Invariant graphs and dynamics of strongly dissipative planar piecewise linear continuous maps

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I will present results obtained together with Anna Cima, Armengol Gasull and Francesc Mañosas of the recent articles:

Cima, Gasull, Mañosas, VM, Invariant graphs and dynamics of a family of continuous piecewise linear planar maps. *Qualitative Theory of Dynamical Systems* **24**:70 (2025), 103 pp.

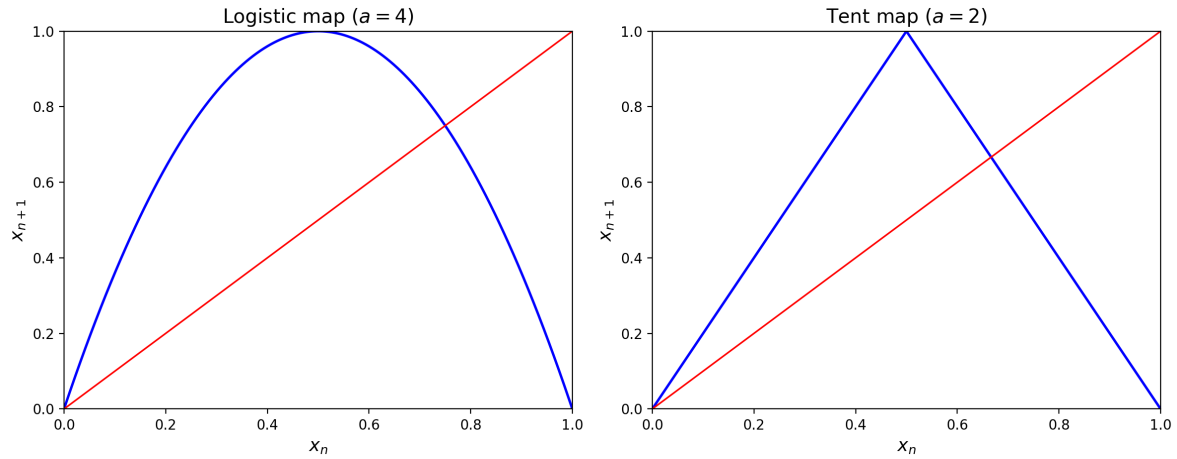
Cima, Gasull, Mañosas, VM, Entropy and ω-limit sets on invariant graphs for a class of planar piecewise linear maps. *Commun Nonlinear Sci Numer Simulat* **163** (2026) 110602, 20pp.

Cima, Gasull, Mañosas, VM, Strongly dissipative planar piecewise linear continuous maps (provisional title). In preparation.

Supported by the Ministry of Science and Innovation–State Research Agency through grants PID2022-136613NB-I00 and PID2025-168432NB-I00

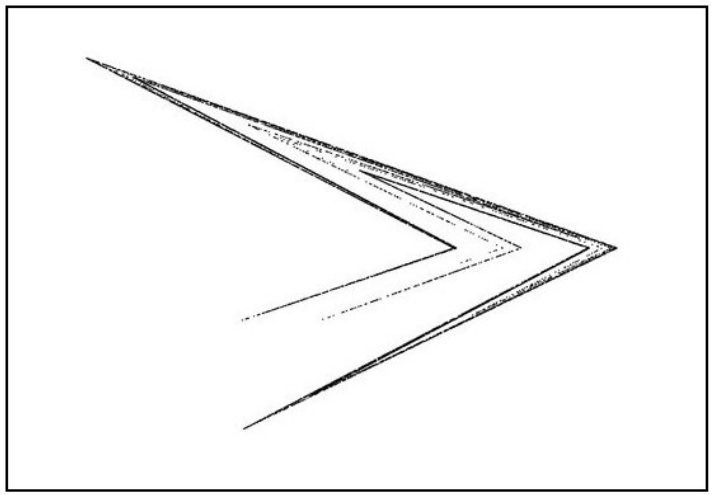


We all know that **discrete piecewise linear** dynamical systems can be very complex (having full **chaos**), even for one-dimensional maps.



logistic map and *tent map*.

In the plane it is also common knowledge:



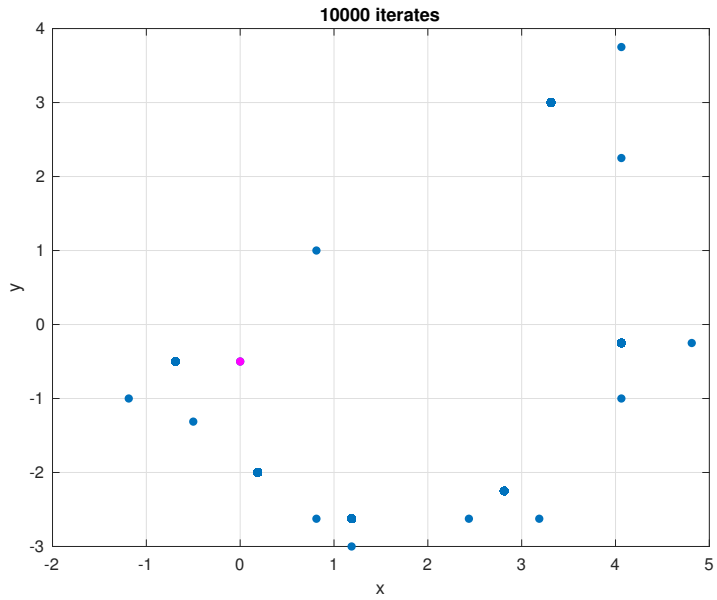
A strange attractor for the Lozi map (1978): $F(x, y) = (1 - \alpha|x| + y, \beta y)$, which a counterpart of the Henon map (1976)

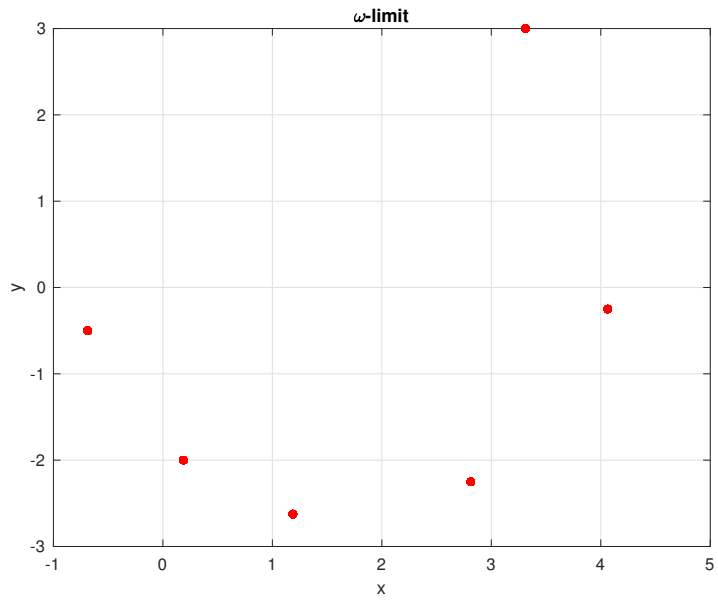
In the last years we have investigated the following family of DDS

$$F(x, y) = (|x| - y + a, x - |y| + b)\,.$$

There are some works of Tikjha, Lapierre & Lenbury..., Bula & Sīle and Aiewcharoen, Boonklurb & Konglawan focusing on periodic and pre-periodic behaviors for some values of the parameters.

Numerically, only periodic and pre-periodic behavior are found.





But there is a hidden complex dynamics.

Main results in a few words

For the map

$$F(x, y) = (|x| - y + a, x - |y| + b),$$

we prove:

Periodic and pre-periodic behavior:

- For $a \geq 0$ all the orbits are pre-periodic and moreover the set of periodic orbits has finite cardinality.

A *(not so-)* novel behavior:

- For $a < 0$ we prove that there exists a compact graph Γ, which is invariant for F, s.t. for all $(x, y) \in \mathbb{R}^2$ there exists $n = n(x, y) \in \mathbb{N}$ such that $F^n(x, y) \in \Gamma$.

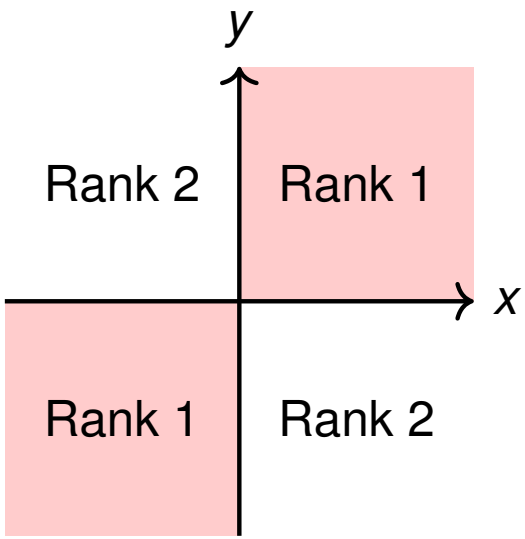
The dynamics is captured in a 1-dimensional set. We call this **strong dissipation**

- We give explicitly all these invariant graphs Γ.
- We characterize the dynamics of the map restricted to each graph. In particular for each (a, b) **we characterize when $F|_\Gamma$ has zero or positive entropy**.

Intermediate dynamical behavior between regular regime and full-plane chaos.

For $a < 0$, we study

$$F(x, y) = (|x| - y + a, x - |y| + b)\,.$$



Theorem

Set $a < 0$. For all $b \in \mathbb{R}$ there exists a compact graph Γ which is invariant under the map F such that for all $\mathbf{x} = (x, y) \in \mathbb{R}^2$ there exists $n_{\mathbf{x}} \in \mathbb{N}$ (that depends on $\mathbf{x}$) such that $F^{n_{\mathbf{x}}}(x, y) \in \Gamma$.

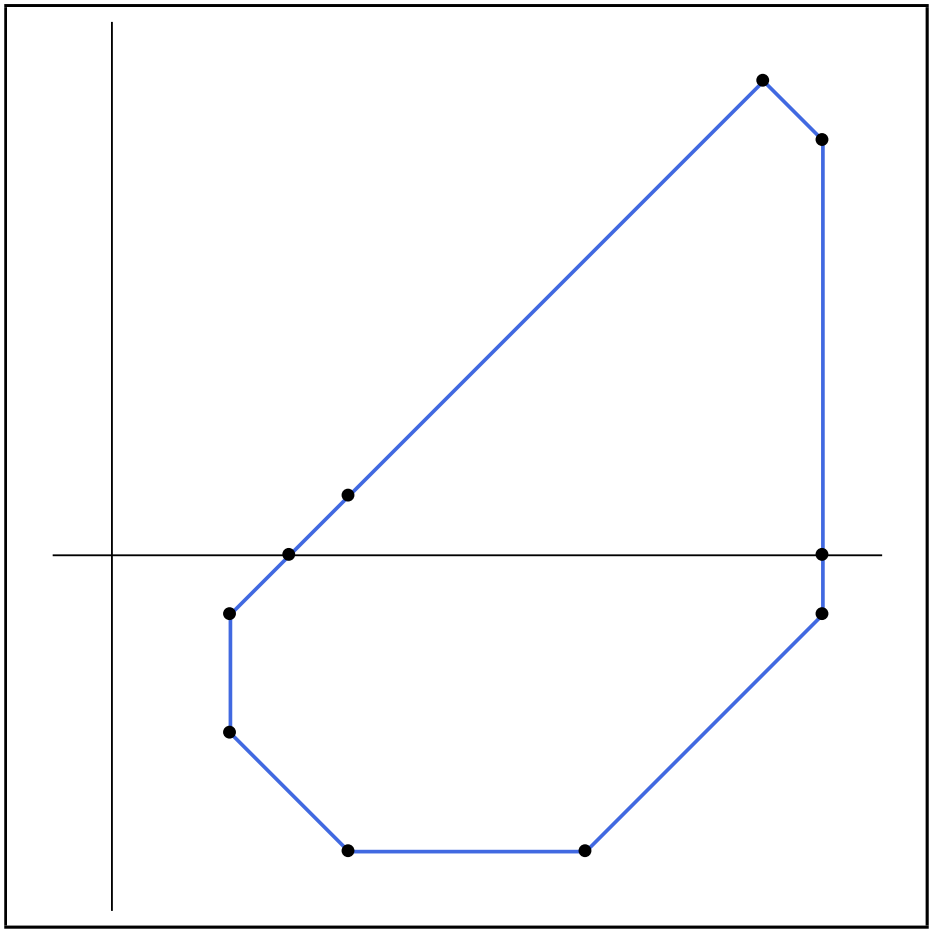
The final dynamics takes place in certain compact graphs, that we will see right now, **however**:

There are many distinct dynamics on each graph.

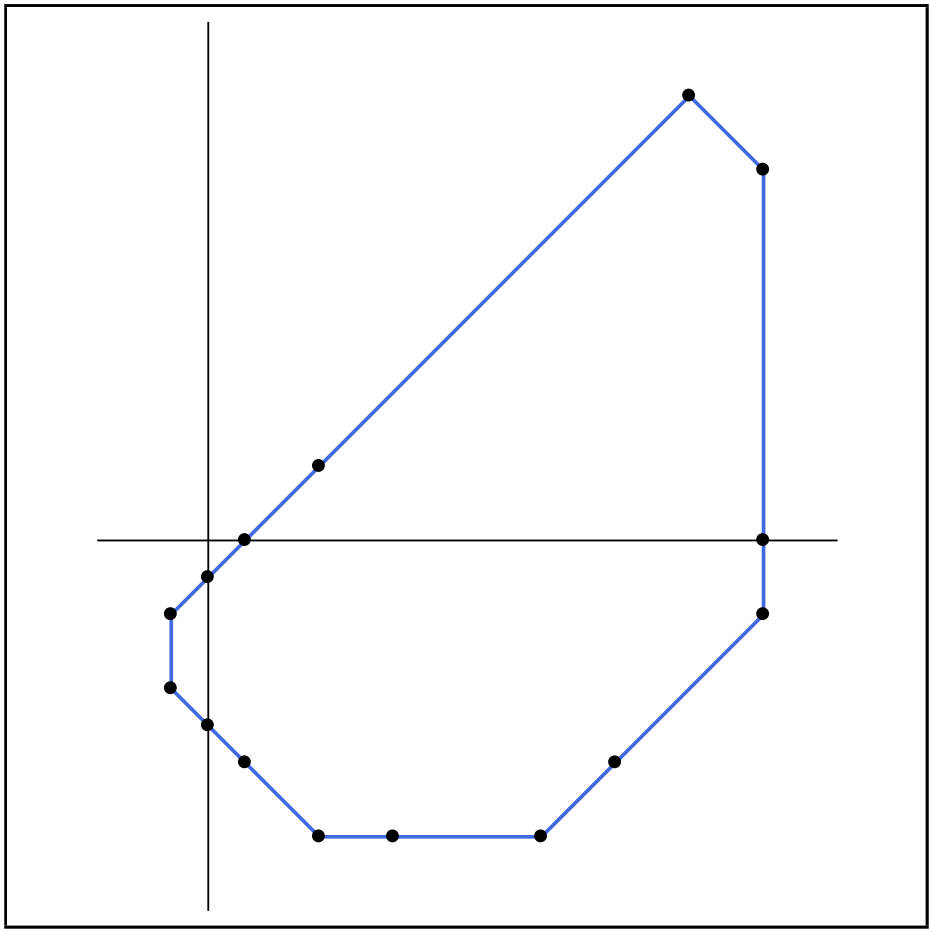
After a rescaling, we can assume that $a = -1$. The map is then conjugate to:

$$F(x, y) = (|x| - y - 1, x - |y| + b).$$

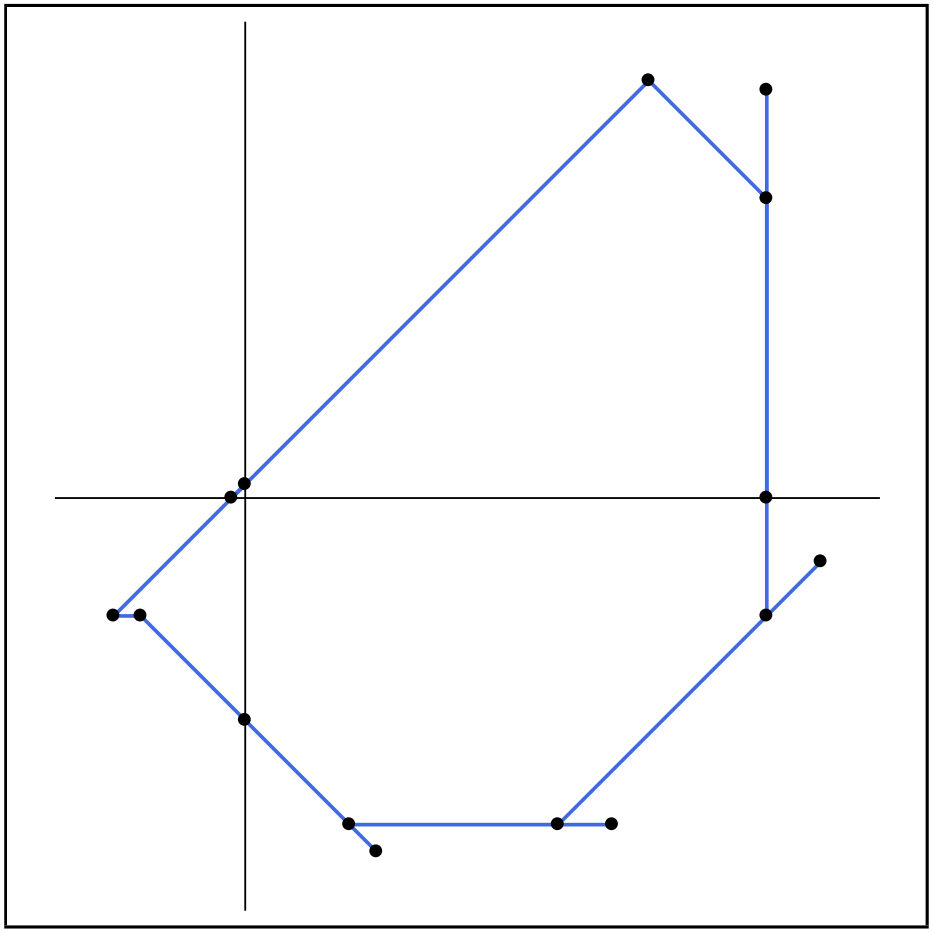
Complete catalog of graphs for the normalized case $a = -1$: $a = -1,\ b \leq -2$



