Bifurcation diagrams of one-dimensional nonlocal elliptic equations

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Introduction

We consider the following one-dimensional nonlocal elliptic equation

$$\begin{cases} -\left(\int_0^1 |u(x)|^p dx + b\right)^q u''(x) = \lambda u(x)^p, & x \in I := (0,1), \\ u(x) > 0, & x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where b, p, q are given constants satisfying

$$b \geq 0, \quad p \geq 1, \quad q > 1 - \frac{1}{p} \tag{1.2}$$

and $\lambda > 0$ is a bifurcation parameter.

Introduction

Problem (1.1) is the model equation of the following nonlocal problem considered in Goodrich [10]:

$$\begin{cases} -a\left(\int_0^1 |u(x)|^p dx\right) u''(x) = \lambda f(x, u(x)), & x \in I, \\ u(x) > 0, & x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $a = a(w)$ is a real-valued continuous function. Let

$$\|u\|_p := \left(\int_0^1 |u(x)|^p dx\right)^{1/p}.$$

If we put $a(\|u\|_p^p) = (\|u\|_p^p + b)^q$ and $f(x, u) = u^p$ in (1.3), then we obtain (1.1).

Introduction

Nonlocal elliptic problems as (1.3) have been studied intensively by many authors, since they arise in various physical models. We refer to

[4] F. J. S. A. Corrêa and D. C. de Morais Filho (2005),

[5] R. Filippucci, R. Ghiselli Ricci and P. Pucci (1994),

[6] R. Filippucci and R. Ghiselli Ricci (1994), [7] R. Filippucci (2007),

[10] C.S. Goodrich (2021),

[11,12] A.A. Lacey (1995),

[14] R. Stańczy (2001).

In particular, [5,6] dealt with the existence of nodal solutions with respect to certain parameter for m-Laplacian case as well as mean curvature equations. In R. Filippucci and R. Ghiselli Ricci [6], a symmetric setting was taken under consideration, and the mean curvature case was considered in R. Filippucci [7].

Purpose

The purpose of this paper is to obtain the global and asymptotic behaviors of bifurcation curves $\lambda = \lambda(\alpha)$ and u_λ as $\lambda \to \infty$ by focusing on the typical nonlocal problem (1.1). Here, u_λ is a solution of (1.1) and $\underline{\alpha := \alpha_\lambda = \|u_\lambda\|_\infty}$ for given $\lambda > 0$.

To state our results, we prepare the following notation. For $p > 1$, let

$$\begin{cases} -W''(x) = W(x)^p, & x \in I, \\ W(x) > 0, & x \in I, \\ W(0) = W(1) = 0. \end{cases} \tag{1.4}$$

We know from B. Gidas, W. M. Ni and L. Nirenberg [9, (1979)] that there exists a unique solution $W_p(x)$ of (1.4).

Theorem 1.1: the case $b = 0$

Theorem 1.1. *Let* $b = 0$ *in (1.1). Then there exists a unique solution* u_λ *of (1.1) for any given* $\lambda > 0$*. Furthermore, the following formulas hold:*
(i) Assume that $p > 1$*. Then*

$$\lambda = 2^{q+1}(p+1)^{1-q}L_p^{2-q}\alpha^{pq-p+1}, \tag{1.5}$$

$$u_\lambda(x) = \lambda^{1/(pq-p+1)} \times \left\{(2^{(2p-1)/(p-1)}(p+1)^{1/(p-1)}L_p^{(p+1)/(p-1)}\right\}^{-q/(pq-p+1)} W_p(x), \tag{1.6}$$

where

$$L_p := \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}ds. \tag{1.7}$$

(ii) Assume that $p = 1$*. Let* $u_\lambda(x) := \alpha \sin \pi x$ *be the solution of (1.1), where* $\alpha > 0$ *is a given constant. Then*

$$\lambda = 2^q \pi^{2-q}\alpha^q. \tag{1.8}$$

Theorem 1.1: the case $b = 0$

We note that if we put $p = 1$ in (1.5) formally, then we obtain (1.8). By Theorem 1.1 (i) and (1.2), we obtain the following qualitative image of the graph of (1.5).

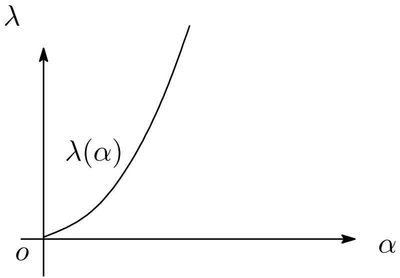

Fig. 1. The graph of $\lambda(\alpha)$

Theorem 1.2: the simple case $b > 0$, $p = 2, q = 1$

We next consider the case $\underline{b > 0}$. To clarify our intention, we start from the simplest case $p = 2$ and $q = 1$. For $p > 1$, we have

$$\|W_p\|_p = 2^{(2p-1)/(p(p-1))}(p+1)^{1/(p(p-1))}L_p^{(p+1)/(p(p-1))}, \quad (1.9)$$

$$\|W_p\|_\infty = (2(p+1))^{1/(p-1)}L_p^{2/(p-1)}. \quad (1.10)$$

We can obtain (1.9) and (1.10) by using time map argument at the end of the next Section 2.

