\} \in \mathbb{R} \cup \{+\infty\}$$

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If $\tau = +\infty \Rightarrow \eta(+\infty) = 1, \beta(+\infty) = 0$

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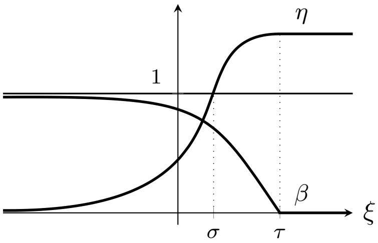

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Proof: Define

$$z(\beta) := g\left(\eta\left(\xi(\beta)\right)\right) h(\beta)\beta'\left(\xi(\beta)\right) \quad 0 < \beta \leq \beta_0$$

where $\xi = \xi(\beta)$ is the inverse function of β

Theorem There exists $\rho_* > 0$ such that, for all $c > \rho_*$ (η, β) is a classical wavefront

Proof: Define

$$z(\beta) := g\left(\eta\left(\xi(\beta)\right)\right) h(\beta)\beta'\left(\xi(\beta)\right) \quad 0 < \beta \leq \beta_0$$

where $\xi = \xi(\beta)$ is the inverse function of β

$\Rightarrow$ z is a solution of

$$\begin{cases} \dot{z}(\beta) = -c - \frac{f(\eta(\xi(\beta)),\beta)g(\eta(\xi(\beta)))h(\beta)}{z(\beta)} \\ z(0) = 0 \end{cases}$$

Comparison techniques

Existence, uniqueness and classification of the wavefront

Theorem There exists $c_0 \in \mathbb{R}$:

- if $c < c_0$ there are no wavefronts
- if $c = c_0$ there is a unique (up to shifts) wavefront which is sharp at 0
- if $c > c_0$ there is a unique (up to shifts) wavefront which is classical

Moreover, denoted

$$\underline{c} := \max \left\{ \sqrt{L_1 M_g \int_0^1 g(1-r)h(r)r \, \mathrm{d}r}, \sqrt{2L_1 M_g \int_0^1 (1-r)g(1-r)h(r)r \, \mathrm{d}r} \right\}$$

and

$$\overline{c} := 2\sqrt{L_2 \max_{s\in[0,1]} g(s) \sup_{r\in(0,1]} \frac{h(r)}{r}}$$

it holds

$$\underline{c} \le c_0 \le \overline{c}$$

Proof Denote

$$B(\eta) := \beta\left(\xi(\eta)\right) \quad 0 < \eta < 1$$

where $\xi = \xi(\eta)$ is the inverse function of η

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$\Rightarrow$ B is a solution of

$$\begin{cases} \dot{B}(\eta) = \dfrac{c^2(1-\eta-B(\eta))}{g(\eta)h(B(\eta))f(\eta, B(\eta))} \\ B(0) = 1 \end{cases}$$

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$$\begin{cases} \dot{B}(\eta) = \dfrac{c^2\left(1 - \eta - B(\eta)\right)}{g(\eta)h(B(\eta))f(\eta, B(\eta))} \\ B(0) = 1 \end{cases}$$

Ad hoc techniques

Thank you for your attention!