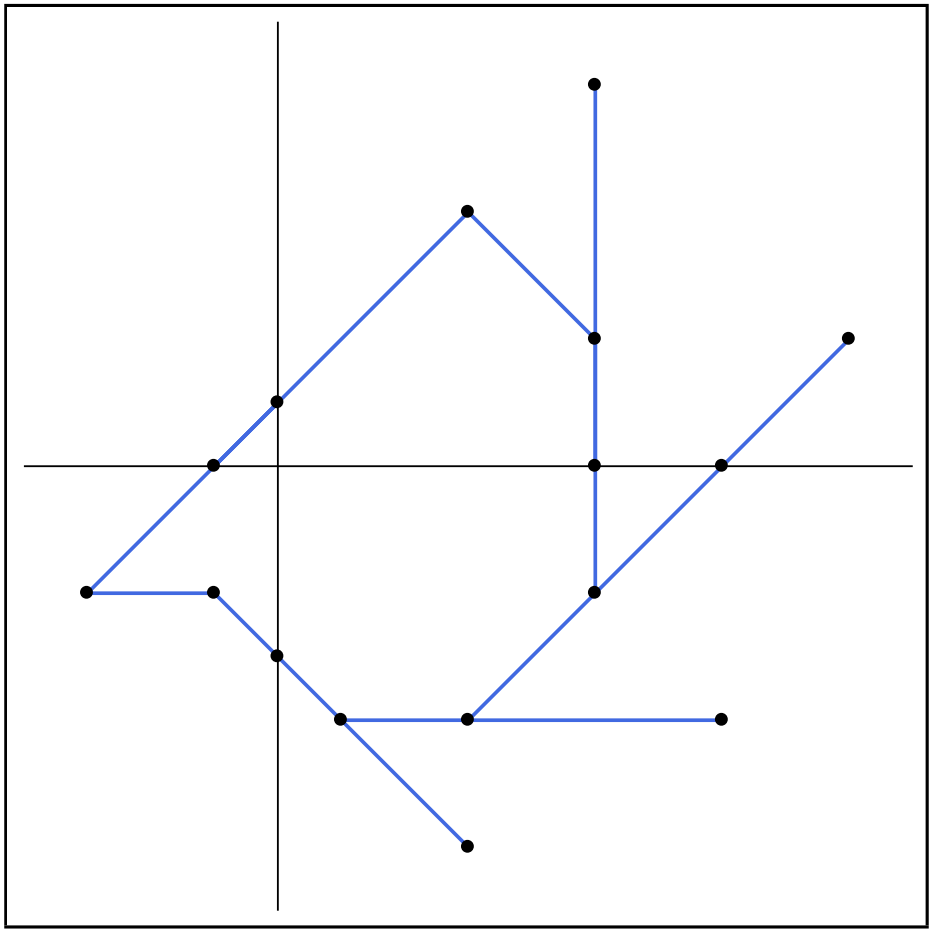
$a = -1,\ -2 < b \leq -1$



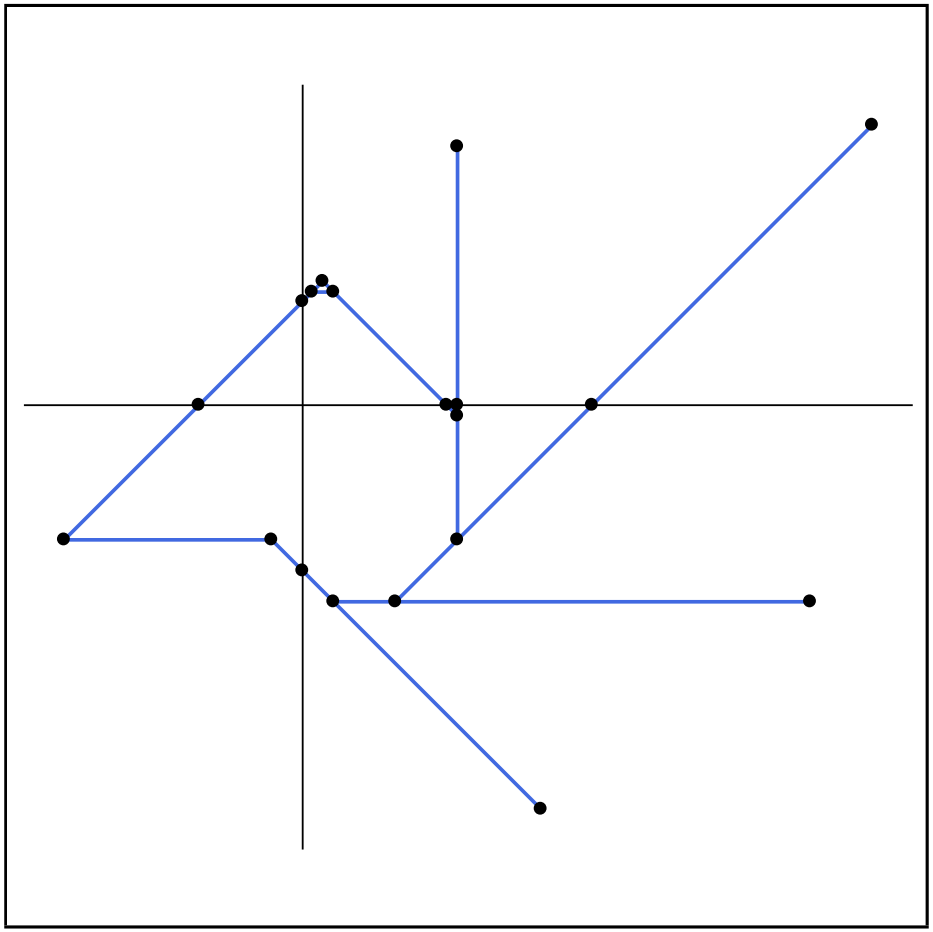
$a = -1,\ -1 < b \leq -3/4$



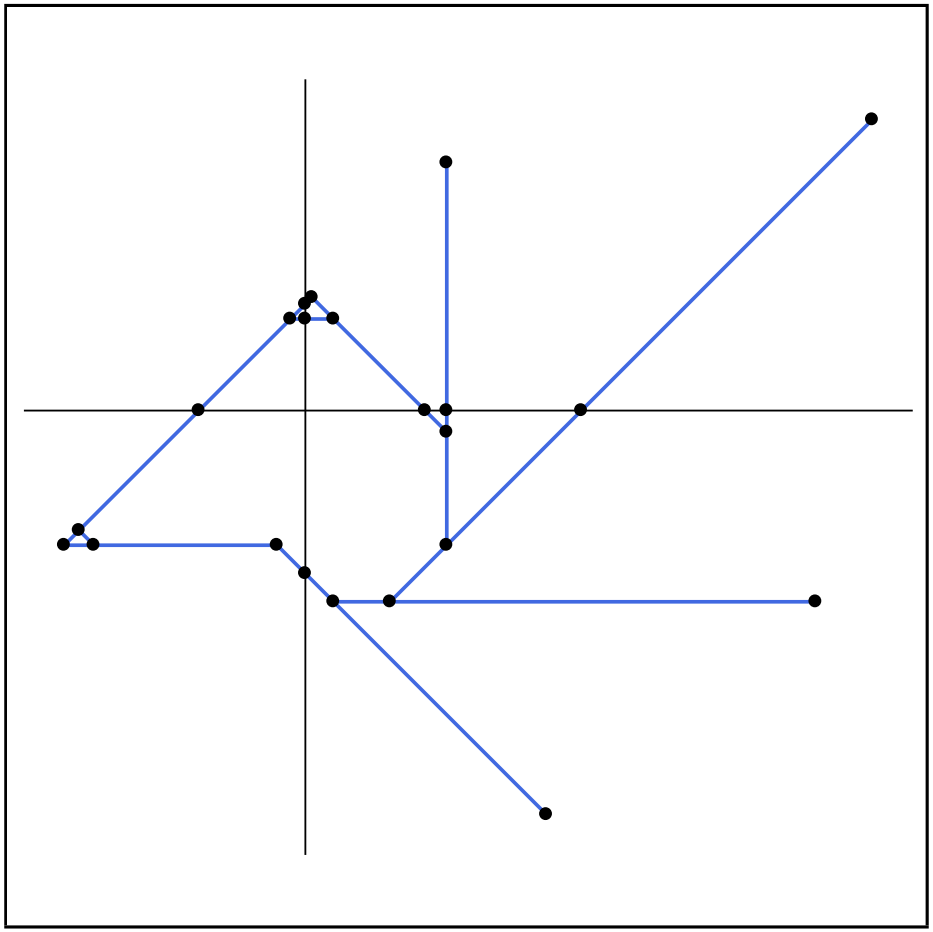
$a = -1,\ -3/4 < b \leq -1/4$



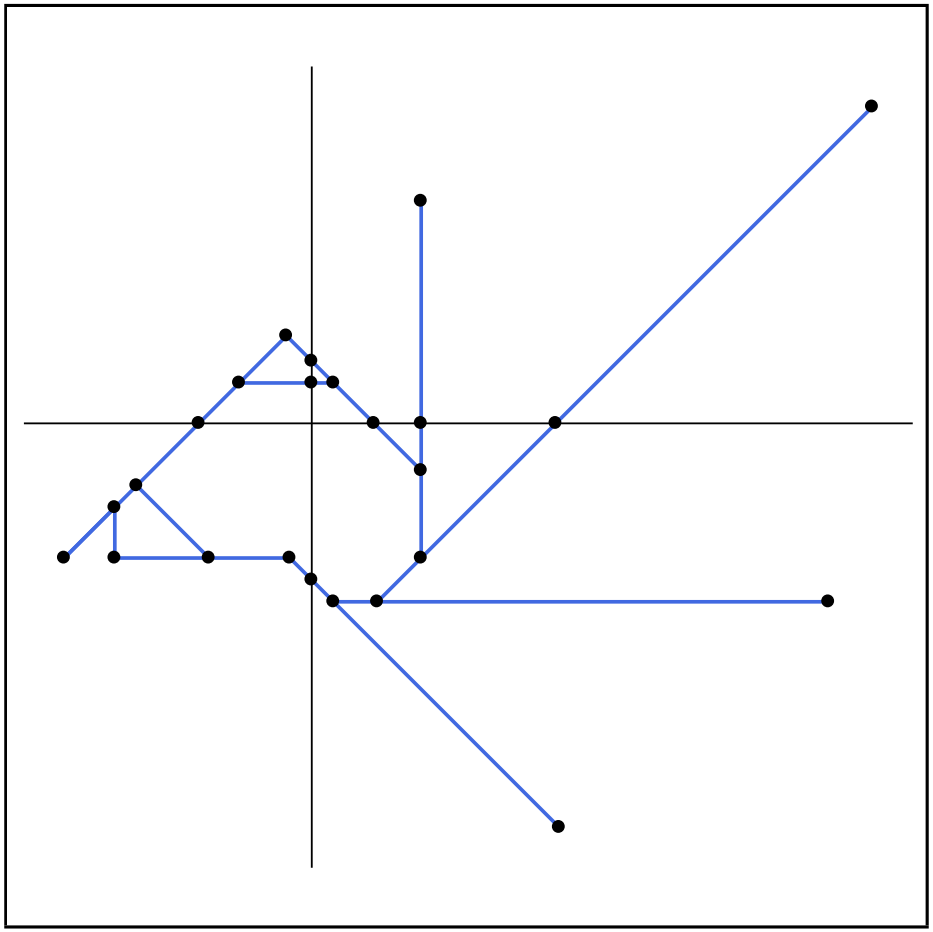
$a = -1,\ -1/4 < b \leq -2/9$



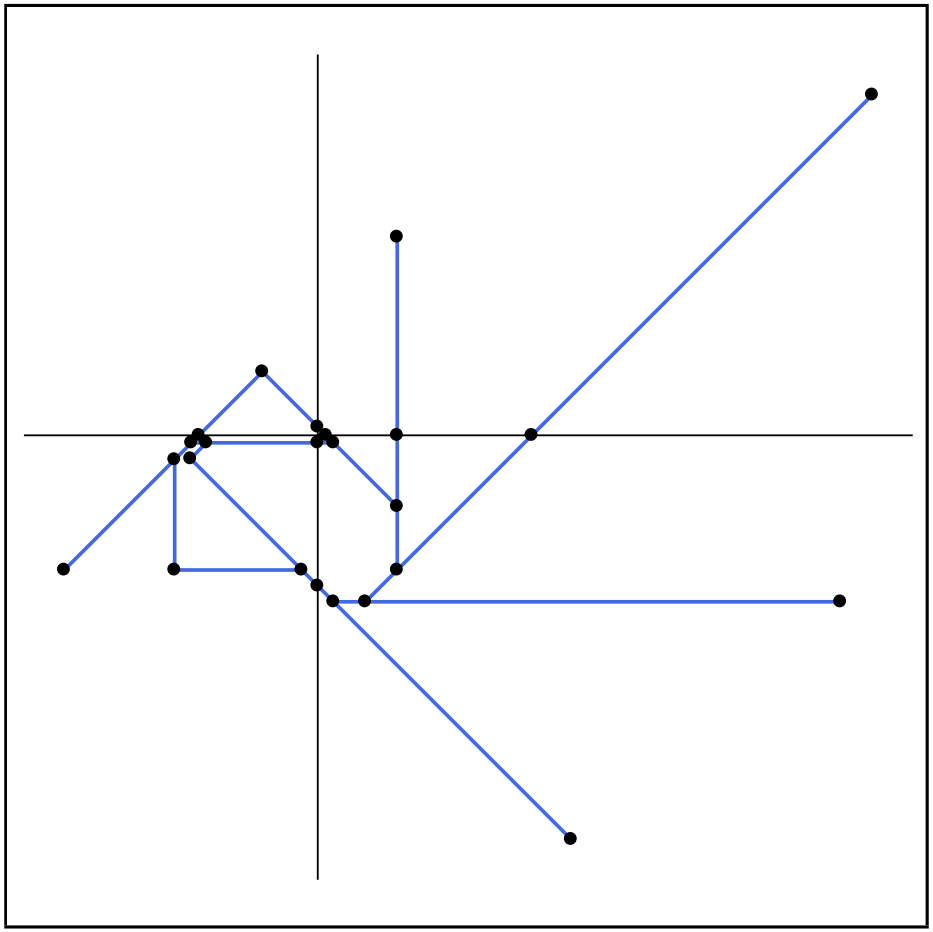
$a = -1,\ -2/9 < b \leq -1/5$



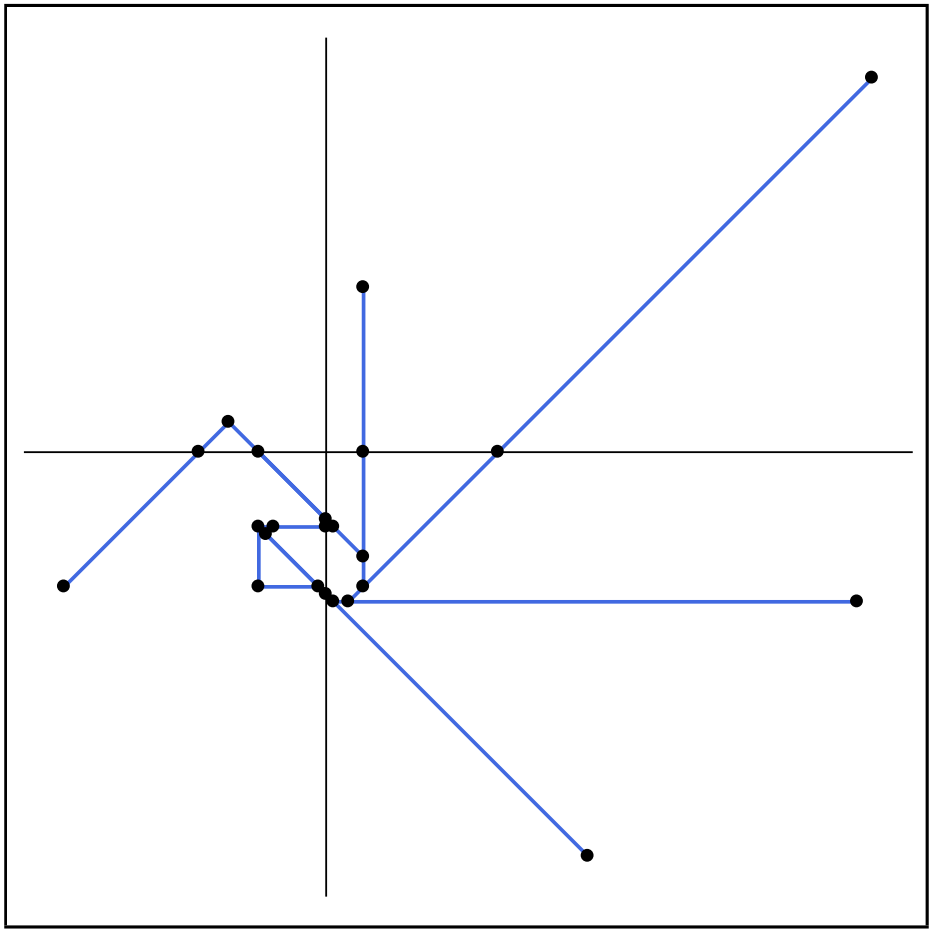
$a = -1,\ -1/5 < b \leq -1/8$



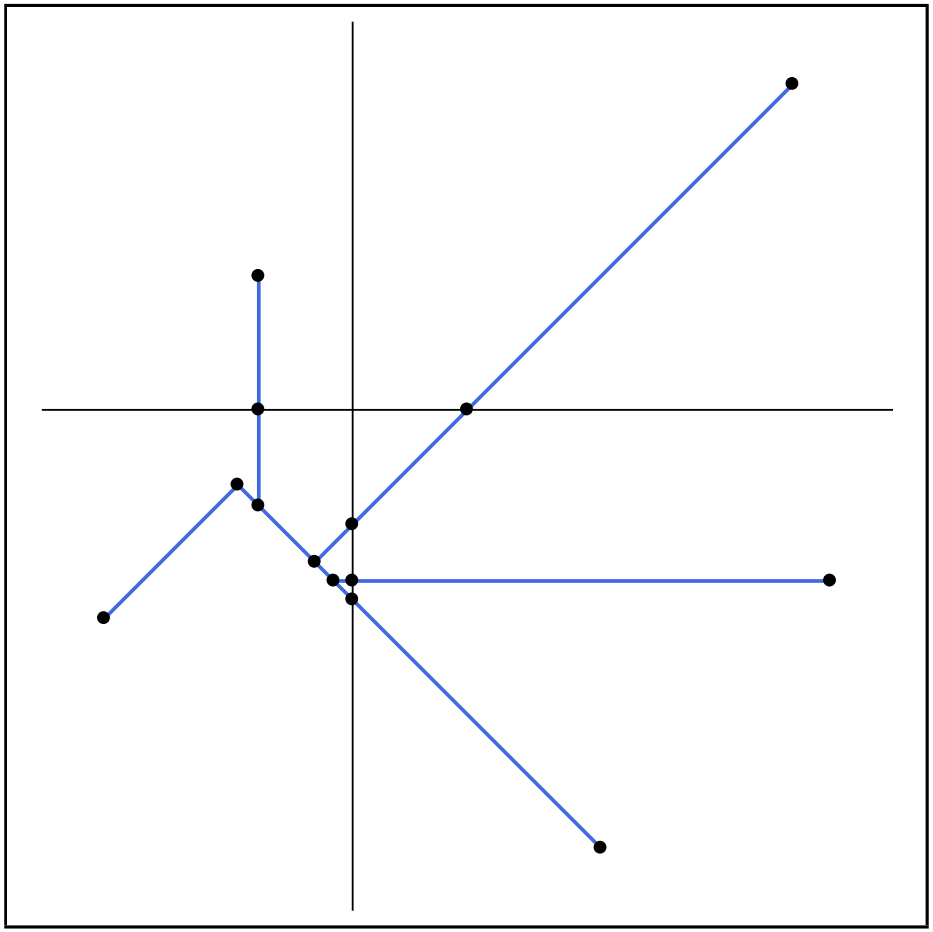
$a = -1,\ -1/8 < b \leq -1/9$



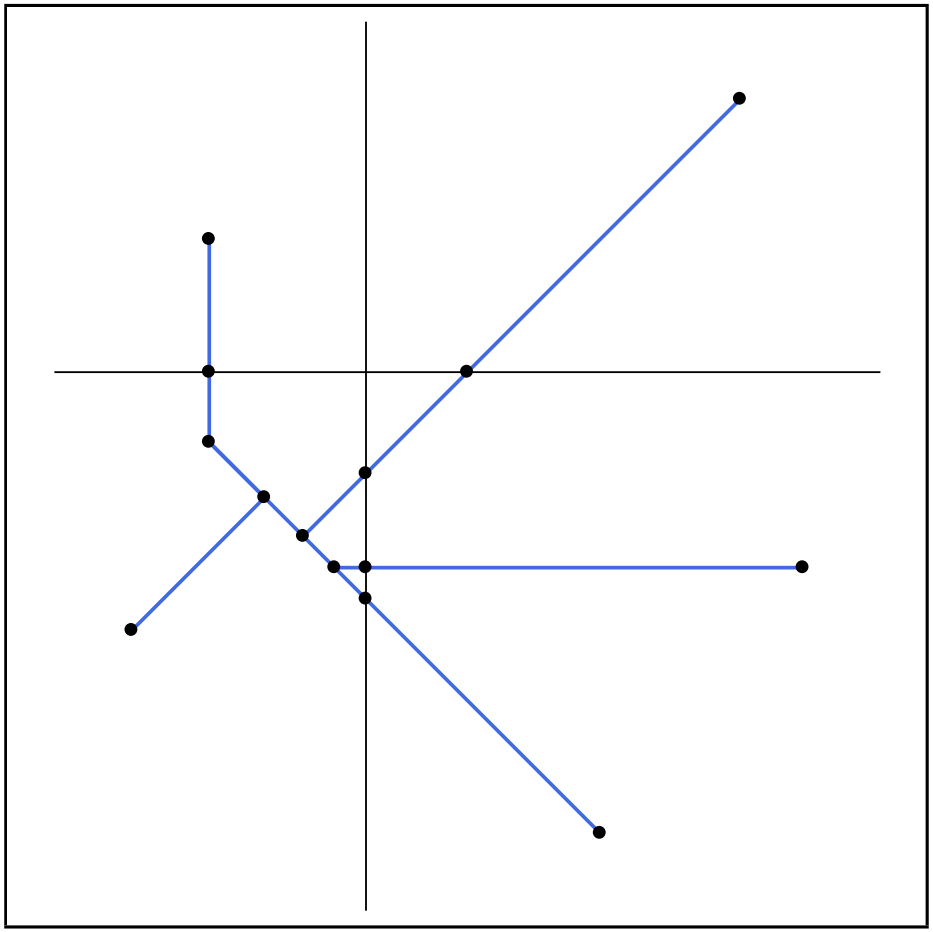
$a = -1,\ -1/9 < b < 0$



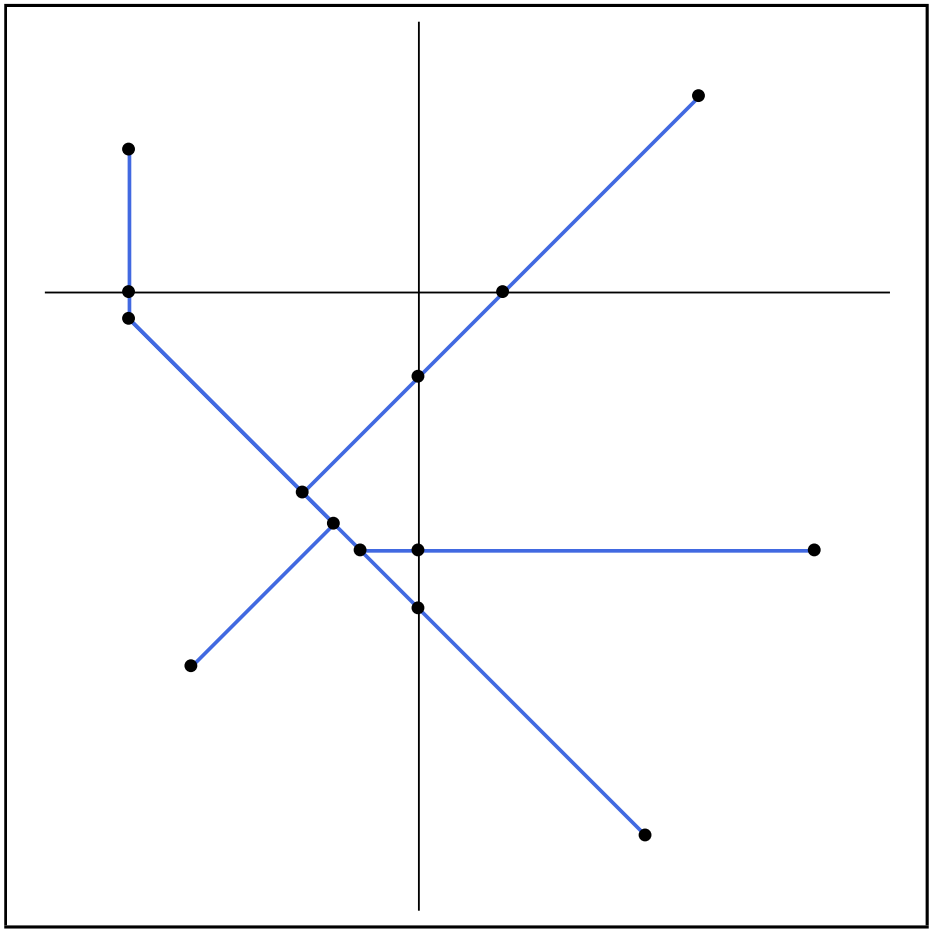
$a = -1,\ 0 \leq b \leq 1/10$



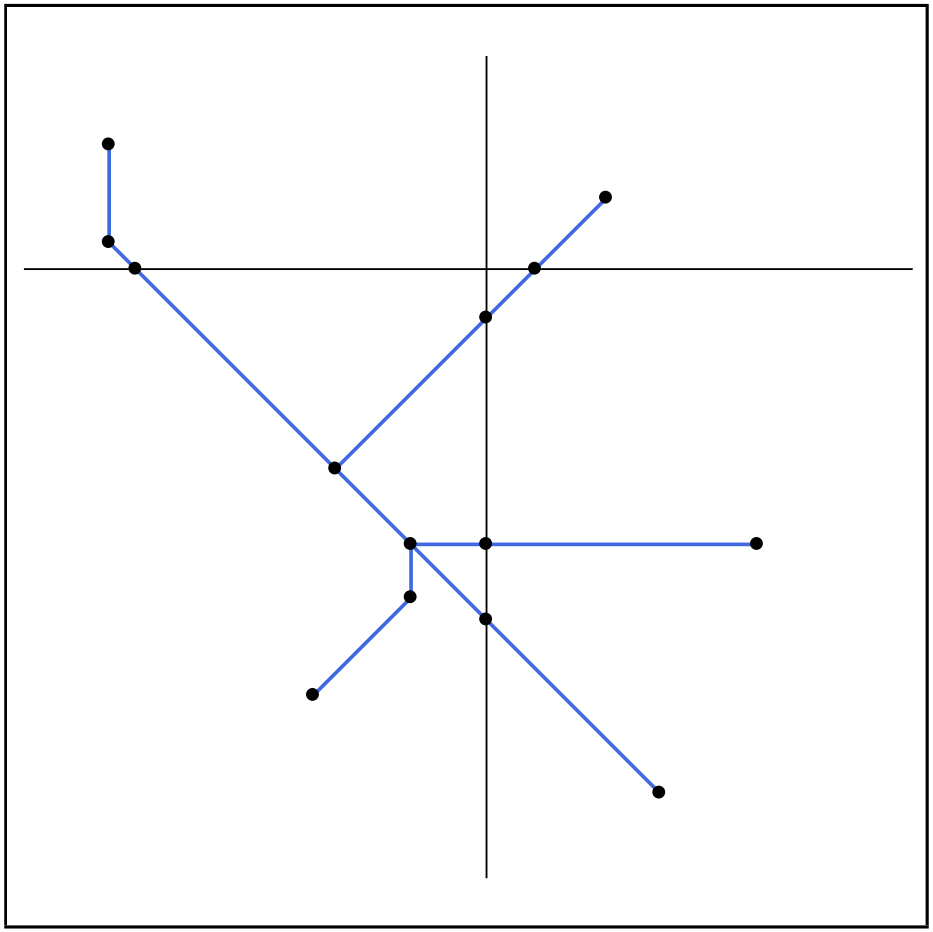
$a = -1,\ 1/10 < b \leq 1/7$



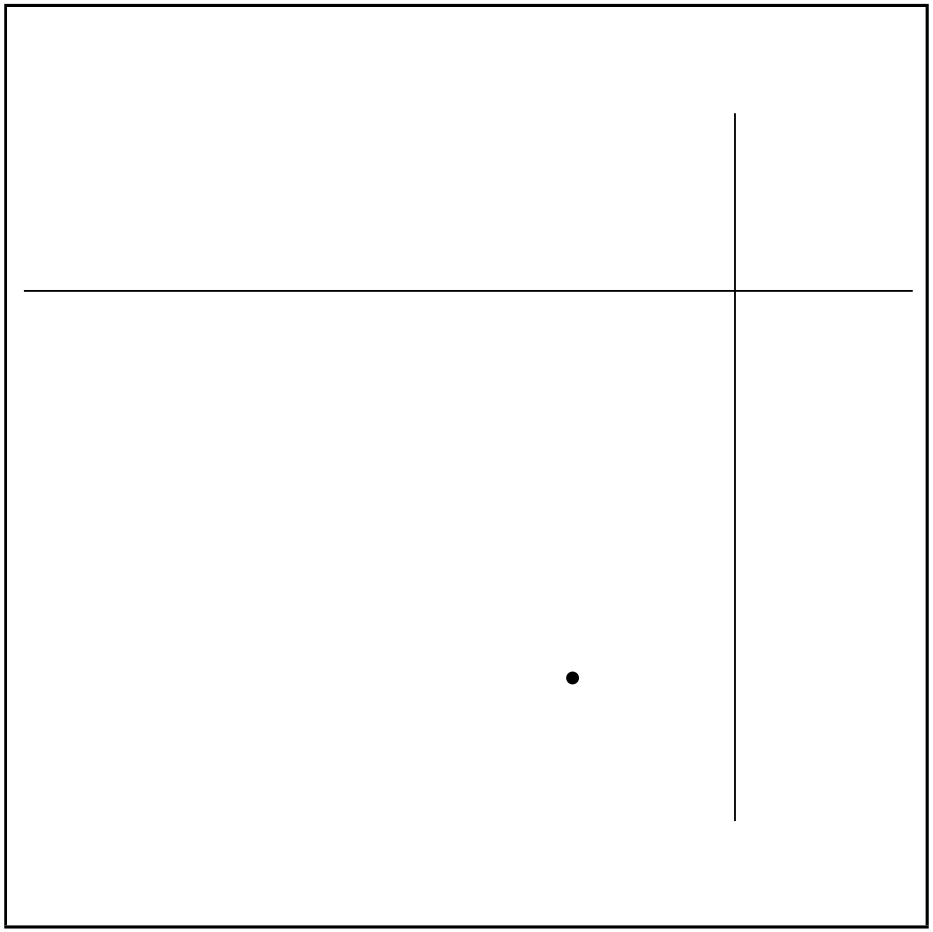
$a = -1,\ 1/7 < b \leq 1/6$



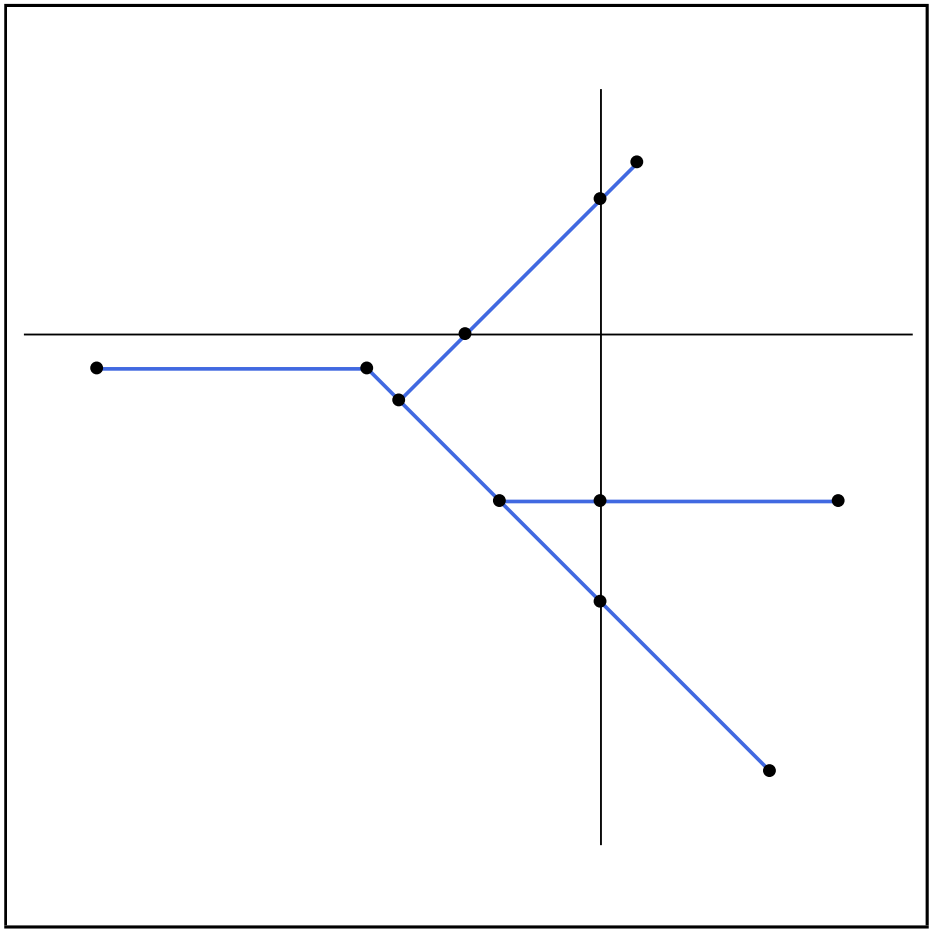
$a = -1,\ 1/6 < b \leq 3/16$



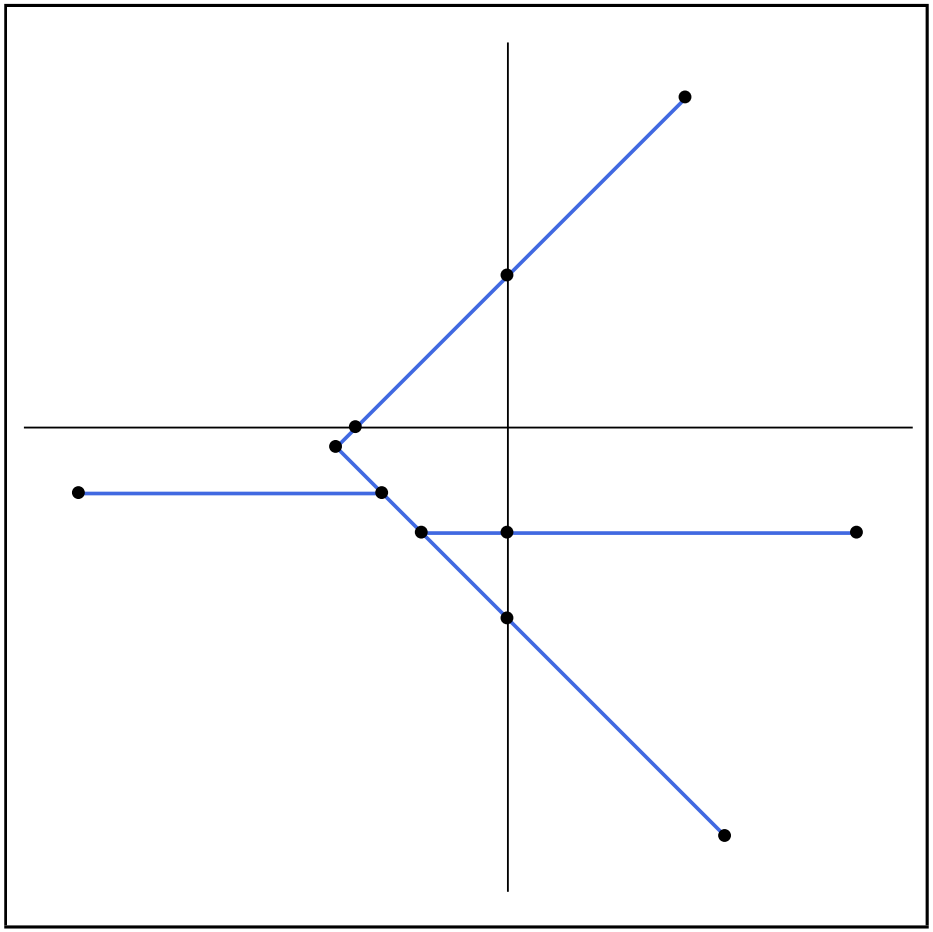
$a = -1,\ 3/16 < b < 4/15$



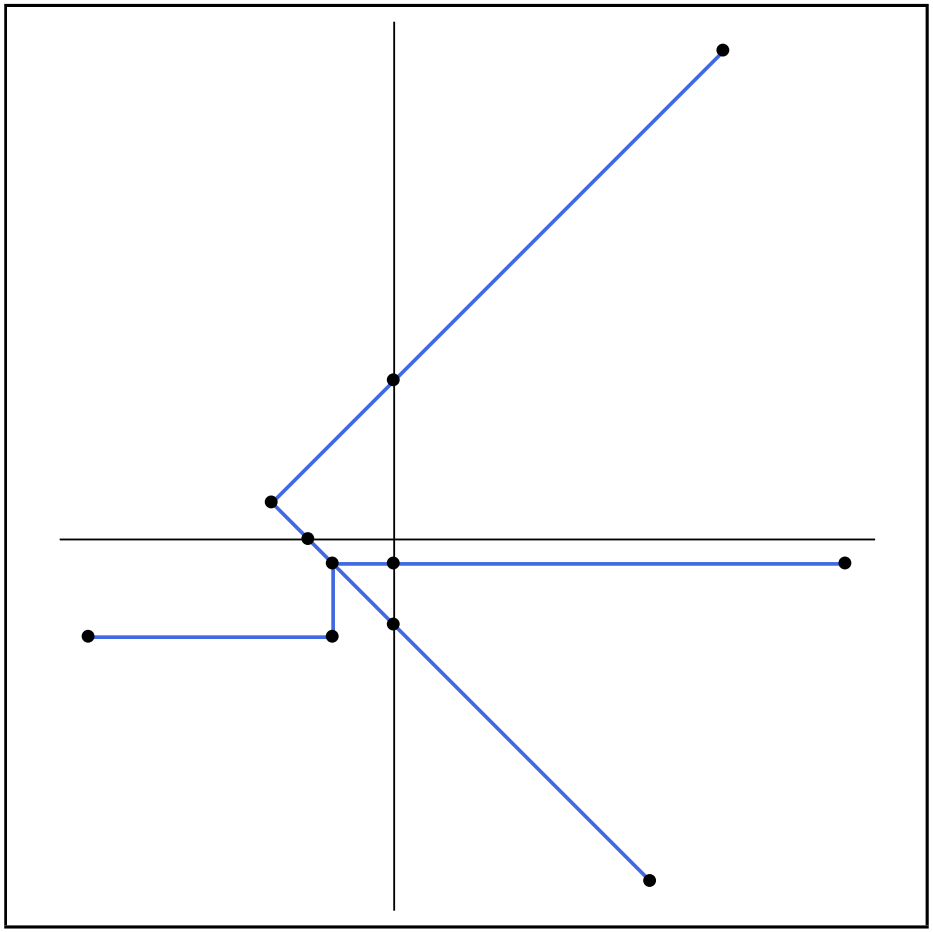
$a = -1,\ 4/15 < b \leq 2/7$



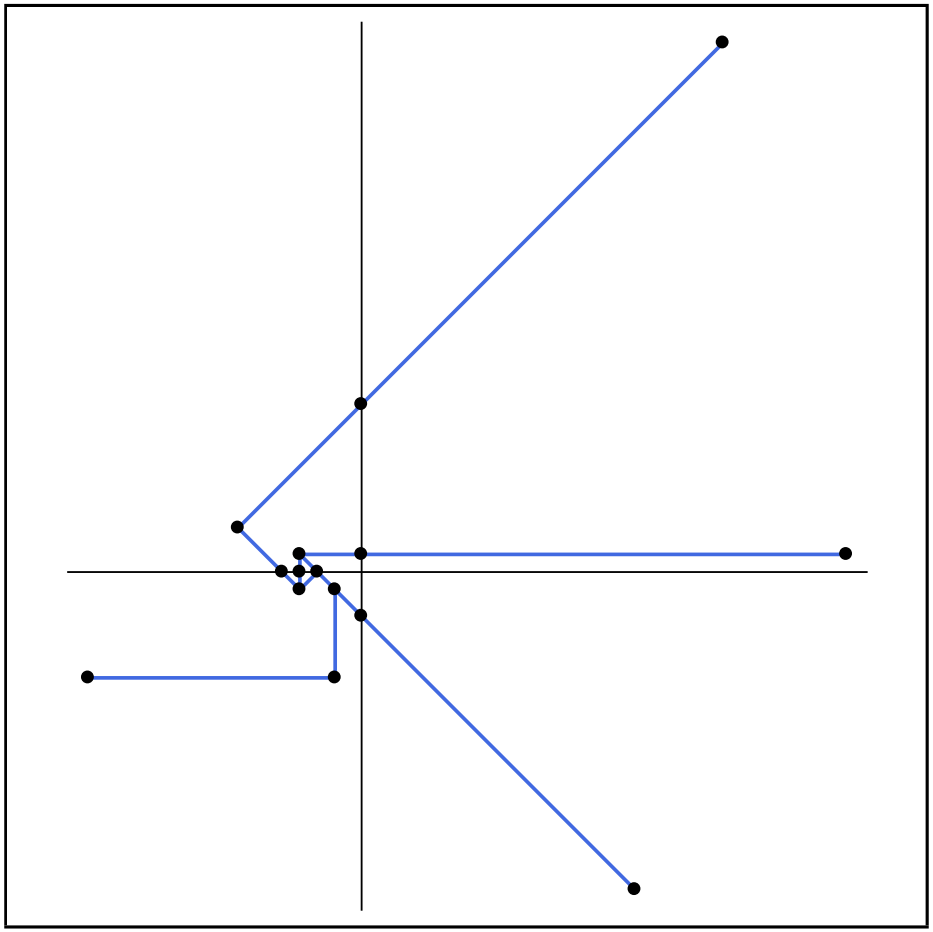