Theorem 1.2. *Let* $b > 0$, $p = 2, q = 1$ *and*

$$\lambda_0 := 2b^{1/2}\|W_2\|_2. \tag{1.11}$$

(i) If $0 < \lambda < \lambda_0$, *then there exists no solution of (1.1).*

(ii) If $\lambda = \lambda_0$, *then (1.1) has a unique solution*

$$u_\lambda(x) = \frac{\lambda_0}{2}\|W_2\|_2^{-2}W_2(x). \tag{1.12}$$

(iii) If $\lambda > \lambda_0$, *then there exist exactly two solutions* $u_{1,\lambda}, u_{2,\lambda}$ *of (1.1) such that*

$$u_{1,\lambda}(x) = \frac{\lambda\|W_2\|_2^{-1} - \sqrt{\lambda^2\|W_2\|_2^{-2} - 4b}}{2}\|W_2\|_2^{-1}W_2(x), \tag{1.13}$$

$$u_{2,\lambda}(x) = \frac{\lambda\|W_2\|_2^{-1} + \sqrt{\lambda^2\|W_2\|_2^{-2} - 4b}}{2}\|W_2\|_2^{-1}W_2(x). \tag{1.14}$$

Theorem 1.2: the simple case $b > 0$, $p = 2, q = 1$

The following Fig. 2 is the qualitative image of $\alpha_j(\lambda) := \|u_{j,\lambda}\|_\infty$ $(j = 1, 2)$ of (1.13) and (1,14).

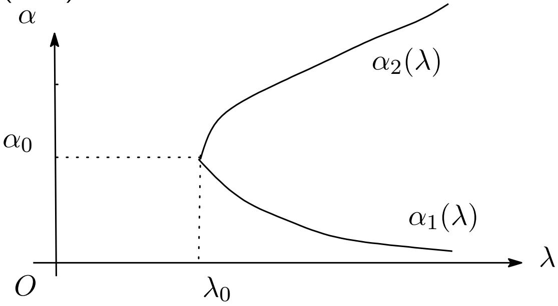

Fig. 2

Indeed, we see from (1.13) and (1.14) that these two curves start from (λ_0, α_0) $(\alpha_0 := \lambda_0 \|W_2\|_2^{-2} \|W_2\|_\infty / 2)$. Further, by Taylor expansion, we see that $\alpha_1(\lambda) = b\|W_2\|_\infty \lambda^{-1}(1 + o(1))$ and $\alpha_2(\lambda) = \|W_2\|_\infty \|W_2\|_2^{-2} \lambda(1 + o(1))$ for $\lambda \gg 1$.

For the case $p > 1$ and $q = 1$ $(p \neq 2)$, it seems difficult to obtain such exact solutions u_λ as in (1.13)–(1.14). Therefore, we try to find the asymptotic shape of solutions u_λ for $\lambda \gg 1$.

Theorem 1.3. *Let $p > 1, b > 0$ and $q = 1$. Put*

$$\lambda_0 := (b(p-1))^{1/p} \frac{p}{p-1} \|W_p\|_p^{p-1}. \tag{1.15}$$

(i) If $0 < \lambda < \lambda_0$, then there exists no solution of (1.1).

(ii) If $\lambda = \lambda_0$, then there exists a unique solution of (1.1).

Theorem 1.3: the case $b > 0$, $p > 1 (p \neq 2), q = 1$

(iii) If $\lambda > \lambda_0$, then there exist exactly two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ of (1.1). Moreover, for $\lambda \gg 1$,

$$\left\{ \begin{array}{l} \lambda := \lambda_1(\alpha) = b\|W_p\|_\infty^{p-1}\alpha^{-(p-1)}\left\{1 + b^{-1}\|W_p\|_p^p\|W_p\|_\infty^{-p}\alpha^p(1+o(1))\right\}, \\ u_{1,\lambda}(x) = b^{1/(p-1)}\lambda^{-1/(p-1)}\left\{1 + \frac{1}{p-1}b^{1/(p-1)}\|W_p\|_p^p\lambda^{-p/(p-1)}(1+o(1))\right\} \end{array} \right. \tag{1.16}$$

$$\left\{ \begin{array}{l} \lambda := \lambda_2(\alpha) = \|W_p\|_p^p\|W_p\|_\infty^{-1}\alpha + b\|W_p\|_\infty^{p-1}\alpha^{1-p} + o(\alpha^{1-p}), \\ u_{2,\lambda}(x) = \left\{\lambda\|W_p\|_p^{1-p} - b\left(\lambda\|W_p\|_p^{1-p}\right)^{1-p}(1+o(1))\right\}\|W_p\|_p^{-1}W_p(x). \end{array} \right. \tag{1.17}$$

Finally, we treat (1.1) under the condition (1.2).