$a = -1,\ 2/7 < b \leq 1/3$



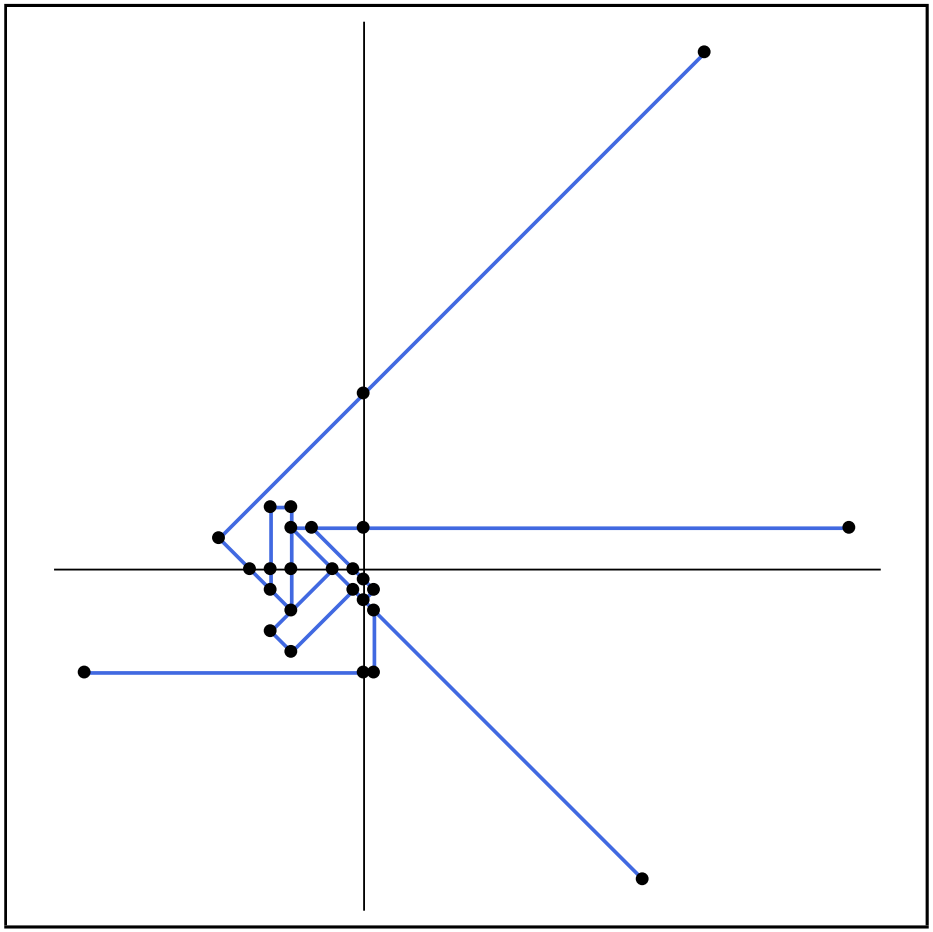
$a = -1,\ 1/3 < b \leq 1/2$



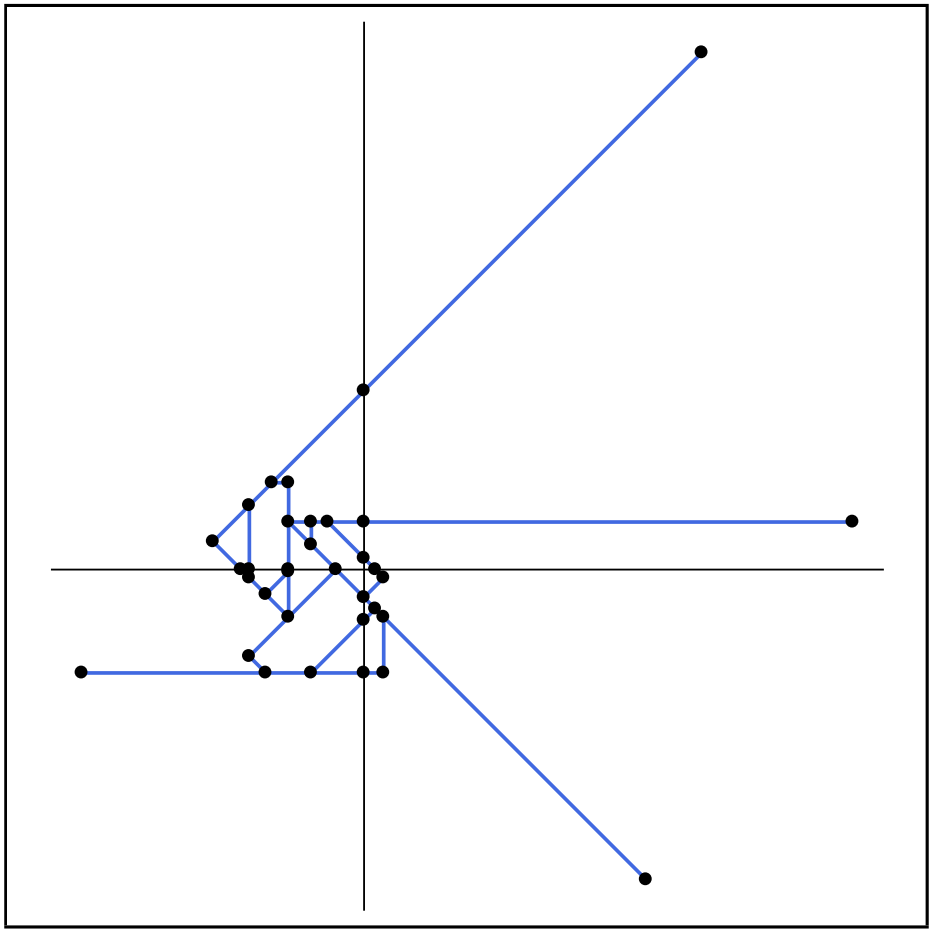
$a = -1,\ 1/2 < b \leq 2/3$



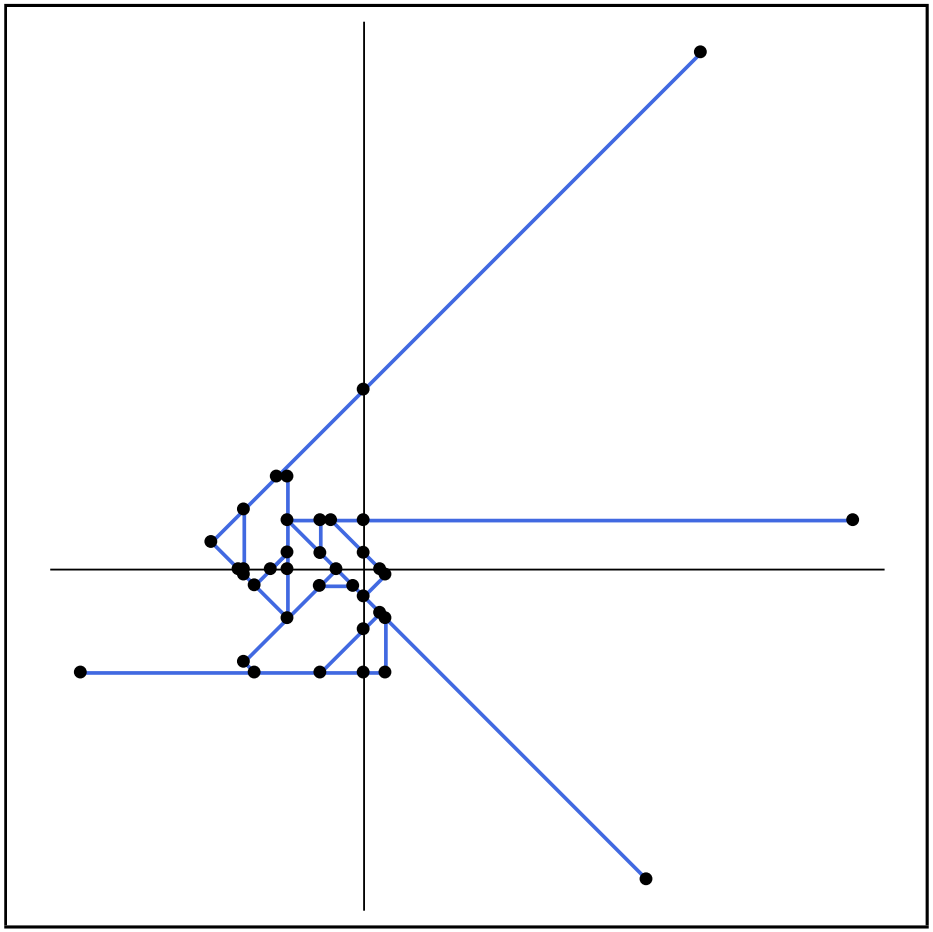
$a = -1,\ 2/3 < b \leq 5/7$



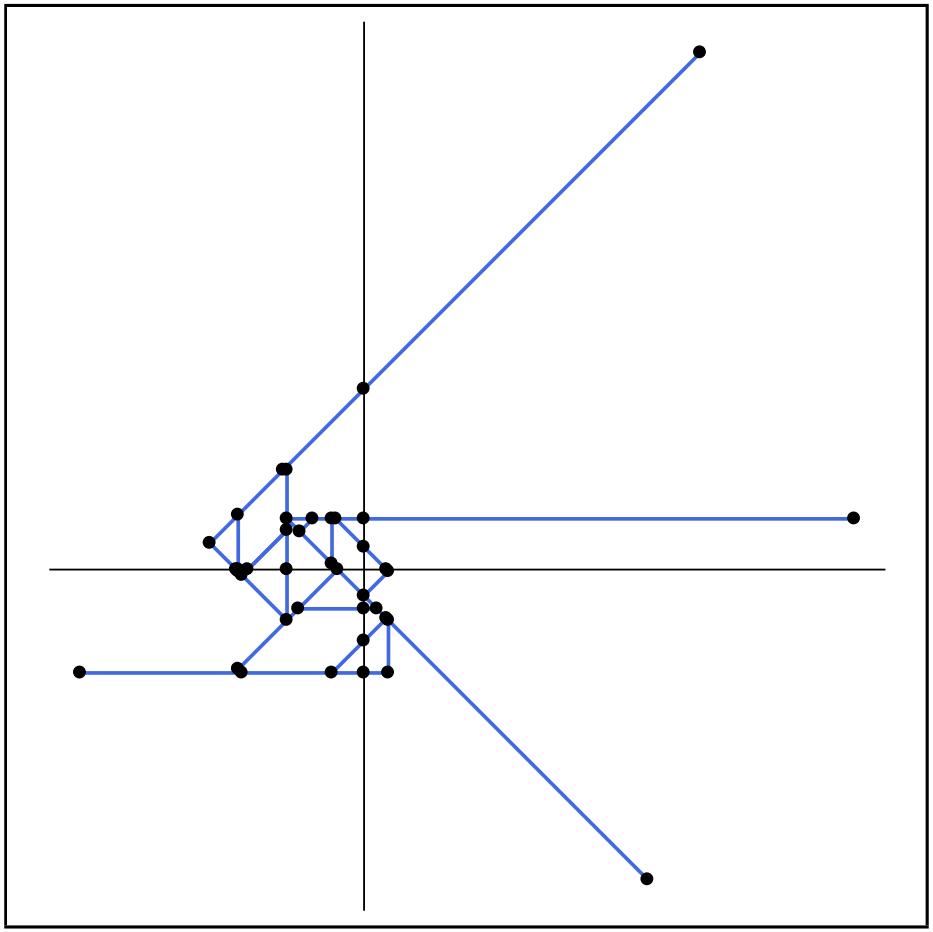
$a = -1,\ 5/7 < b \leq 19/26$



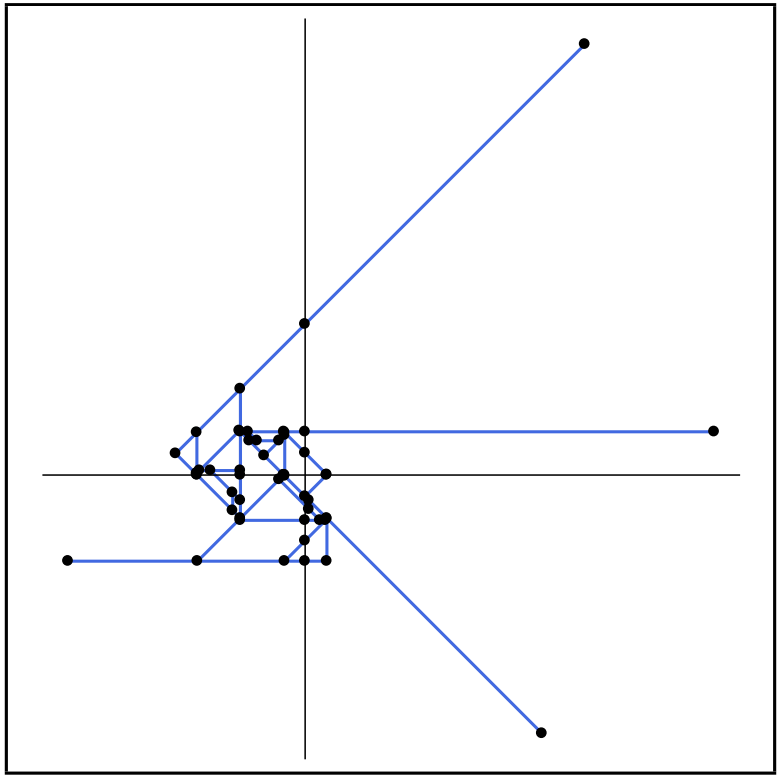
$a = -1,\ 19/26 < b \leq 20/27$



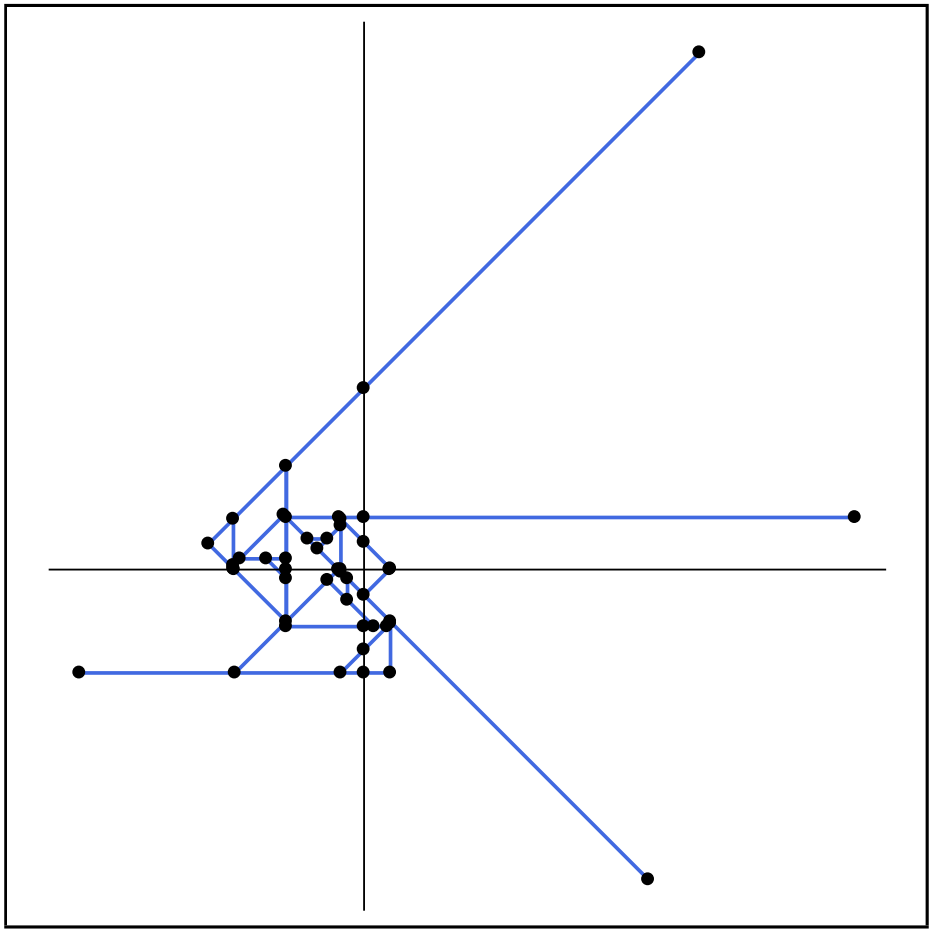
$a = -1,\ 20/27 < b \leq 3/4$



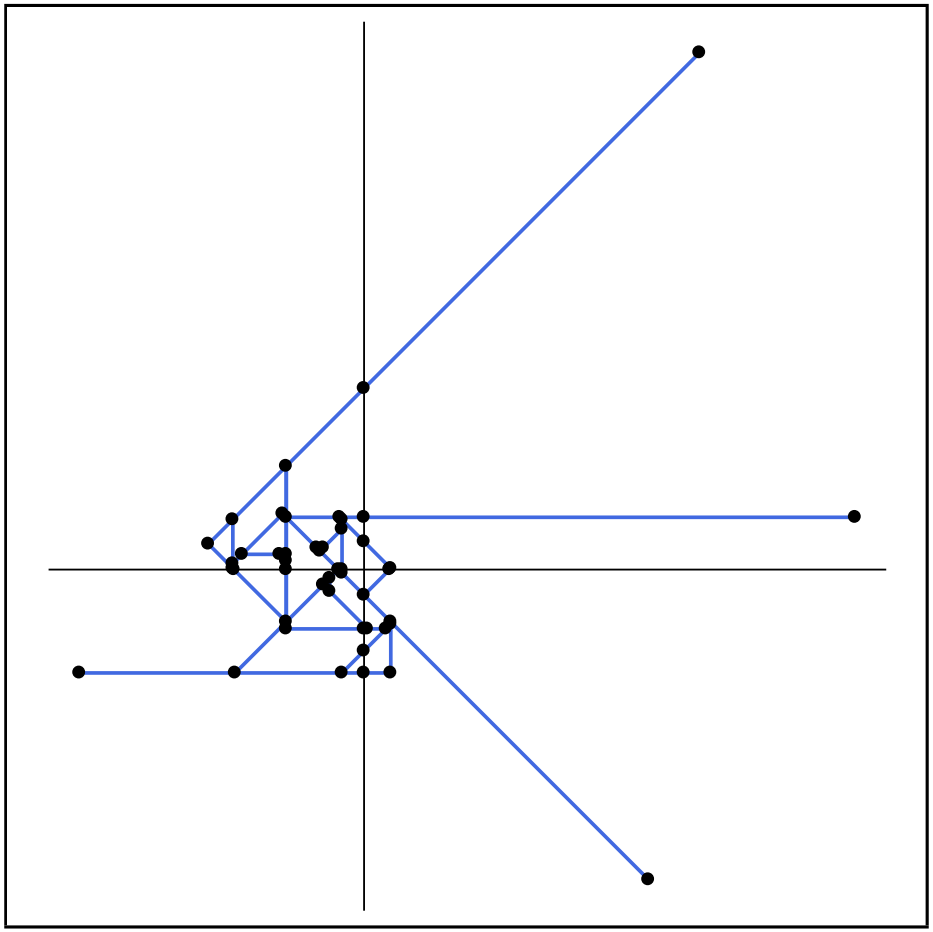
$a = -1,\ 3/4 < b \leq 154/205$



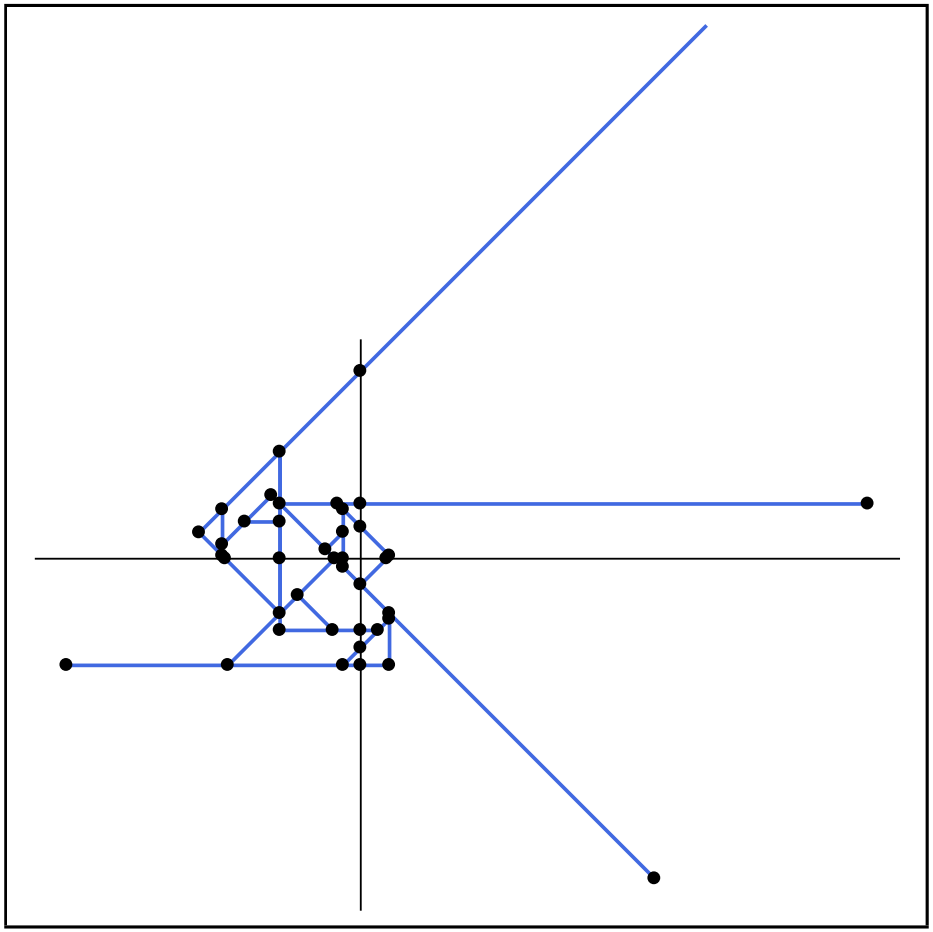
$a = -1,\ 154/205 < b \leq 155/206$



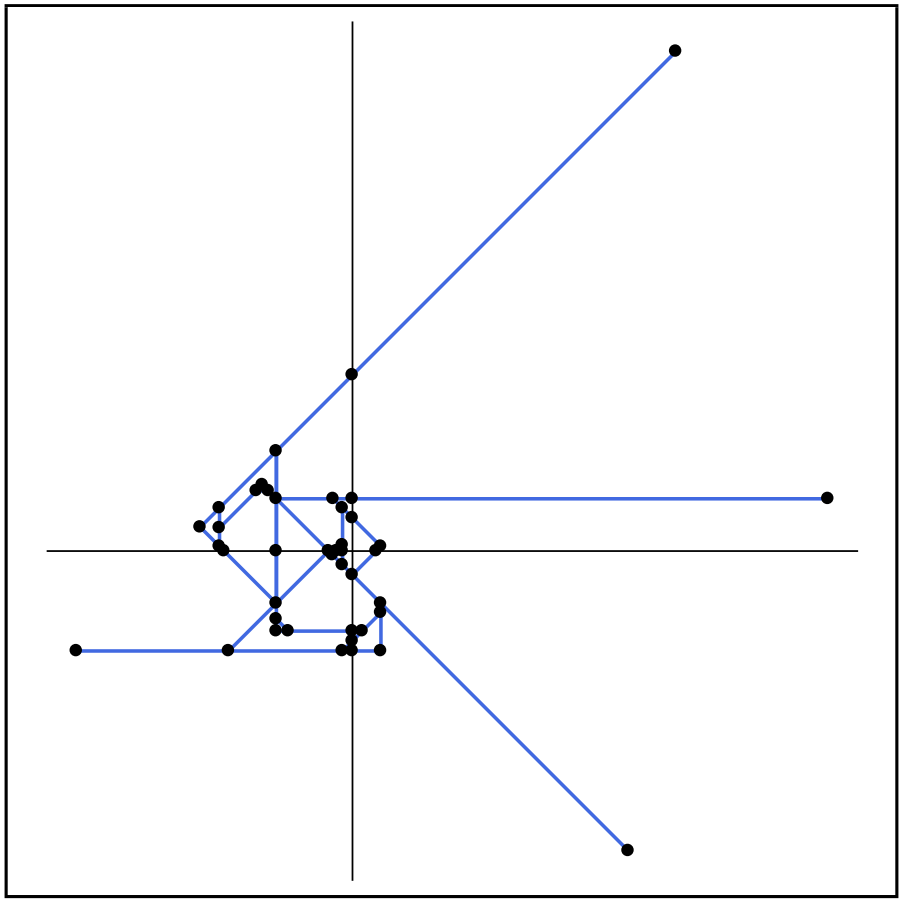
$a = -1,\ 155/206 < b \leq 58/77$



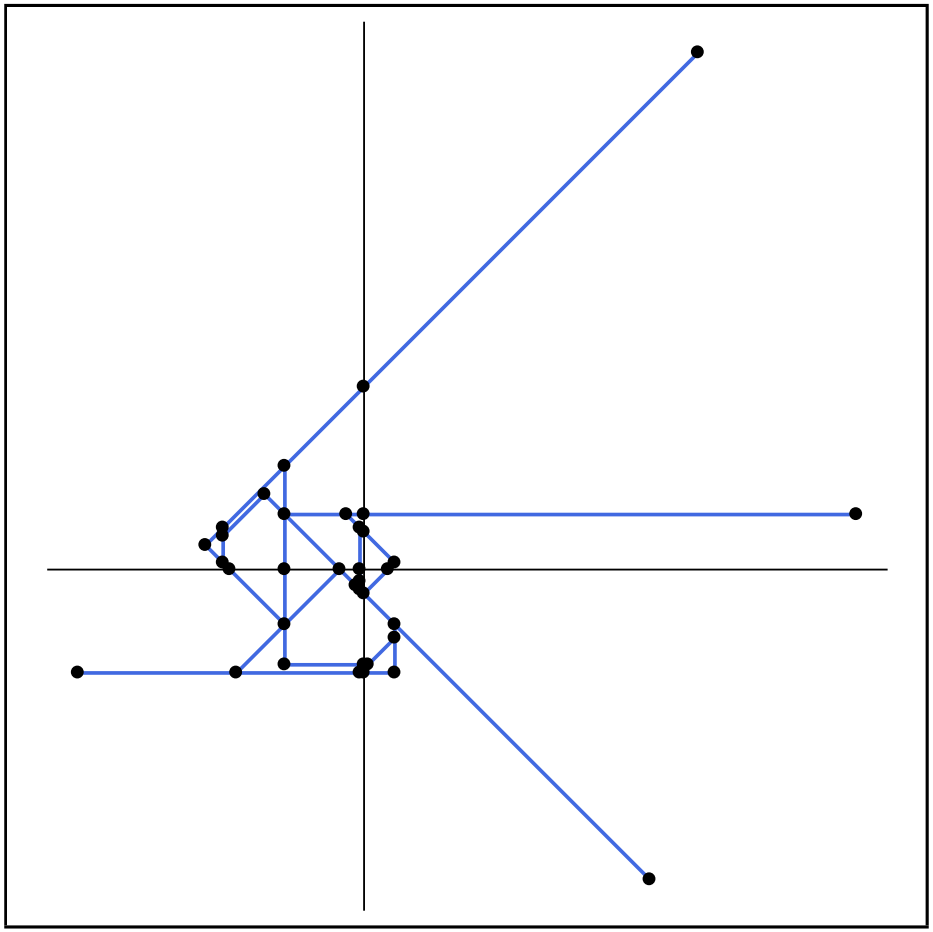
$a = -1,\ 58/77 < b \leq 19/25$



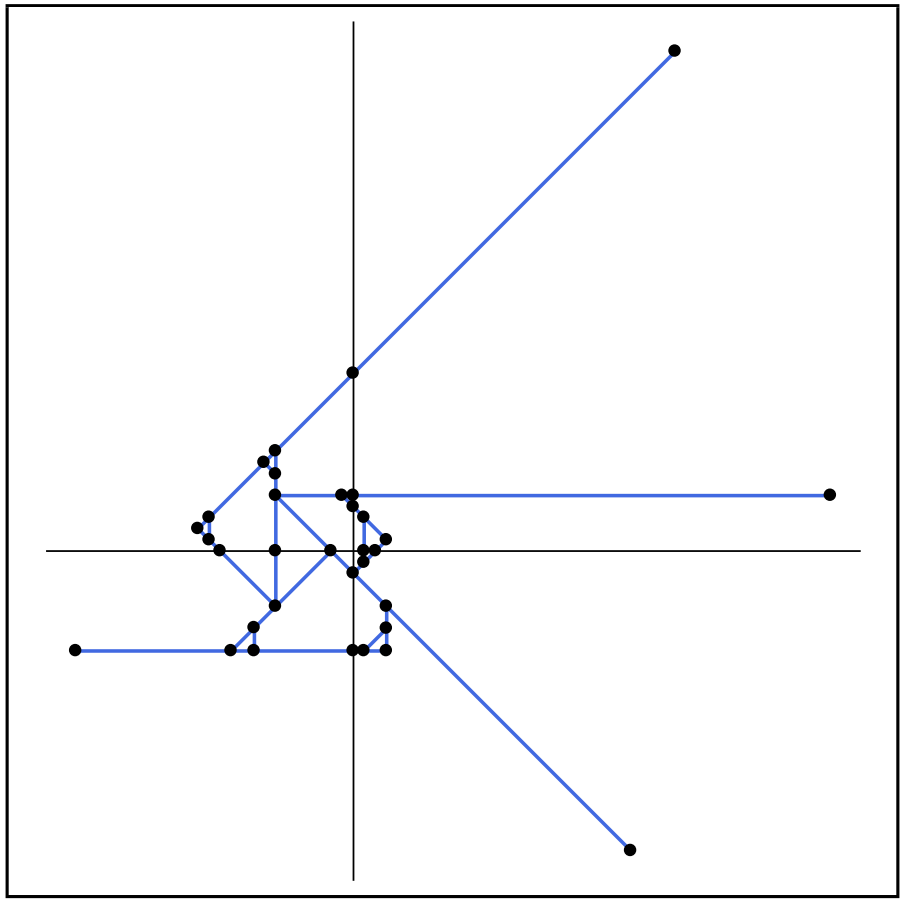
$a = -1,\ 19/25 < b \leq 29/38$



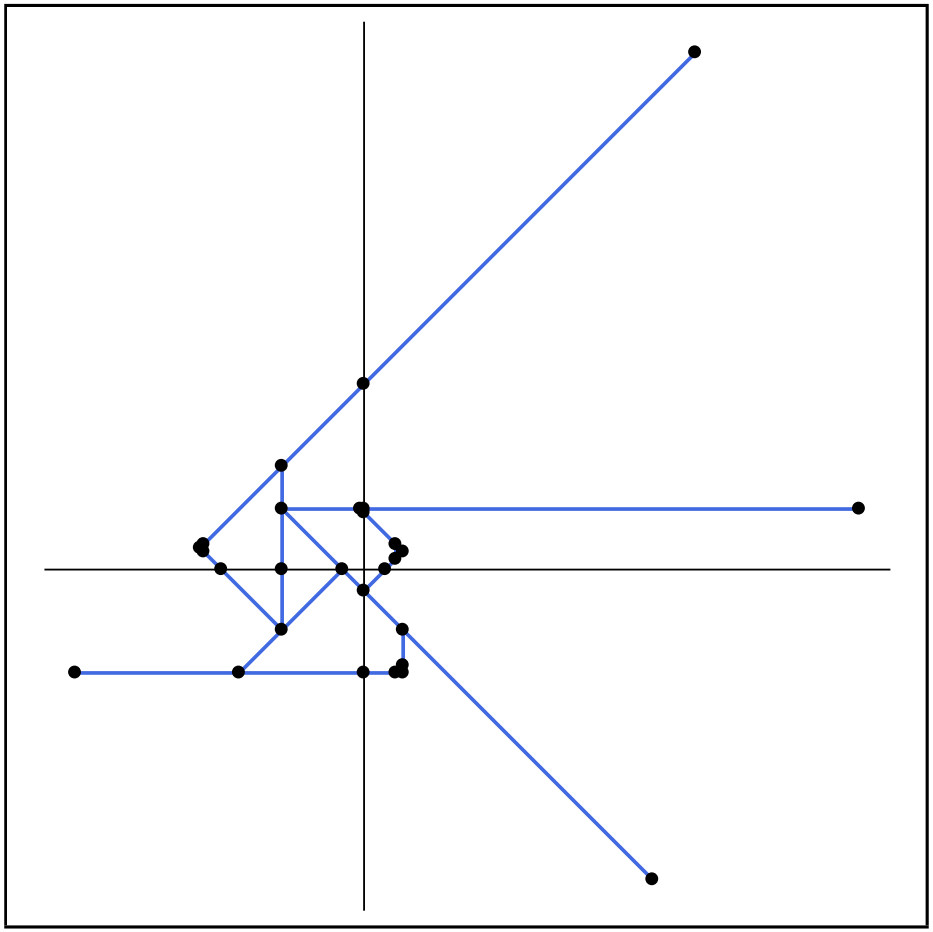
$a = -1,\ 29/38 < b \leq 10/13$



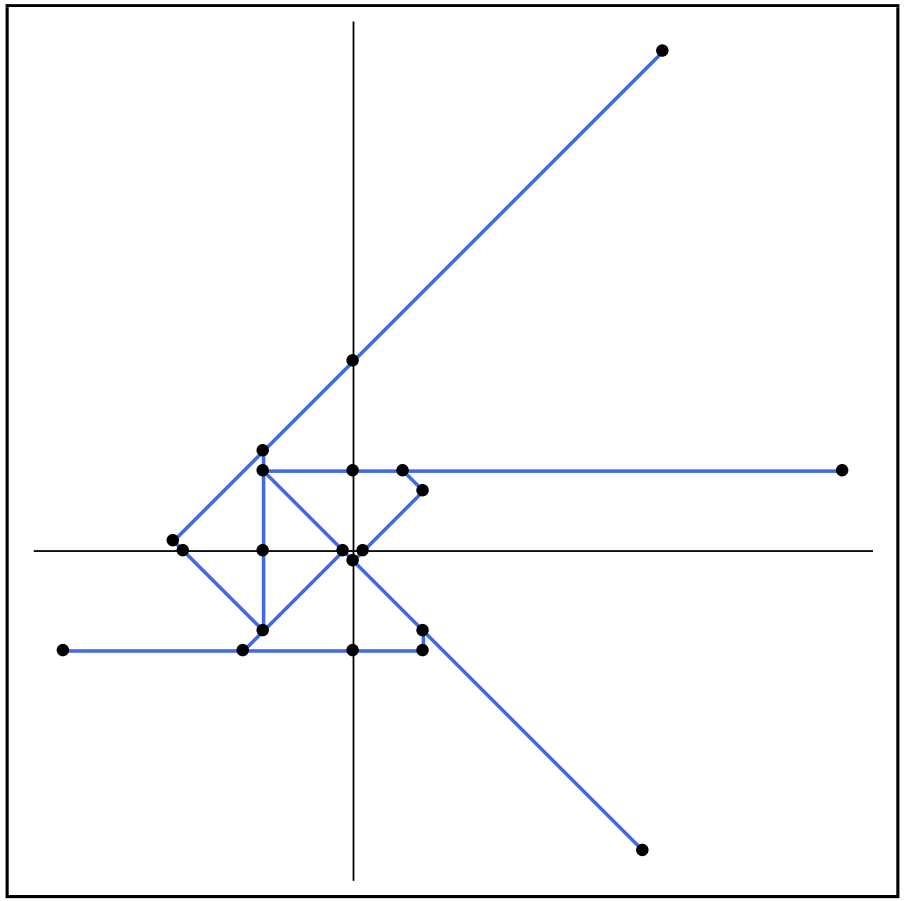
$a = -1,\ 10/13 < b \leq 11/14$



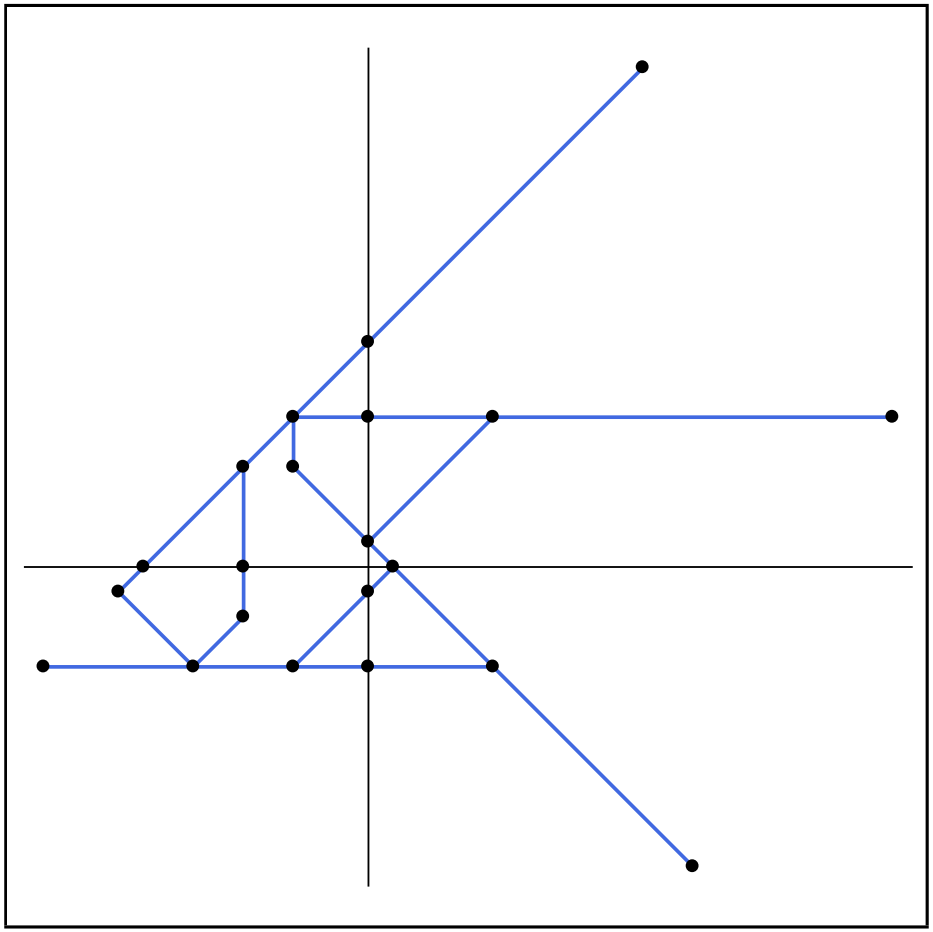
$a = -1,\ 11/14 < b \leq 4/5$



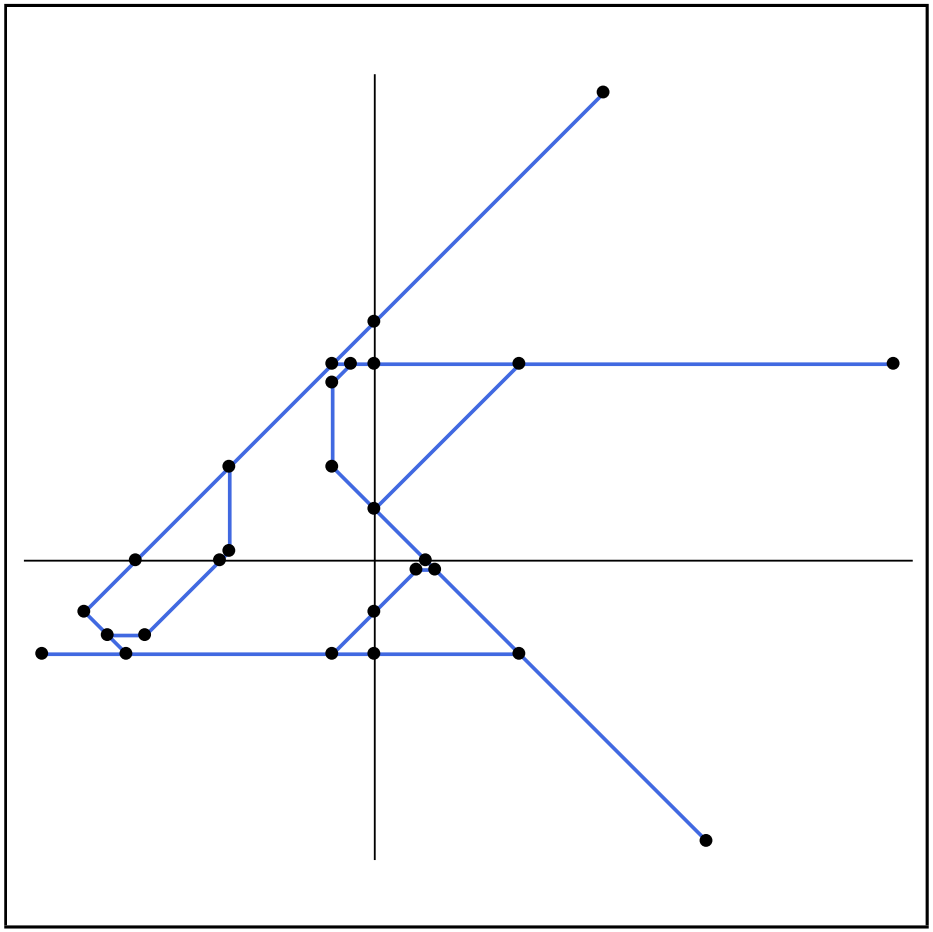
$a = -1,\ 4/5 < b \leq 1$



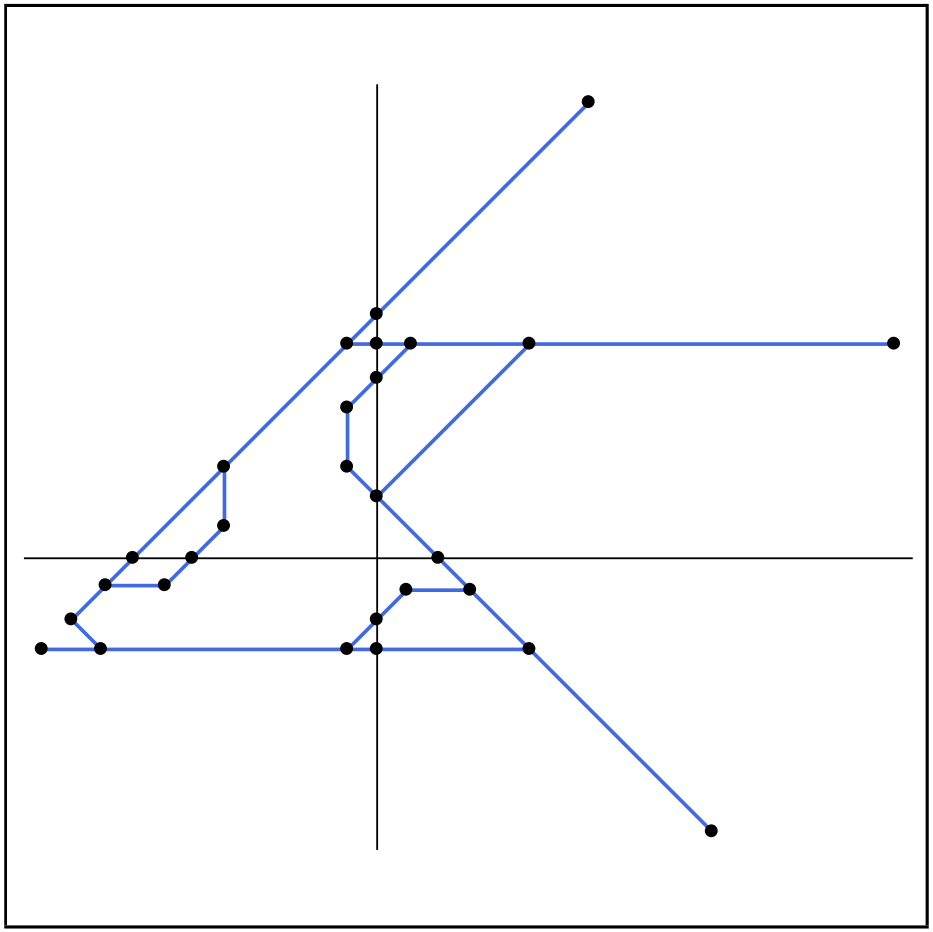
$a = -1,\ 1 < b \leq 3/2$



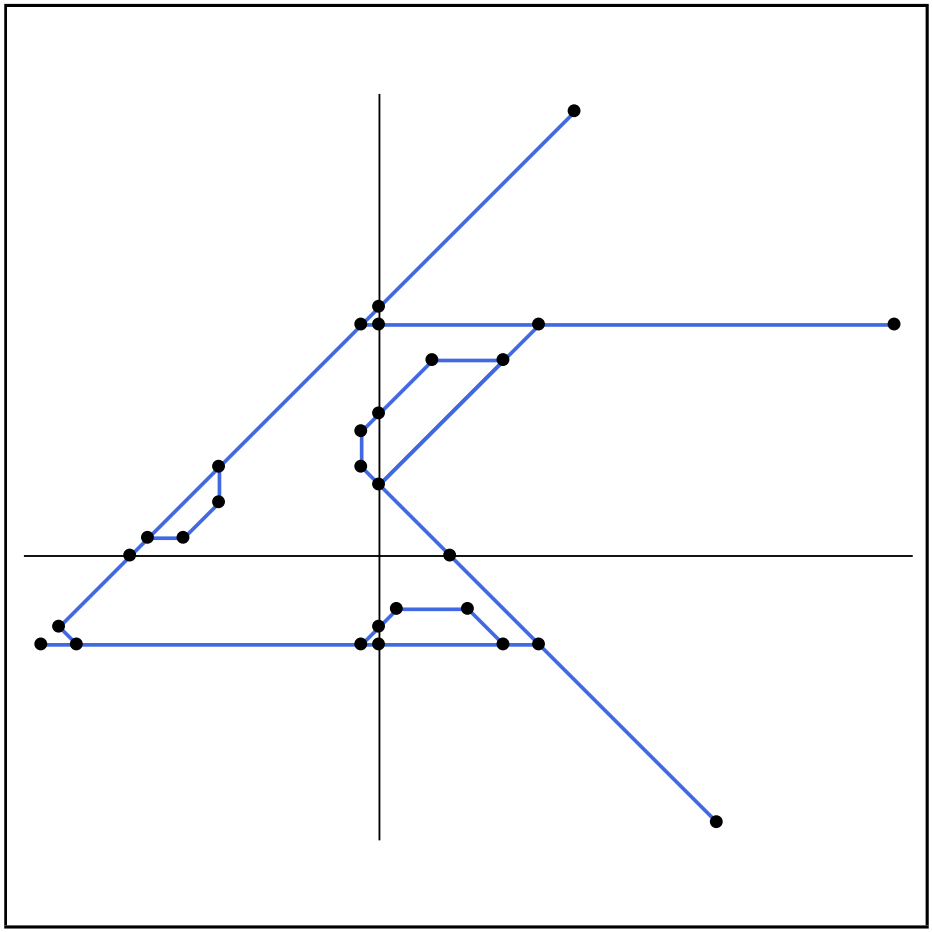
$a = -1,\ 3/2 < b \leq 8/5$



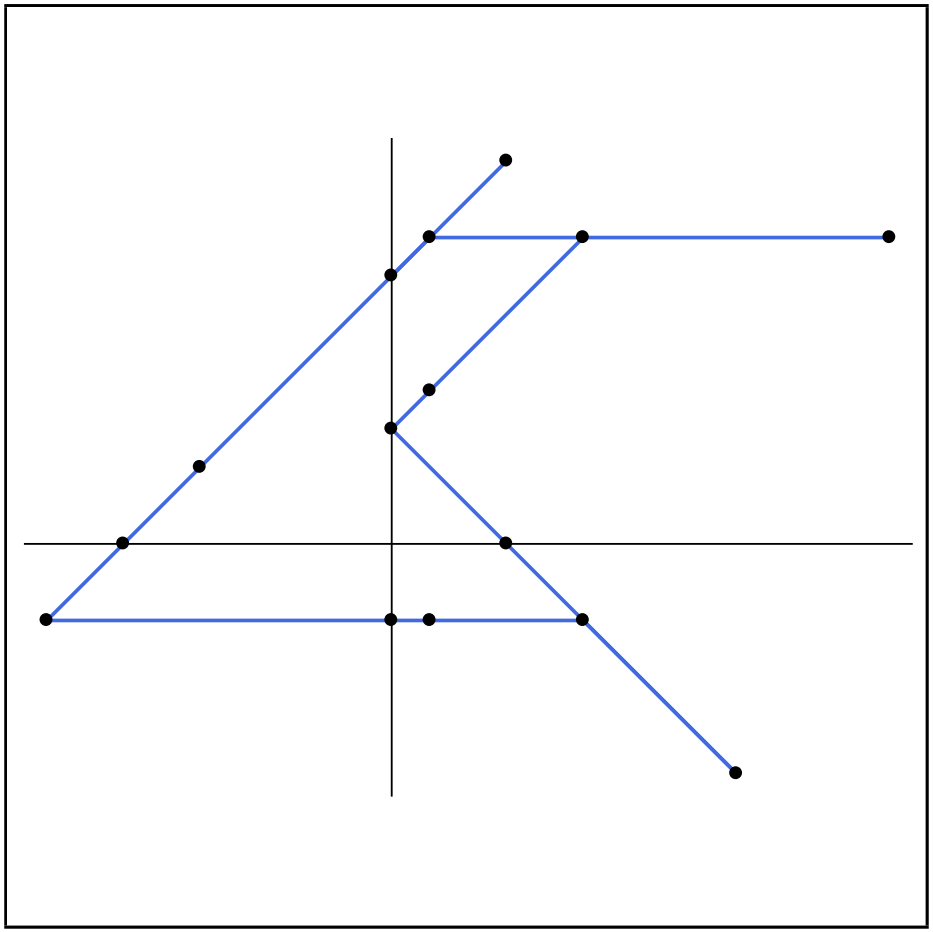
$a = -1,\ 8/5 < b \leq 7/4$



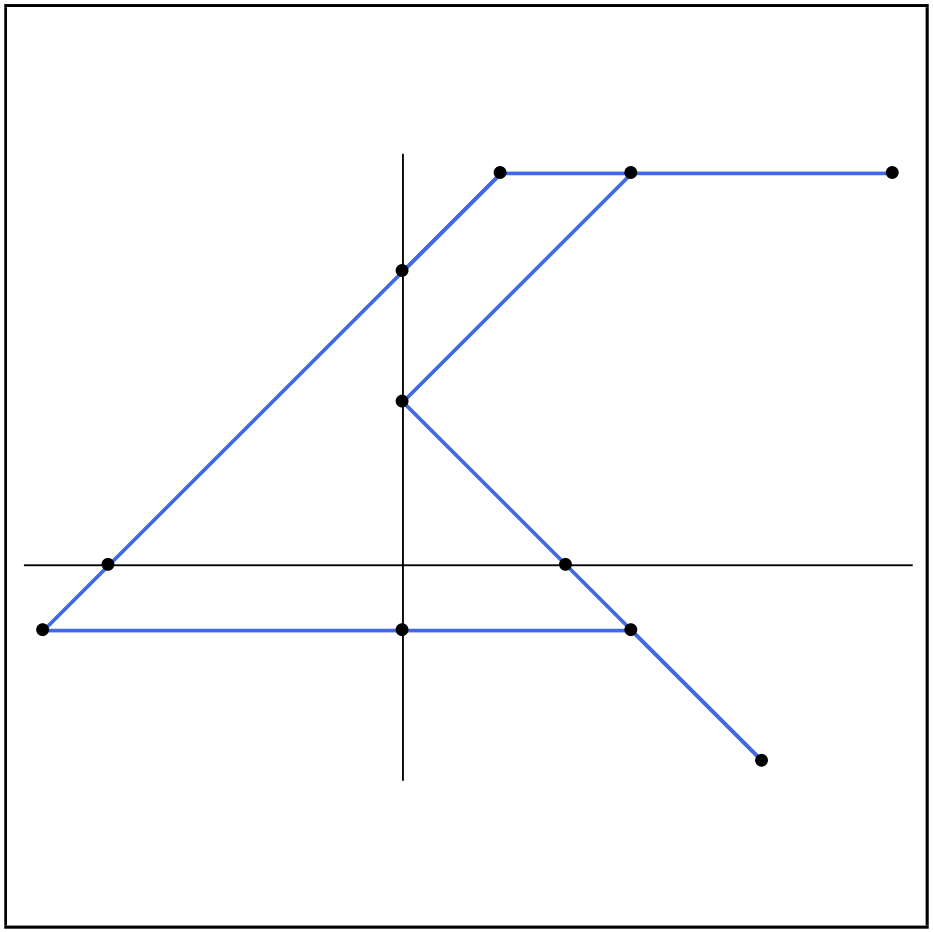
$a = -1,\ 7/4 < b < 2$



$a = -1,\ 2 \leq b < 3$



$a = -1,\ b \geq 3$



Why not so novel:

In the context of border-collision bifurcations in 2-zone piecewise-linear maps, Parui and Banerjee (2002) gave numerical evidence for the existence of one-dimensional attractors composed of line segments. Later Kowalczyk (2005) proved the existence of these kind of chaotic attractors for a certain maps.

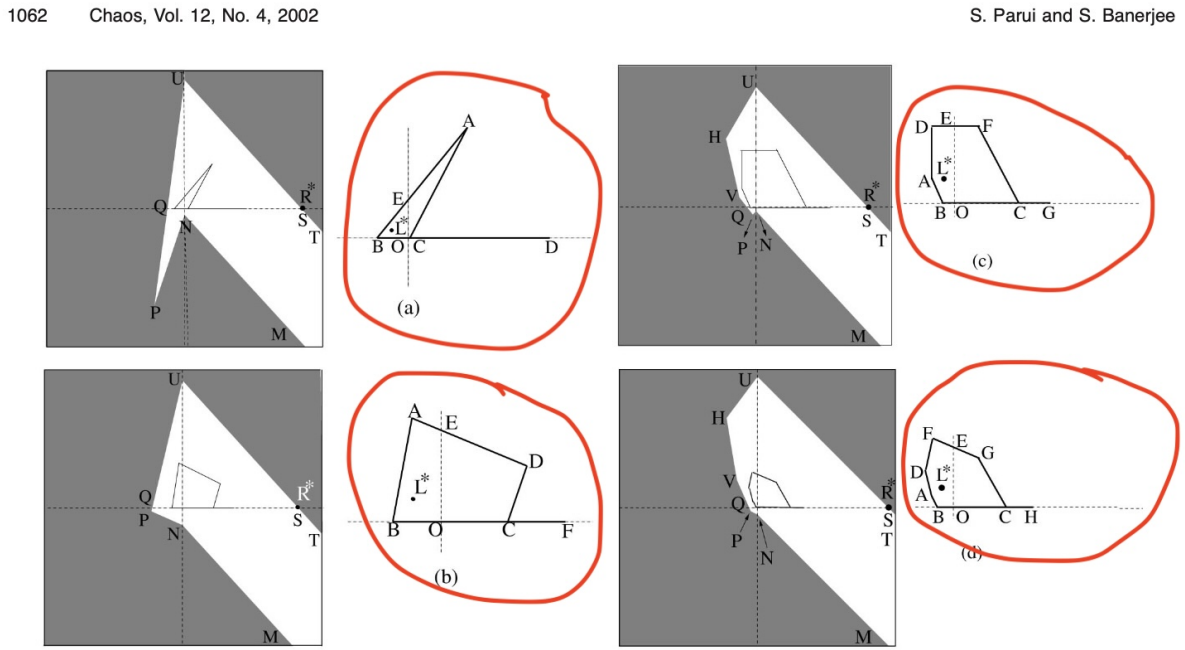
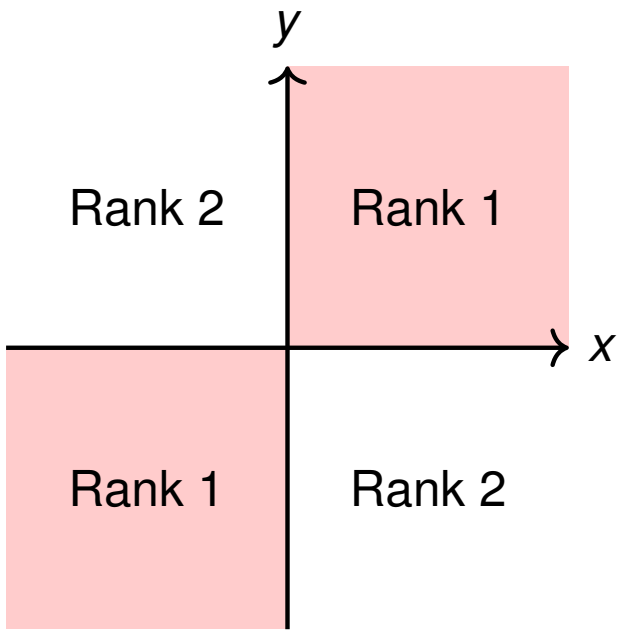


FIG. 13. The chaotic attractors for different combinations of the parameters for Region 6 in positive determinant case: (a) $\tau_L=-3.5$, $\delta_L=4$, $\tau_R=1.1$, $\delta_R=0$, and $\mu=-0.1$. (b) $\tau_L=-0.5$, $\delta_L=4$, $\tau_R=1.1$, $\delta_R=0$, and $\mu=-0.1$. (c) $\tau_L=2.0$, $\delta_L=4$, $\tau_R=1.04$, $\delta_R=0$, and $\mu=-0.1$ (d) $\tau_L=2.7$, $\delta_L=4$, $\tau_R=1.01$, $\delta_R=0$, and $\mu=-0.1$.

These maps are normal forms for grazing-sliding bifurcations in Filippov-type systems. This suggests that one-dimensional chaotic attractors of this kind may also arise in realistic models.

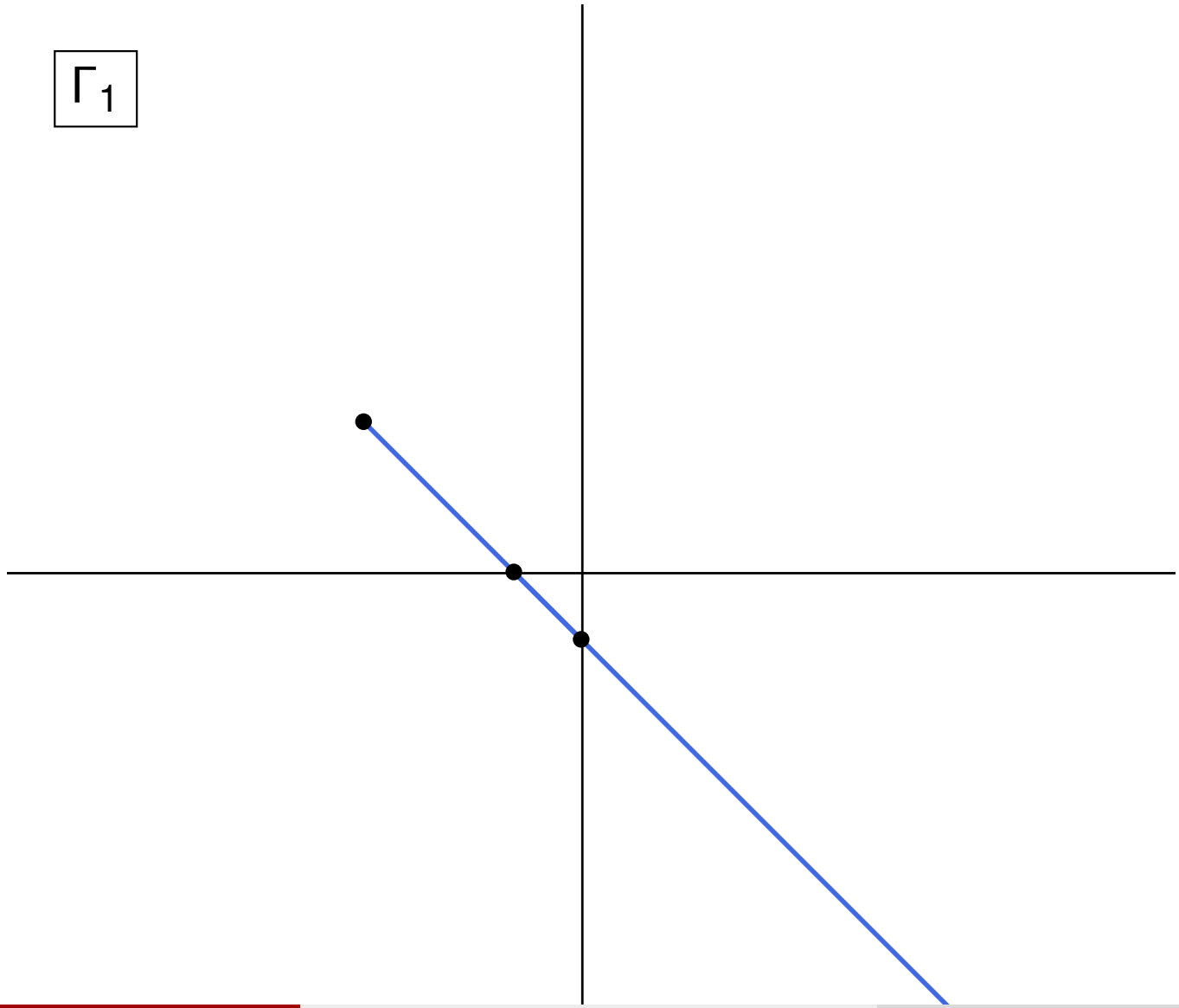
To prove the Theorem we must:

1. **Step 1 I'll skip:** Show that for every b, the orbits of all points in $\mathbb{R}^2$ (except fixed points and very specific 3-periodic orbits) reach the set $Q_1 \cup Q_3$ at some time.

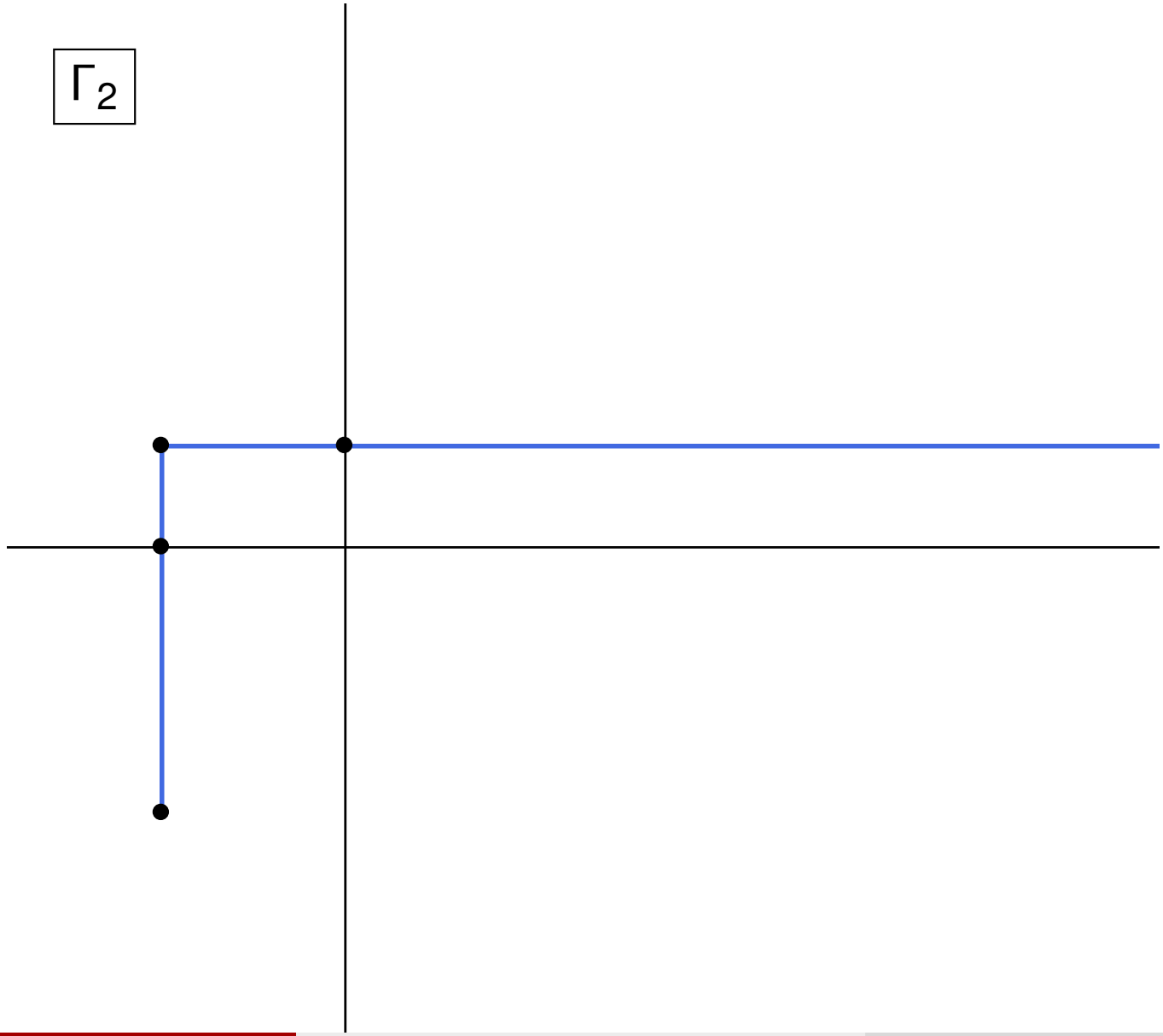


2. **Step 2:** For each $b \in \mathbb{R}$:
   - 2.1 We need to characterize the graph Γ (analytically).
   - 2.2 We need to prove its invariance.
3. **Step 3:** Show that for every b, every orbit with initial conditions in $Q_1 \cup Q_3$ reaches the invariant graph Γ (in finite time: at most 11 iterates).

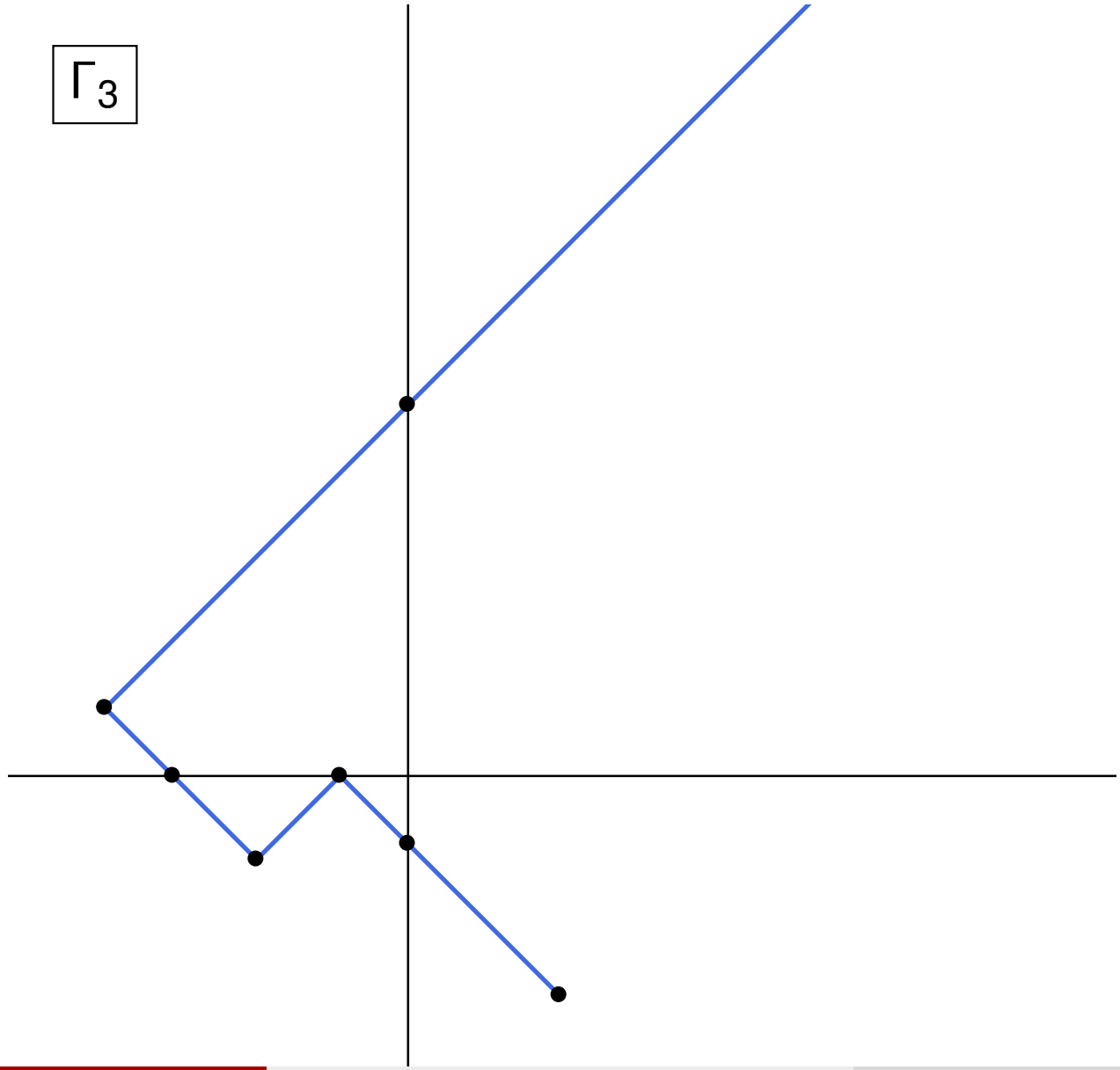
We will follow the iterates of Q_3.



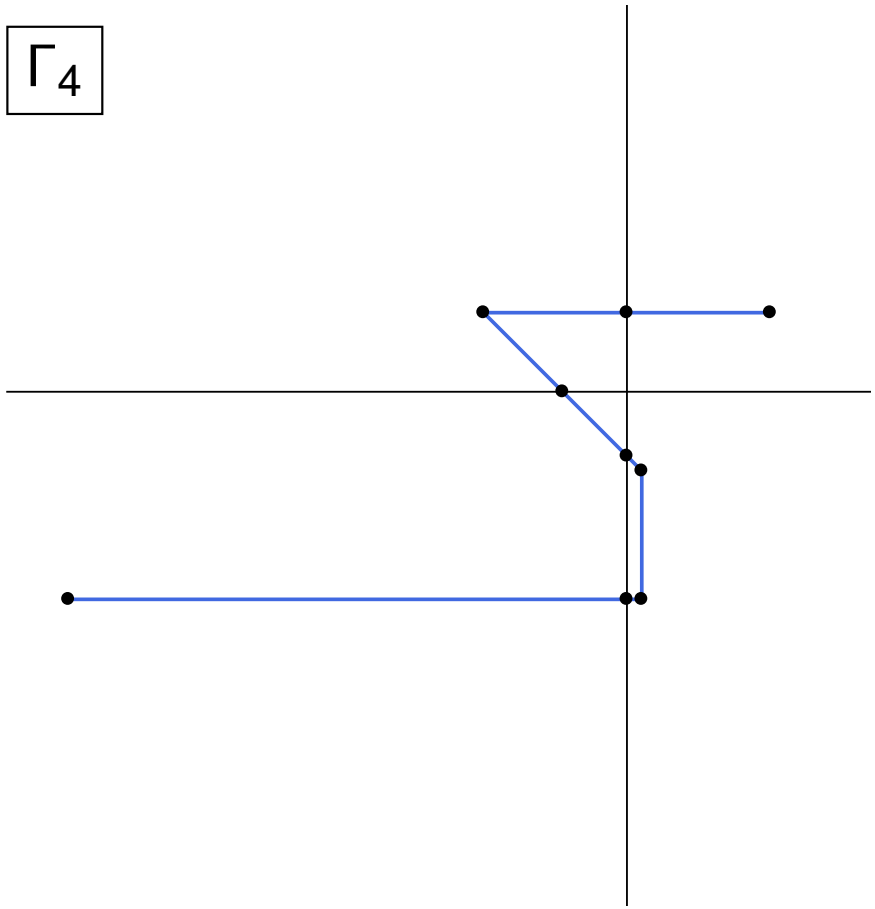
Second iterate of Q_3



Third iterate of Q_3



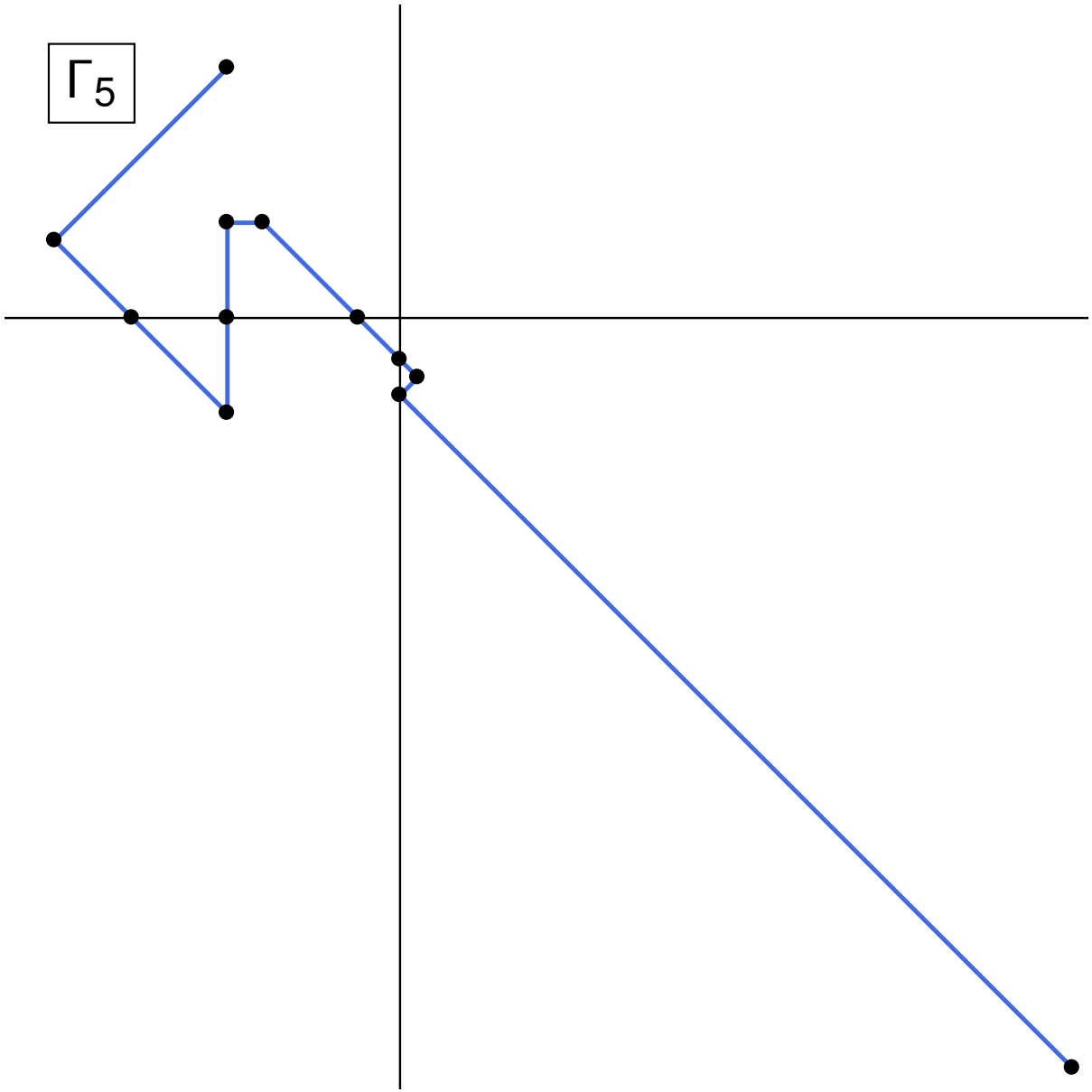
Fourth iterate of Q_3



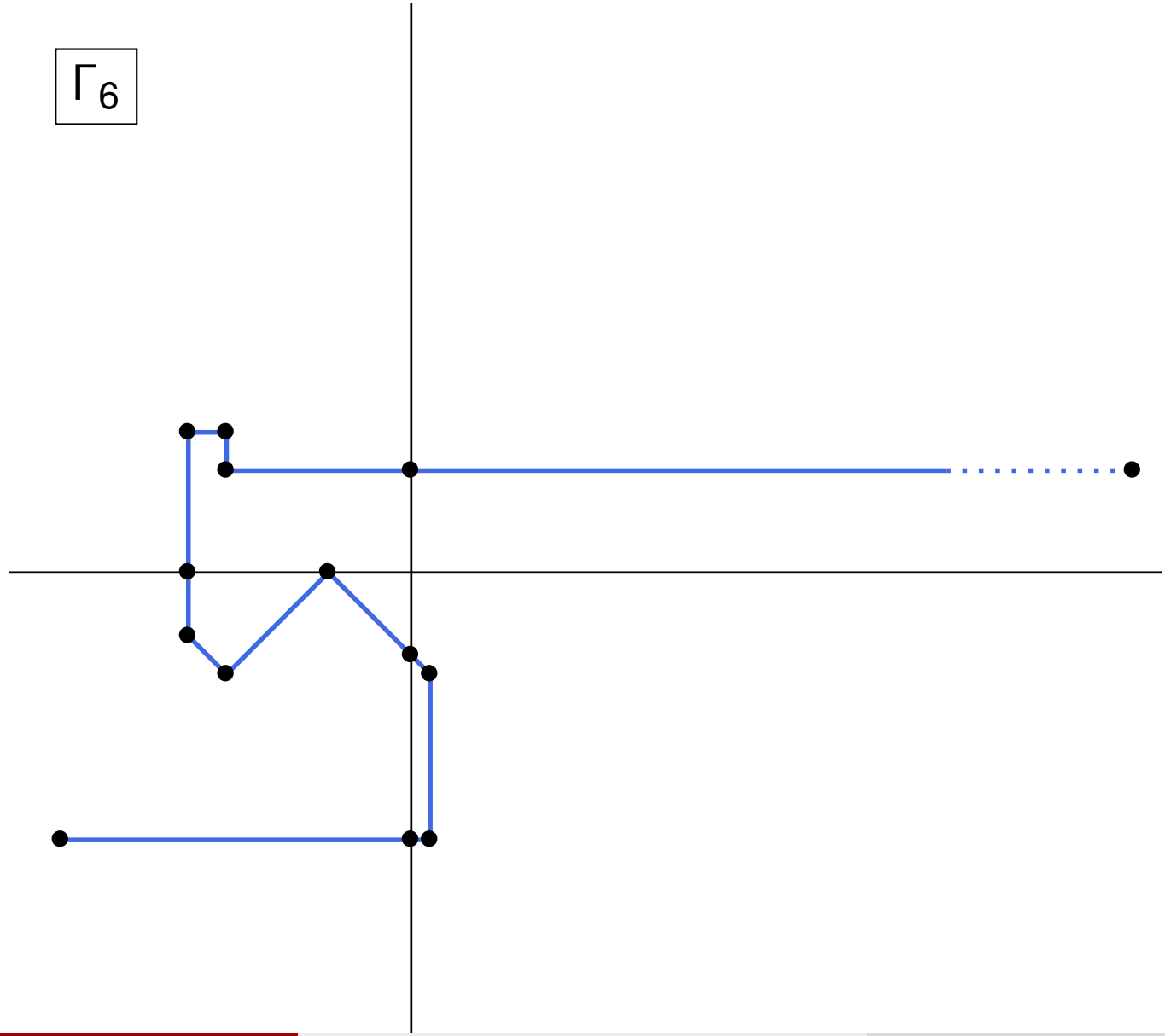
For the first time the image is compact.

This is because the collapse of the unbounded edge.

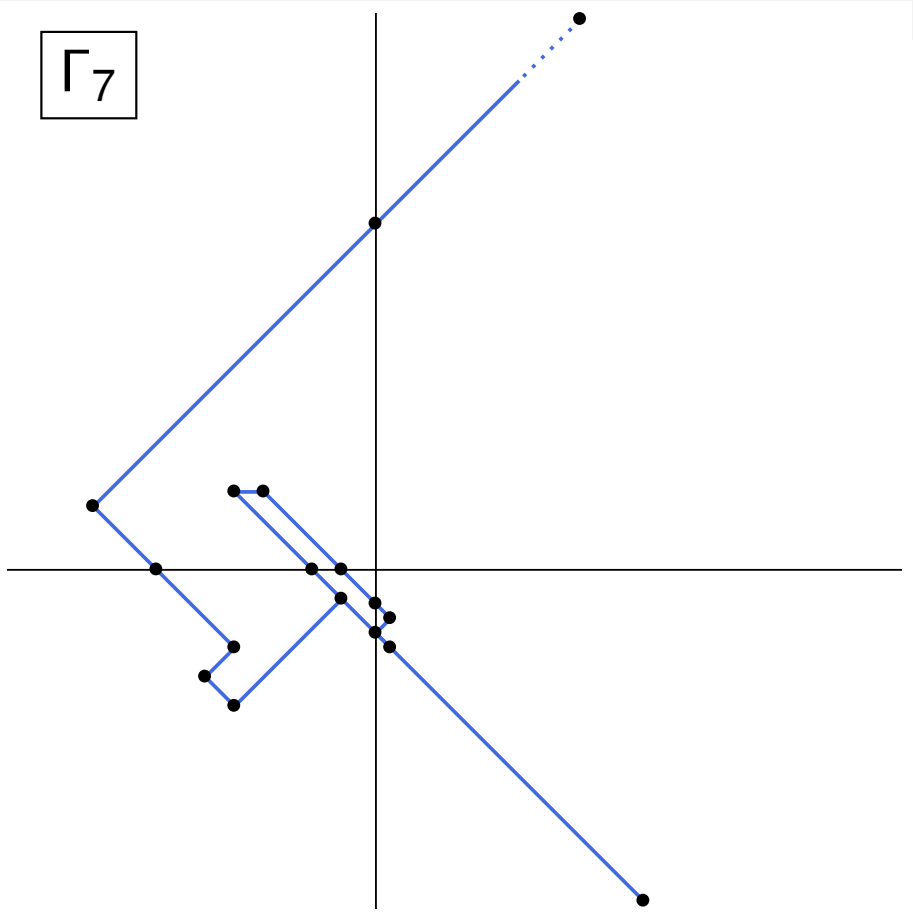
Fifth iterate of Q_3



Sixth iterate of Q_3



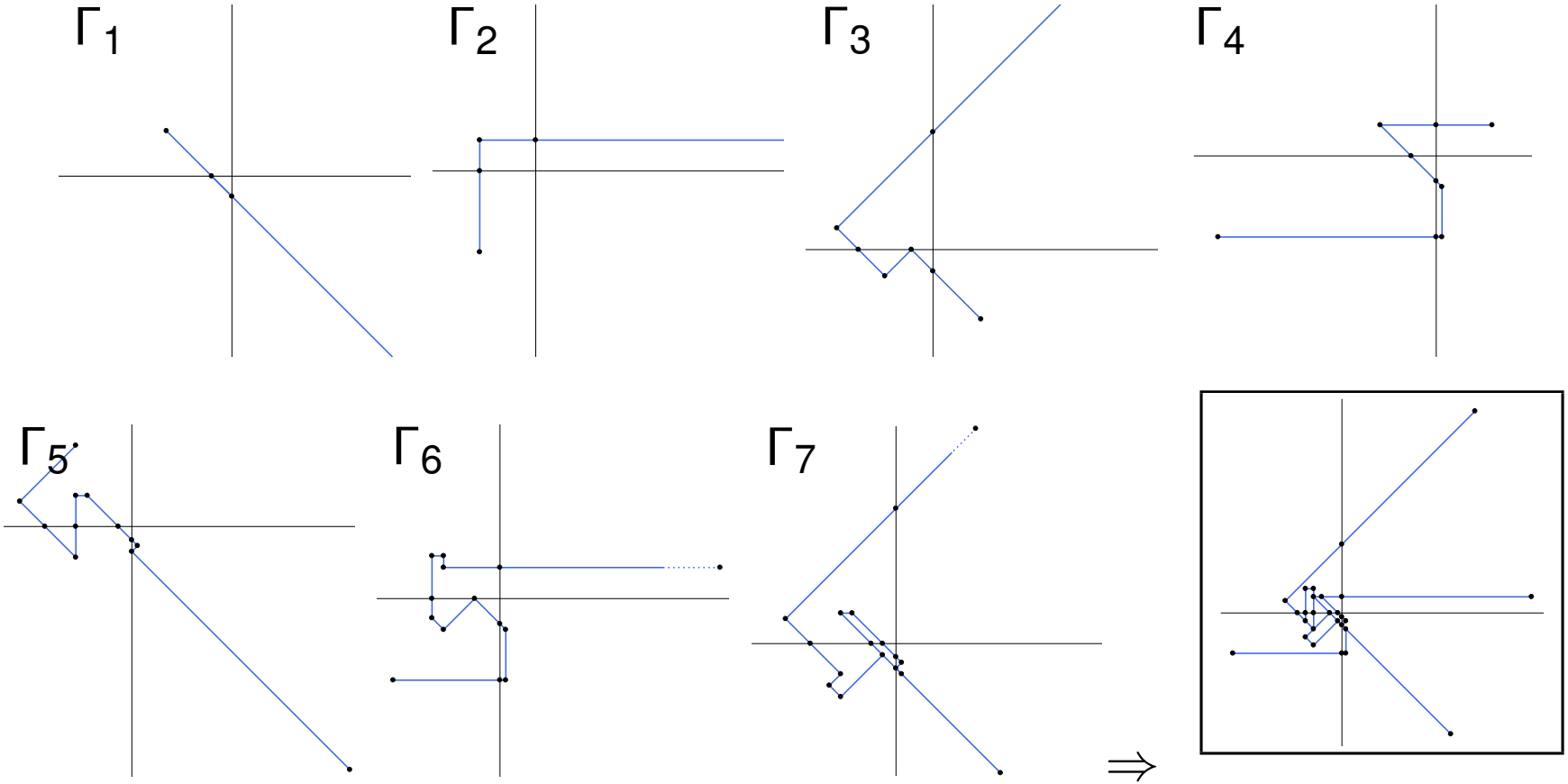
Seventh iterate of Q_3

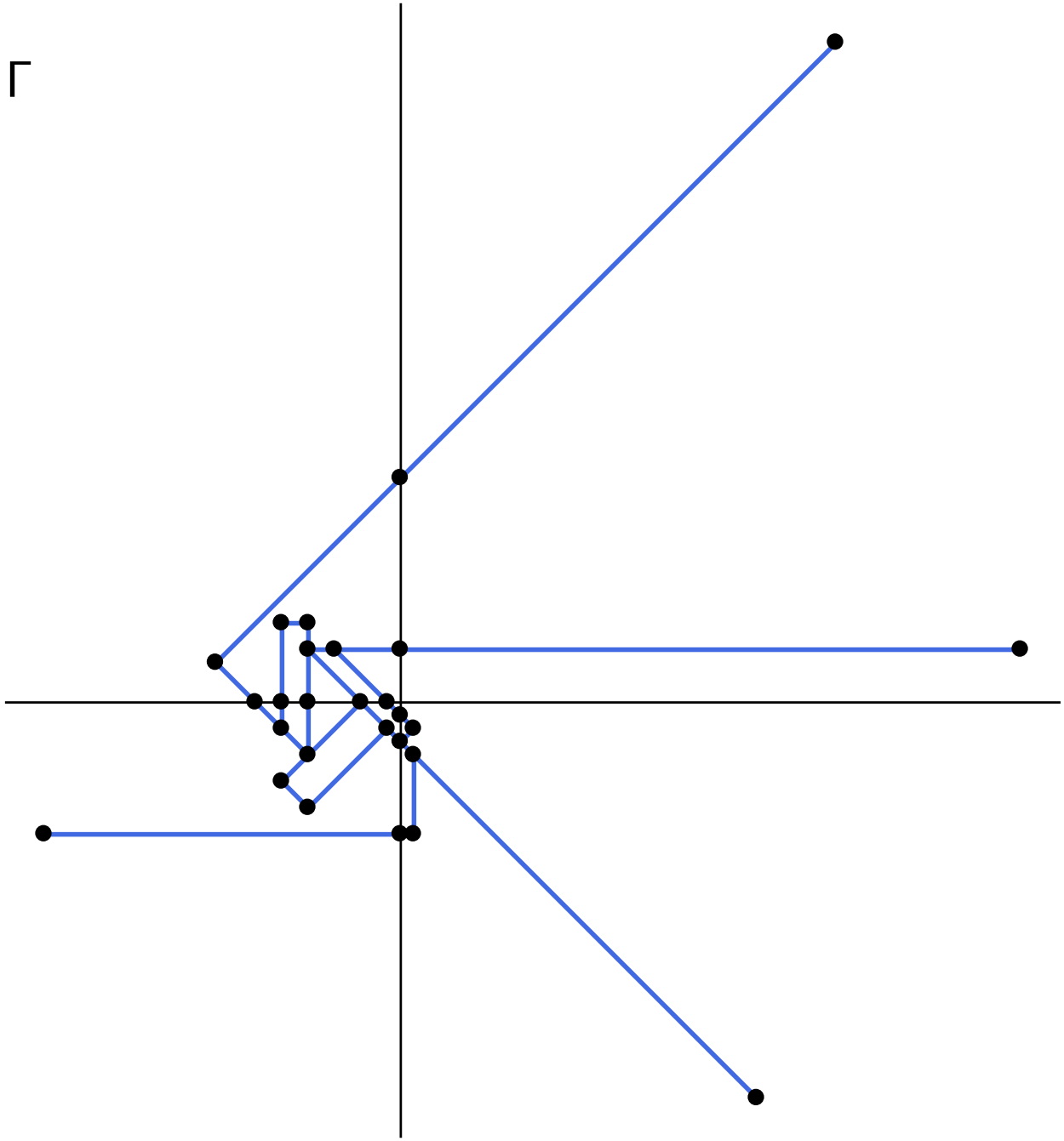


At this point we realize that the following iterates remain in

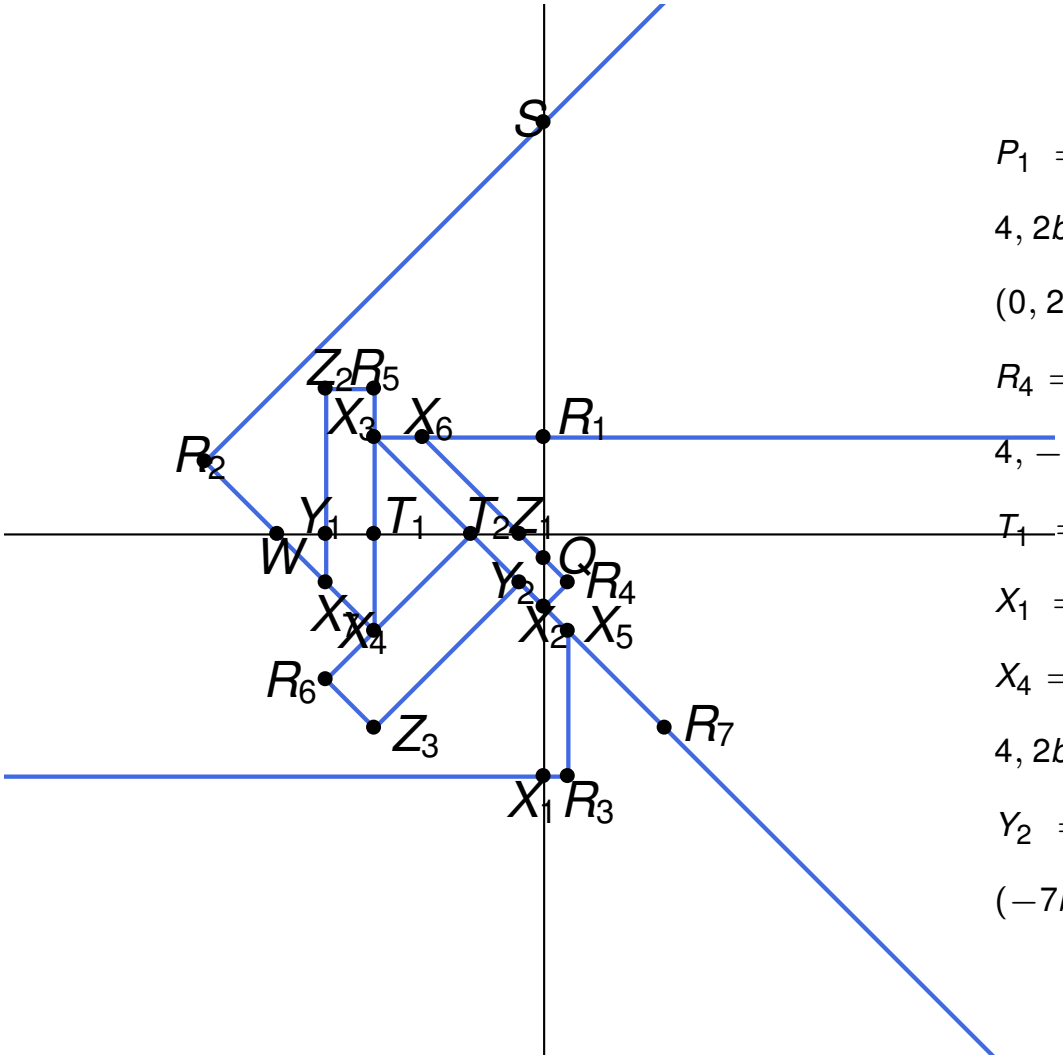
$$\Gamma = \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \Gamma_7$$

Summary of the process:




Γ

We prove the invariance by following the image of all edges



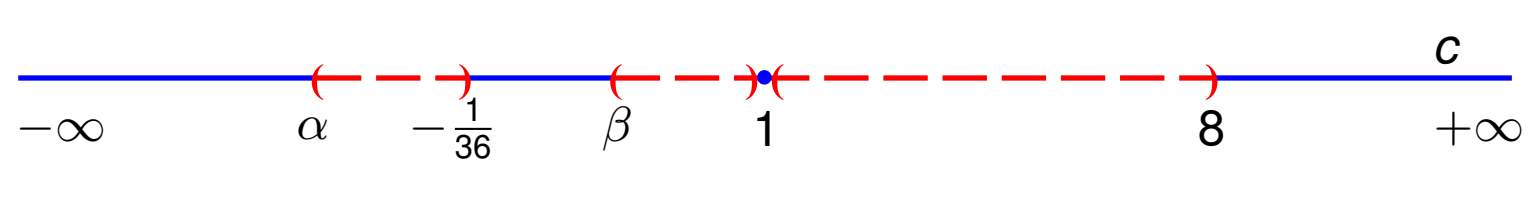
$P_1 = (-b-2,-1)$, $P_2 = (b+2,-3)$, $P_3 = (b+4, 2b-1)$, $P_4 = (-b+4,5)$, $Q = (0, 7b-5)$, $R_1 = (0, 2b-1)$, $R_2 = (-2b, 1-b)$, $R_3 = (3b-2,-1)$, $R_4 = (3b-2, 4b-3)$, $R_5 = (-b, 8b-5)$, $R_6 = (-7b+4, -8b+5)$, $R_7 = (15b-10, -14b+9)$, $S = (0, b+1)$, $T_1 = (-b, 0)$, $T_2 = (b-1, 0)$, $W = (1-3b, 0)$, $X_1 = (0,-1)$, $X_2 = (0, b-1)$, $X_3 = (-b, 2b-1)$, $X_4 = (-b, 1-2b)$, $X_5 = (3b-2, 1-2b)$, $X_6 = (5b-4, 2b-1)$, $X_7 = (-7b+4, 4b-3)$, $Y_1 = (-7b+4, 0)$, $Y_2 = (7b-5, -6b+4)$, $Z_1 = (7b-5, 0)$, $Z_2 = (-7b+4, 8b-5)$ and $Z_3 = (-b, 9-14b)$.

Step 3: In this case $F^4(Q_3) = \Gamma_4 \subset \Gamma$ and $F^5(Q_1) \subset \Gamma$. In general the maximum *arrival time* of $Q_1 \cup Q_3$ is 11 iterates.

Theorem

Set $a < 0$. For each $b \in \mathbb{R}$ consider the map F restricted to the corresponding invariant graph Γ. Set $\mathbf{c = -b/a}$, then there exists α and β such that:

1. $F|_\Gamma$ has positive entropy if and only if $c \in (\alpha, -1/36) \cup (\beta, 1) \cup (1, 8)$.
2. $F|_\Gamma$ has zero entropy if and only if $c \in (-\infty, \alpha] \cup [-1/36, \beta] \cup \{1\} \cup [8, \infty)$.



Furthermore:

- Rational bounds for α and β can be found with arbitrarily high precision.

$$\alpha = -0.817001660127394075579379106922368833240\ldots,$$
$$\beta = 0.689932428204574286700488912950781738705 26\ldots,$$

- The entropy is discontinuous at $c = -1/36$, but continuous at $c \in \{\alpha, \beta, 1, 8\}$.

Topological entropy was introduced by Adler, Konheim & McAndrew (1965).

Take a partition $\mathcal{P} = I_1, \ldots, I_n$ s.t. $F(I_i)$ is an interval and $F|_{I_i}$ is continuous and *monotonic*.

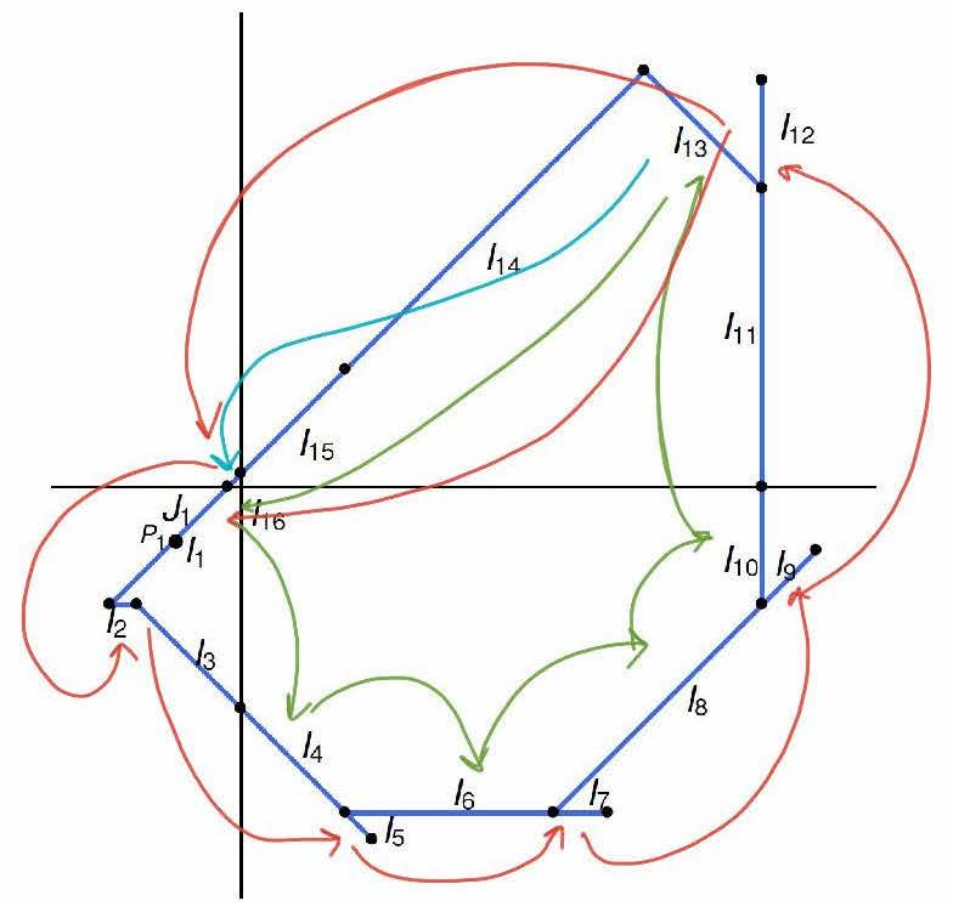
It gives the exponential growth rate of the number of itineraries in this partition when the number of iterates increase.

The topological entropy is

$$h(f) := \ln\left(\lim_{m\to\infty} \sqrt[m]{N(F,\mathcal{P},m)}\right).$$

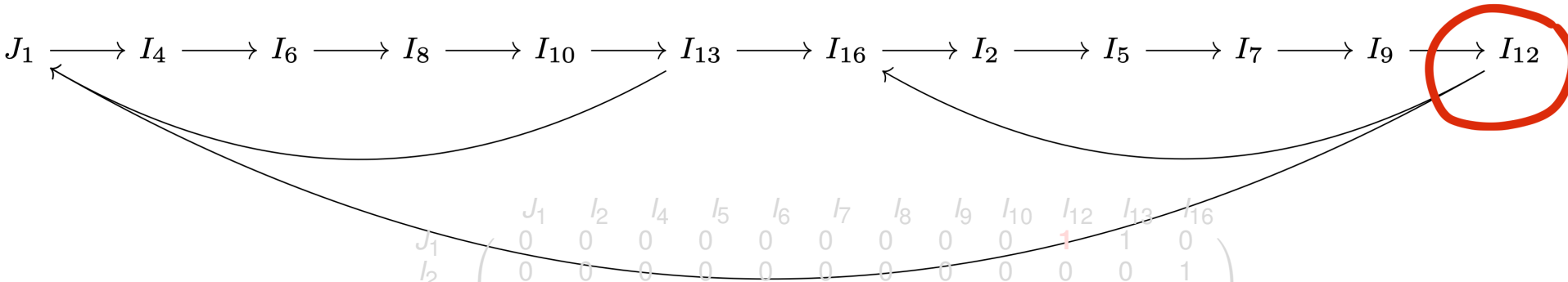
Where $N(F,\mathcal{P},m)$ be the number of different itineraries of length m.

It is independent of the monopartition.



$$-7/9 \leq b \leq -3/4$$

For $-7/9 < b \leq -3/4$ **at least** we have:



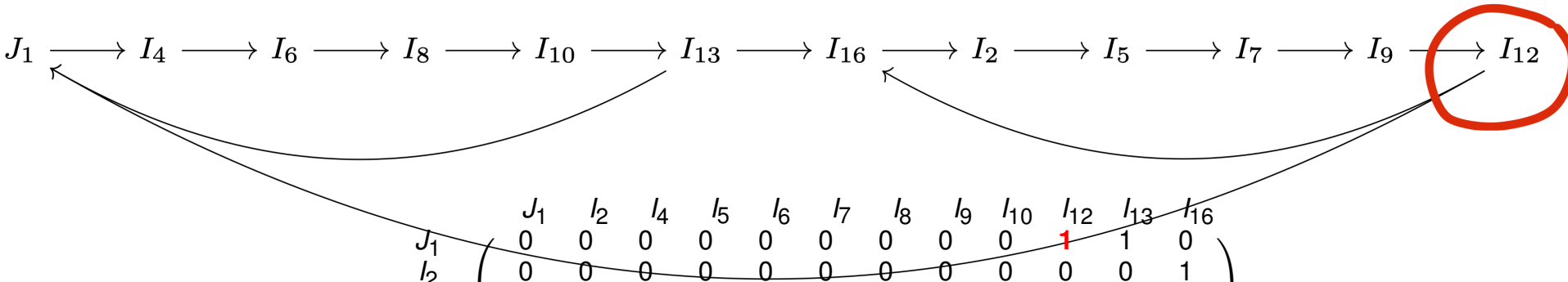
$$M(F,\mathcal{P}) = \begin{array}{c} \\ J_1 \\ I_2 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \\ I_{10} \\ I_{12} \\ I_{13} \\ I_{16} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} J_1 & I_2 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{12} & I_{13} & I_{16} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

When the Partition is a Markov one: $\mathbf{h(F)} = \ln(\mathbf{r(M)})$, where $r(M)$ is the spectral radius (by the *Perron-Frobenius* Theorem a positive eigenvalue)

The characteristic polynomial is $\lambda^6(\lambda^6 - 2)$. The spectral radius is $r = \sqrt[6]{2}$, hence

$$h(F) \geq \ln(r) = \ln(\sqrt[6]{2}) \approx 0.1155245298 > 0.$$

For $-7/9 < b \leq -3/4$ **at least** we have:



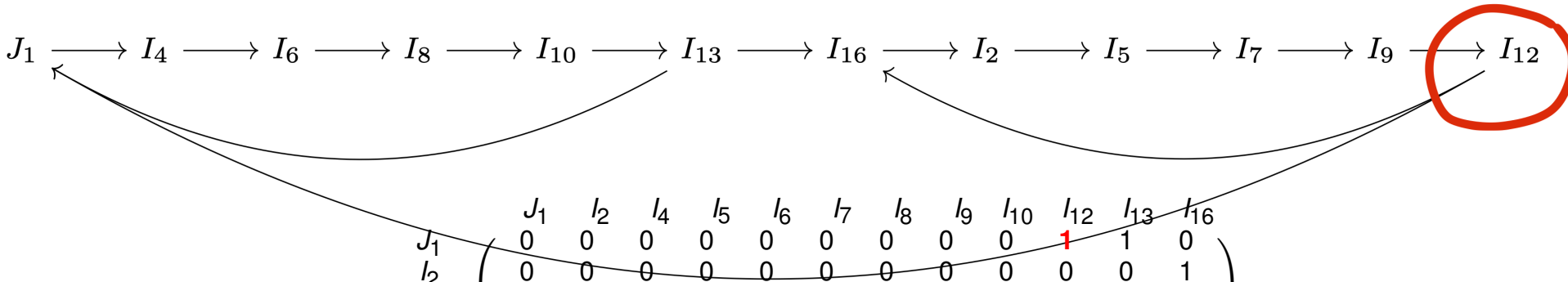
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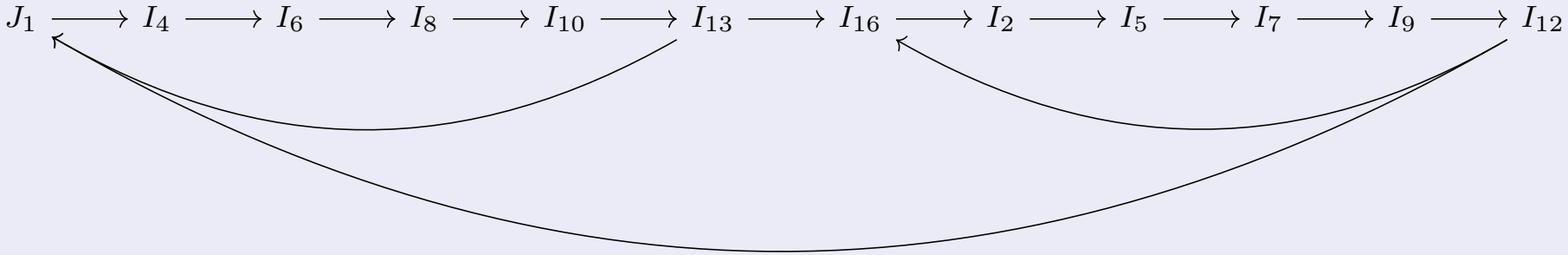
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A quick way to compute the entropy. (SKIP)

The factor giving rise the spectral radius of the Markov Matrix can be computed using only the **Romes**, taking into account the length of its cycles and the itineraries connecting them.



The intervals $\{J_1, I_{16}\}$ form **a Rome**.

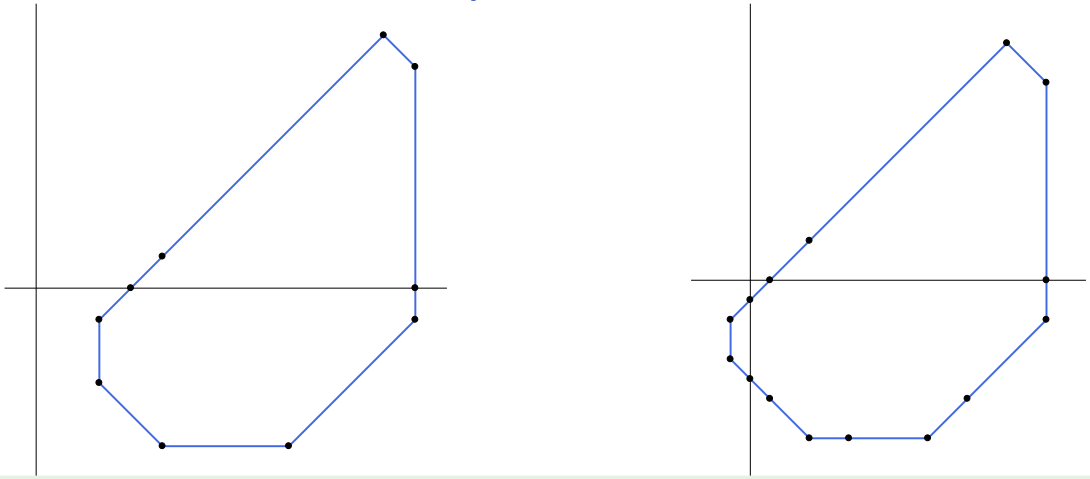
$$\begin{array}{cc} & \begin{array}{cc} J_1 & I_{16} \\ \downarrow & \downarrow \end{array} \\ \begin{array}{l} J_1 \to \\ I_{16} \to \end{array} & \left| \begin{array}{cc} \lambda^{-6} - 1 & \lambda^{-6} \\ \lambda^{-6} & \lambda^{-6} - 1 \end{array} \right| \end{array} = 1 - 2\lambda^{-6}$$

Hence the entropy is $h_F(b) \geq \ln(\sqrt[6]{2}) \approx 0.115524$.

In blue: length of the first cycle. **In green**: length of the second cycle. **In maroon**: length of the path connecting them.

What happens in the case of **zero entropy**?

1. In the majority of cases there is a finite number of **Periodic orbits (repelling or attracting), the dynamics is pre-periodic in finite time**.
2. For $b \leq -1$, $F|\Gamma$ is a degree one map of the circle (it turns around once).



Given a degree 1 map g of $\mathbb{S}^1$ it is characterized by the **rotation number** (which is the "average angle modulus 2π"):

(a) g has periodic orbits if and only if $\rho(g) \in \mathbb{Q}$.

(b) If $\rho(g) = p/q$ with $(p, q) = 1$ all the periodic orbits of g have period q.

(c) If $\rho(g) \in \mathbb{R} \setminus \mathbb{Q}$ then the ω-limit is the same for every point an it is either the whole $\mathbb{S}^1$ or a Cantor set.

Proposition. An example: **circle map and 0-entropy**.

(a) If $b \leq -15/8$, then $\rho = 1/7$. $\exists$ a fixed point and two 7-Periodic orbits (explicit).

If $-7/4 \leq b \leq -1$, then $\rho = 1/6$. $\exists$ a fixed point and two 6-Periodic orbits (explicit).

Every orbit reaches one of these orbits in a finite number of iterates.

(b) If $-15/8 \leq b \leq -7/4$. then ρ **varies continuously from** $1/7$ **to** $1/6$.

- For the irrational values, there is a fixed point an the ω-limit of the other points is a **Cantor set** of Γ.
- When $\rho = p/q$, $\exists$ two Periodic orbits q-periodic.

The set of periods is are **all the natural numbers** *except:* $2-5$, $8-12$, $14-17$, 18, $21-24$, 26, $28-30$, 35, 36, $38-40$, 42, 50, 52, 54, 57, 60, $64-66$, 78, 96, 100, 102, 138 and 220.

- In the transitions from 0 to positive entropy, using Brucks, Misiurewicz and Tresser (1991) theory we have proved that there are invariant sets of the graphs such that in there ***the map* shows all the situations displayed by the orbits of the maxima of unimodal maps (logistic, tent) in the classical route to chaos in these maps.**
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- In the cases with positive entropy there is chaos in the sense of Li and Yorke.

What happens in the case of **positive entropy**?

When the topological entropy is positive, $F_{|\Gamma}$ has periodic orbits with infinitely different periods and the orbits have different combinatorial behaviors.

Since, for all b, Γ is compact, when $F|_\Gamma$ has positive entropy, then it is chaotic in the sense of Li and Yorke.

Chaos in the sense of Li and Yorke

A map f is said to be chaotic in the sense of Li and Yorke if:

1. It has periodic points with arbitrarily large periods and,
2. There exists a uncountable set $\mathcal{S}$ (***scrambled set***), so that for any $p, q \in \mathcal{S}$ and each periodic point r of f we have
   (a) $\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0$,
   (b) $\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0$,
   (c) $\limsup_{n\to\infty} |f^n(p) - f^n(r)| > 0$.

Theorem

For all $a < 0$ and $b \in \mathbb{R}$, **for an open and dense set of initial conditions in** Γ. there are at most three possible ω-limit sets
Moreover if $b/a \in \mathbb{Q}$ **these** ω**-limit sets are periodic orbits.**

When $a < 0$ and $b/a \in \mathbb{Q}$, **thus generically**, the ω-limit sets are periodic orbits.

However: **genericity does not explains numerics**: the generic set in the above result must have full Lebesgue measure in Γ.

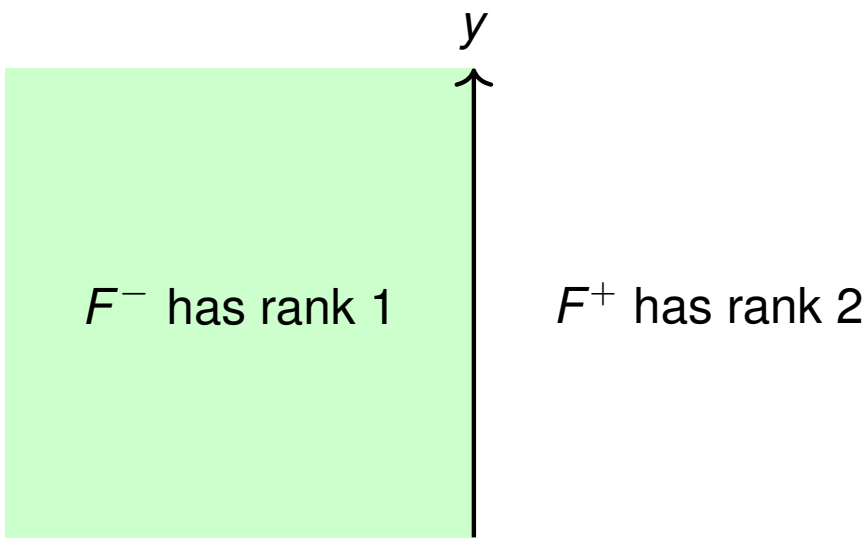
Theorem

Set $a < 0$. For $-b/a < -2$ and for $-b/a \in [-112/137, -13/16] \cup [603/874, 563/816]$ $\exists$ **a full Lebesgue measure set of initial conditions in** Γ, such that there are at most three possible ω-limit sets. Moreover, if $b/a \in \mathbb{Q}$ these ω-limit sets are periodic orbits.

Guiding question

How general is this graph-capturing mechanism for strongly dissipative planar piecewise linear maps?

To understand better why such behaviors do appear we study families of planar piece-wise linear continuous maps defined in two zones of the form:



Important: The image of the green region is simply a line: L.

We consider continuous maps of the form

$$F(x,y) = \begin{cases} F^+(x,y), & x \geq 0, \\ F^-(x,y), & x \leq 0, \end{cases}$$

where

$$F^{\pm}(x,y) = A^{\pm}(x,y)^T + b, \qquad A^{\pm} \in \mathbb{R}^{2\times 2}, \quad b \in \mathbb{R}^2.$$

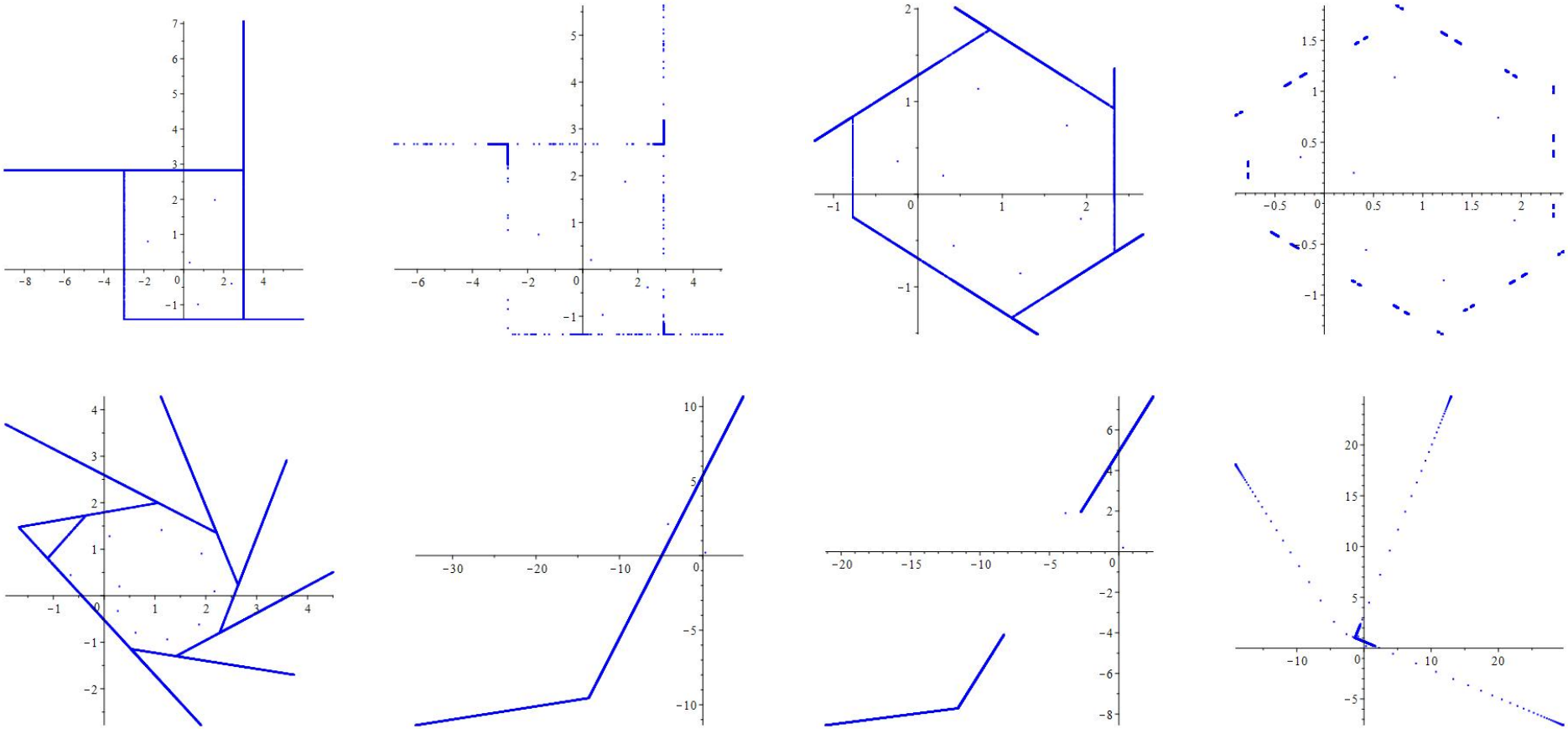
The two regions are

$$H^+ = \{(x,y) : x > 0\}, \qquad H^- = \{(x,y) : x \leq 0\}.$$

The main assumptions are that $\det(A^-) = 0$ and $\det(A^+) \neq 0$.

The fixed point of F^+ is **virtual** if it lies in H^-, and **real or admissible** otherwise.

Some simulations with 10000 points of a single orbit



This numerical behavior is different than the one observed in the map of Part I.

- I. When do almost all orbits visit the collapse region? ✓

  Not always. We will characterize the cases.

  The cases where this property does not happen have simple (linear) dynamics.

- II. When do invariant graphs that capture the interesting dynamics do exist? ✓

  Not always. We will study many cases where they exist and show an example where they do not exist.

- III. When do other invariant **compact** sets that capture the interesting dynamics do exist? ✓

  We will describe these invariant compacts.

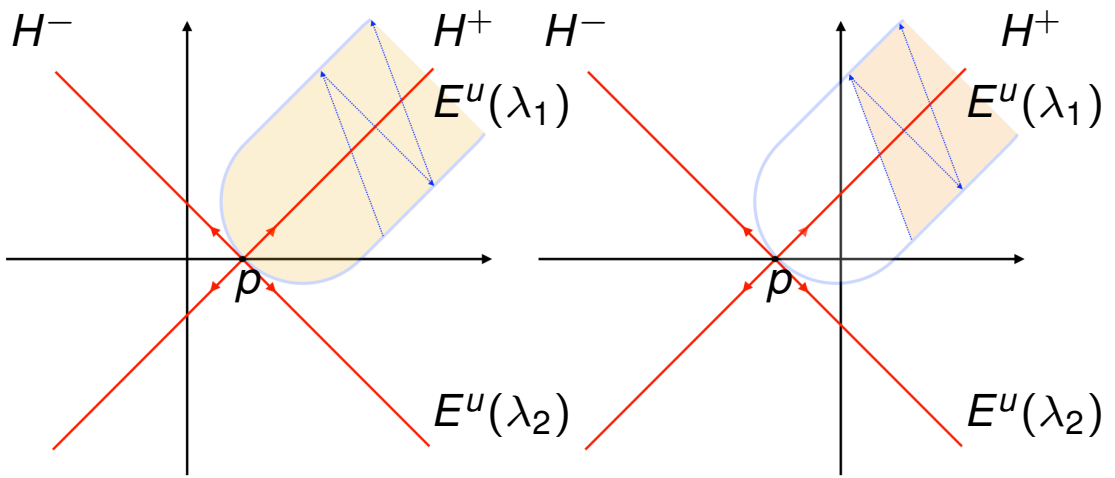
- IV. Which tools can be used to study the complexity of the dynamics on these objects? ✓

  Markov partitions, topological entropy and rotation intervals of circle maps.

Proposition: Main criterion

Except possibly for initial conditions in a finite graph, all orbits in H^+ eventually enter H^-, after a finite number of iterations, precisely in the following cases:

- virtual attractor;
- real or virtual repelling focus;
- repelling node with the appropriate negative dominant direction, except some non generic situations;
- saddle with a negative repelling direction, except some non generic situations.



A real node case

A virtual node case

A key necessary condition for strong dissipation is

$$\text{rank}(A^-) = 1, \quad \text{then} \quad \det(A^-) = 0.$$

Then the image of the half-plane H^-, coincides with the image of the y-axis, Y, and collapses onto a line, say L.

The natural candidate invariant set for concentrate all the dynamics is $\Omega = \bigcup_{i\geq 0} F^i(L)$.

Central question

When is the global invariant set

$$\Omega = \bigcup_{n\geq 0} F^n(L)$$

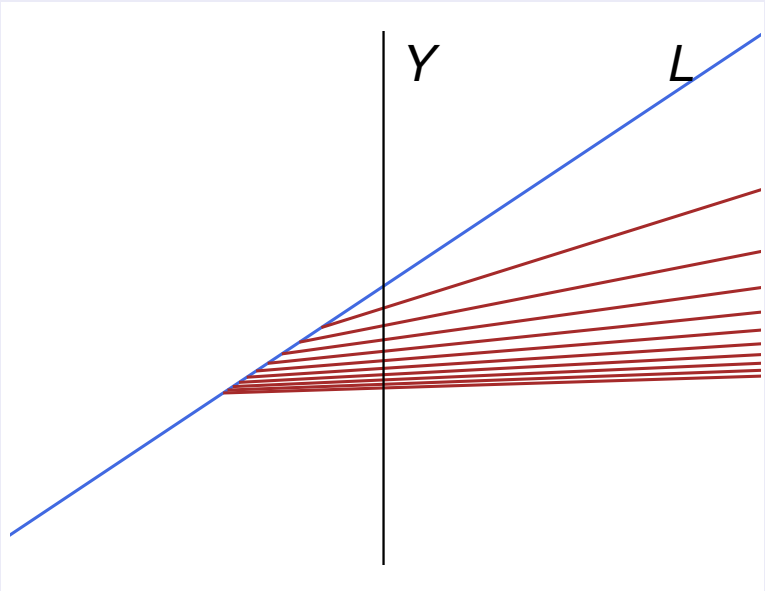
a finite graph, and how does its construction bifurcate as the parameters vary?

In this talk, a graph means a closed, connected, piecewise-linear set, obtained as a finite union of segments, lines and rays.

Recall that for us a graph means a closed, connected, piecewise-linear set, obtained as a **finite union of segments**, lines and rays. Notice that, now, *our* graphs are not necessarily bounded.

Proposition: Existence of graphs

- For the case of **real or virtual focus** the graph always exists.
- There are case of node for which the set $\Omega = \bigcup_{n\geq 0} F^n(L)$ is not a graph.



Theorem

If the map F_+ has a **focus** (virtual or real repelling) or a **virtual center**, then:

(a) $\Omega = \bigcup_{n\geq 0} F^n(L)$ is a finite graph and "all" orbits reach Ω.

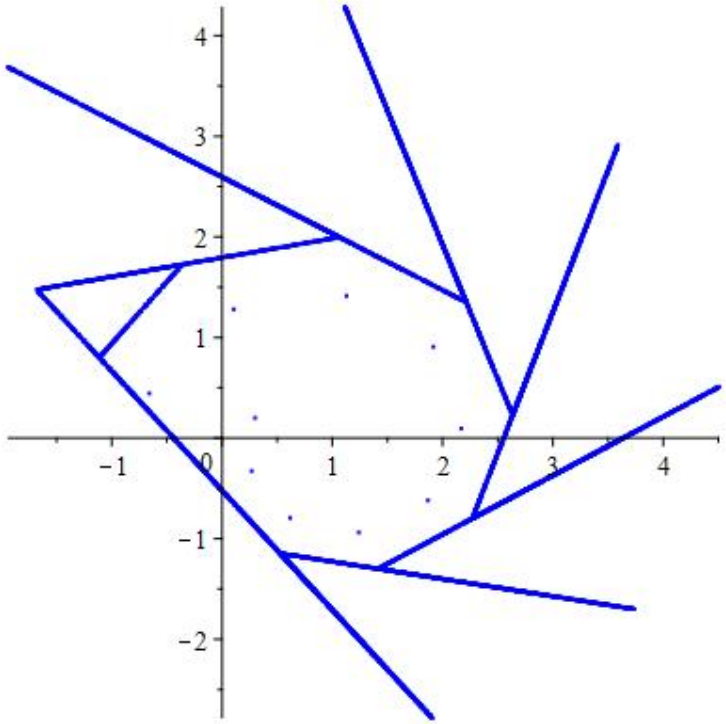
(b) One of the following alternatives holds

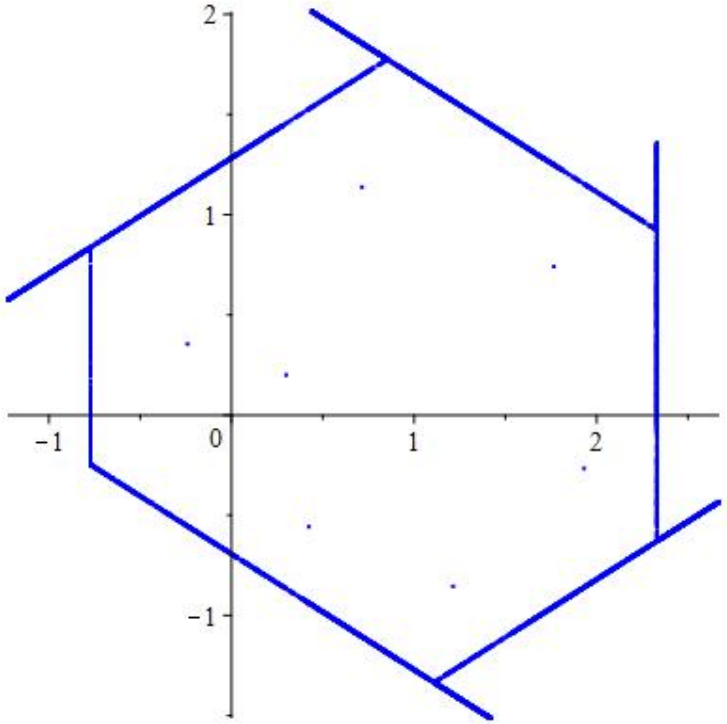
1rst: There exists an invariant compact subgraph $\Gamma \subset \Omega$ that collects all the interesting dynamics of $F|_\Omega$.

2nd: There exists a non-invariant compact subgraph $\Gamma \subset \Omega$ such that $\lim |F^n(x)| = \infty$ for all $x \in \Omega \setminus \Gamma$ and there exists a compact invariant subset $\Delta \subset \Gamma$ that collects all non-escaping dynamics of $F|_\Omega$.

I show some shapes of Γ and Δ.

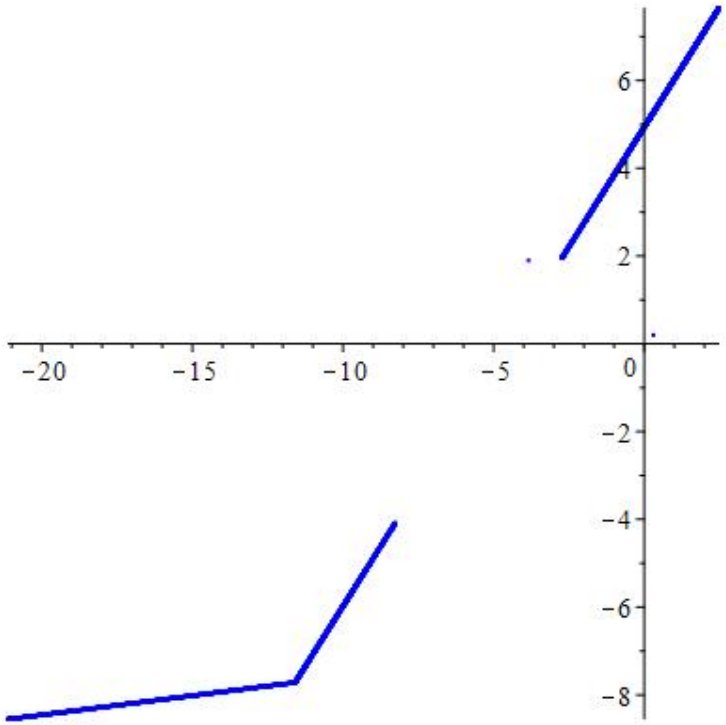
1rst alternative: two examples with a compact subgraf $\Gamma \subset \Omega$.

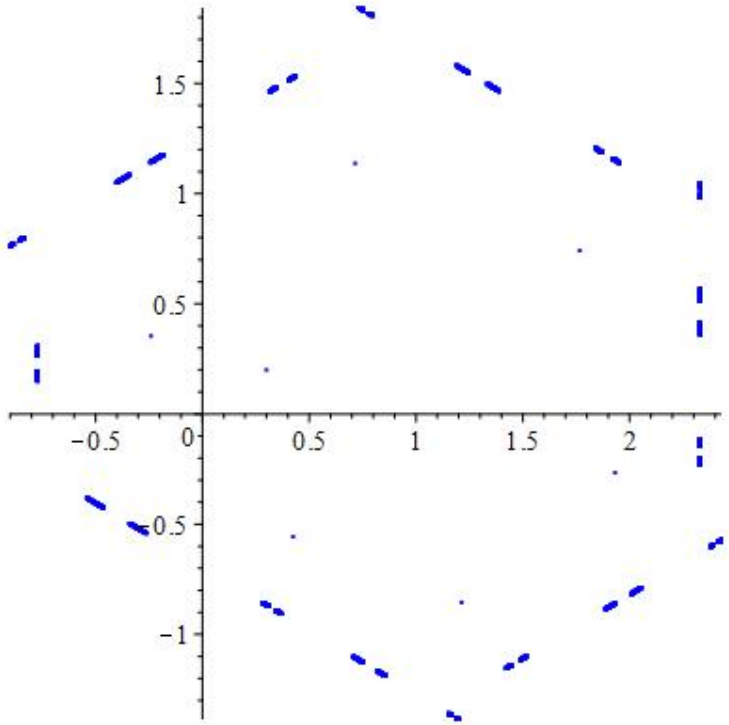




These pictures are obtained by iteration of a single orbit.

2nd alternative: two examples with a compact invariant subset $\Delta \subset \Gamma \subset \Omega$.





After a linear change of coordinates preserving Y, the real repelling focus is placed at $(1,0)$.

$$F(x,y) = \begin{cases} F_+(x,y), & x \geq 0, \\ F_-(x,y), & x < 0, \end{cases}$$

where $r > 1$, $\lambda \in \mathbb{R}$, and $\theta \in (0,\pi)$. The map in H^+ and in H^- is

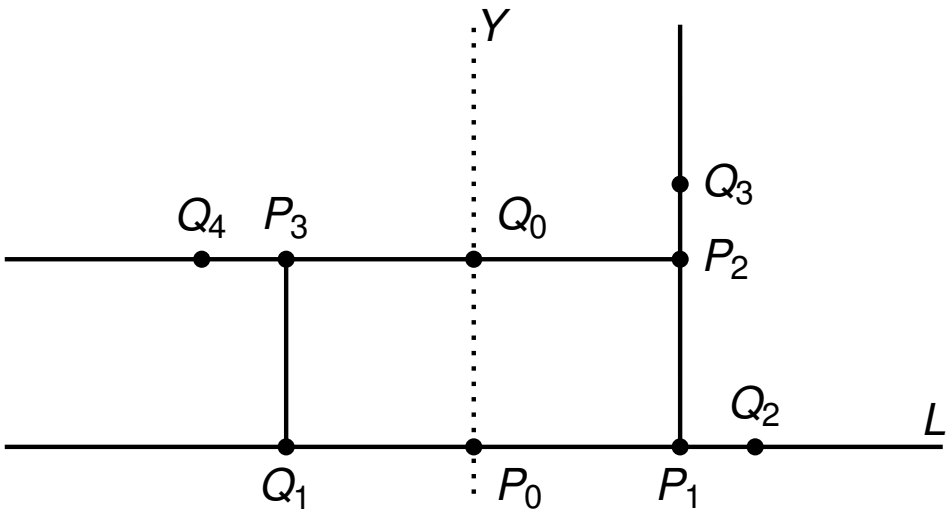
$$F_+(x,y) = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - r\cos\theta \\ -r\sin\theta \end{pmatrix}.$$

and

$$F_-(x,y) = \begin{pmatrix} -\lambda r\sin\theta & -r\sin\theta \\ \lambda r\cos\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - r\cos\theta \\ -r\sin\theta \end{pmatrix}.$$

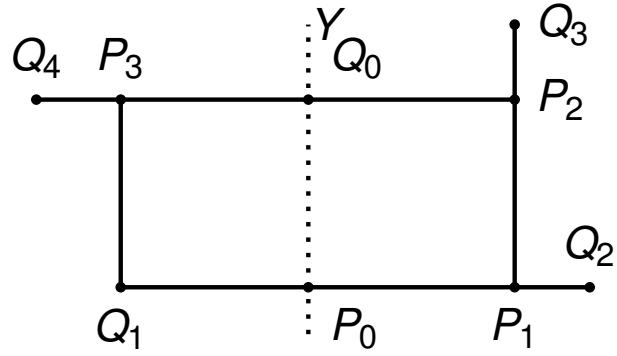
Its linear part has rank one.

As an example we choose $(\theta, r) = (\pi/2, \sqrt{2})$.



Graph Ω

$\Longrightarrow$

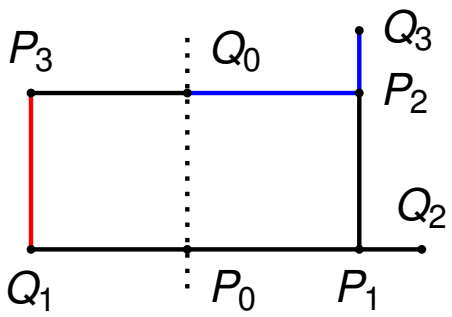


Compact graph $\Gamma_1 \subset \Omega$

To simplify the exposition we fix $\lambda = 1/\sqrt{2}$. This gives $Q_5 = F(Q_4) = Q_2$, which forces the existence of a compact invariant subgraph.

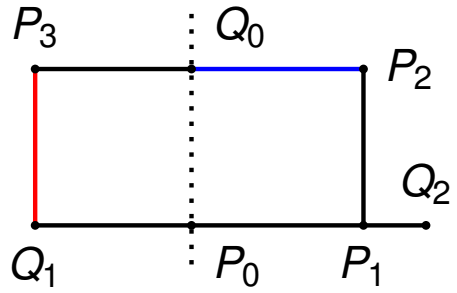
The escaping ray can be removed, leaving a compact graph Γ_1 which still contains the relevant bounded dynamics.

We identify points with the same image $x \sim y \Leftrightarrow F(x) = F(y)$

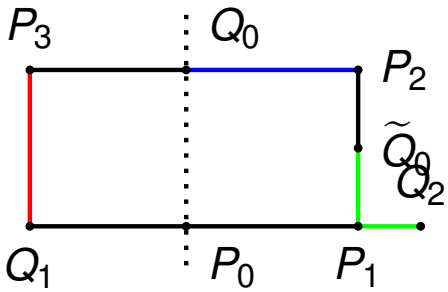


Graph Γ_2

$\Longrightarrow$

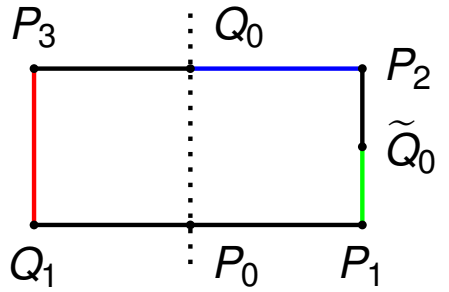


Graph Γ_3



Graph Γ_3

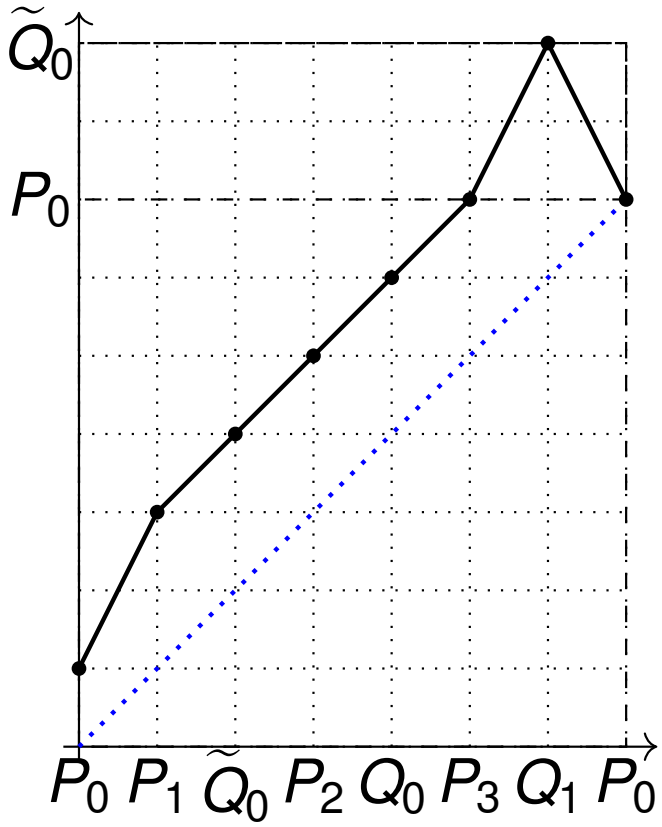
$\Longrightarrow$

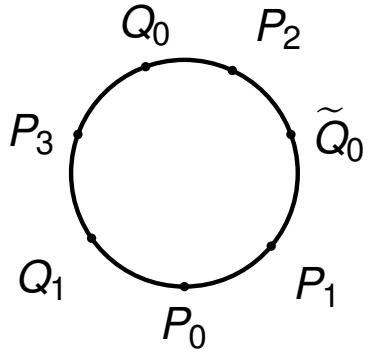


Graph $\Gamma_4 \cong \mathbb{S}^1$

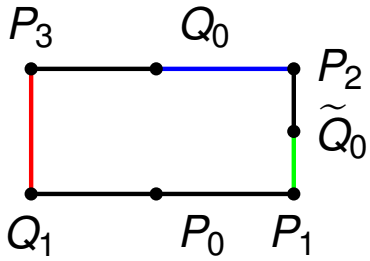
by *projection* we can define maps F_i on each of these graphs. The entropy decreases under these identifications, but it remains positive at the final step:

$$h(F) \geq h(F_2) \geq h(F_3) \geq h(F_4) > 0.$$





$\Omega_4 \cong \mathbb{S}^1$



Γ_4

The circle map F_4, has two P.O. that give two different rotation numbers $1/4$ and $1/3$.

$$P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow P_0 \text{ and } Q_0 \longrightarrow Q_1 \longrightarrow \tilde{Q}_0 \longrightarrow Q_0.$$

- By Ito's theorem, $\exists$ of the rotation interval $\mathcal{I} := [1/4, 1/3] \Rightarrow$ **the entropy is positive**.
- By a result of Misiurewicz, $\exists$ Periodic orbits with rotation number for all $p/q \in \mathring{\mathcal{I}}$ with minimal period kq, for $k \geq 1$.

Thank you very much!



A fifth co-author?