Theorem 1.4. *Assume (1.2) with $p > 1, b > 0$ and $\lambda \gg 1$. Then there exist exactly two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ of (1.1). Moreover, for $\lambda \gg 1$,*

$$u_{1,\lambda}(x) = b^{q/(p-1)}\lambda^{-1/(p-1)} \times \left\{1 + \frac{q}{p-1} b^{(pq-p+1)/(p-1)} \|W_p\|_p^p \lambda^{-p/(p-1)}(1+o(1))\right\} W_p(x), \tag{1.18}$$

$$u_{2,\lambda}(x) = \left\{m\lambda^{1/(pq-p+1)} - \frac{bqm^{1-p}}{pq-p+1}\lambda^{(1-p)/(pq-p+1)}(1+o(1))\right\} \times \|W_p\|_p^{-1} W_p(x), \tag{1.19}$$

where $m := \|W_p\|_p^{(1-p)/(pq-p+1)}$.

Proof of Theorem 1.1: the case $b = 0$

In this section, let $b = 0$ in (1.1).

Lemma 2.1. *For any $\lambda > 0$, (1.1) has a unique solution u_λ.*

Proof. We apply the argument used in C. O. Alves, etal (2005) to (1.1).
For a given $\lambda > 0$, we consider

$$\begin{cases} -w''(x) = \lambda w(x)^p, \ \ x \in I = (0,1), \\ w(x) > 0, \ \ x \in I, \\ w(0) = w(1) = 0. \end{cases} \tag{2.1}$$

Then it is clear that

$$w_\lambda(x) := \lambda^{1/(1-p)} W_p(x) \tag{2.2}$$

is the unique solution of (2.1).

Existence

For $t > 0$, we put

$$g(t) := t^{pq/2} - \|w_\lambda\|_p^{1-p} t^{(p-1)/2}. \tag{2.3}$$

Then it is known from C. O. Alves, etal (2005) [1,Theorem 2] that if $\underline{g(t_\lambda) = 0}$, then

$$u_\lambda := \gamma w_\lambda \quad (\gamma := t_\lambda^{1/2} \|w_\lambda\|_p^{-1})$$

satisfies (1.1). Indeed, by (2.1) and (2.3), we have

$$\begin{aligned} -\left(\int_0^1 u_\lambda(x)^p dx\right)^q u_\lambda''(x) &= -\|u_\lambda\|_p^{pq} u_\lambda''(x) \\ &= -(\|\gamma w_\lambda\|_p)^{pq} \gamma w_\lambda''(x) = t_\lambda^{pq/2} \gamma \lambda w_\lambda(x)^p \\ &= (t_\lambda^{1/2} \|w_\lambda\|_p^{-1})^{p-1} \gamma \lambda w_\lambda(x)^p \\ &= \lambda(\gamma w_\lambda(x))^p = \lambda u_\lambda(x)^p. \end{aligned} \tag{2.4}$$

Existence

On the other hand, assume that u_λ satisfies (1.1). Then we see that $u_\lambda = \|u_\lambda\|_p^{pq/(p-1)} w_\lambda$. Then we have $\|u_\lambda\|_p^{(p-1-pq)/(p-1)} = \|w_\lambda\|_p$. We put $t_\lambda^{1/2} := \|u_\lambda\|_p = \|w_\lambda\|_p^{(1-p)/(pq-p+1)}$. Then by (2.3), we have $g(t_\lambda) = 0$. Consequently, the solutions $\{t_{1,\lambda}, t_{2,\lambda}, \cdots, t_{k,\lambda}\}$ of $g(t) = 0$ correspond to the solutions $\{u_{1,\lambda}, u_{2,\lambda}, \cdots, u_{k,\lambda}\}$. Therefore, if $g(t) = 0$ has a unique soution, then (1.1) also has a unique solution. By (1.2) and (2.3), we see that

$$g(t) = t^{(p-1)/2}\left(t^{(pq-p+1)/2} - \|w_\lambda\|_p^{1-p}\right) = 0 \tag{2.5}$$

has a unique positive solution $t_\lambda = \|w_\lambda\|_p^{2(1-p)/(pq-p+1)}$. Thus the proof is complete. □

Proof of Theorem 1.1. (i) Let $t_\lambda = \|w_\lambda\|_p^{2(1-p)/(pq-p+1)}$. By Lemma 2.1 and (2.2), we have

$$u_\lambda(x) = t_\lambda^{1/2}\|w_\lambda\|_p^{-1}w_\lambda(x) = t_\lambda^{1/2}\|W_p\|_p^{-1}W_p(x). \tag{2.6}$$

By (2.2) and (2.5), we have

$$t_\lambda = \|w_\lambda\|_p^{2(1-p)/(pq-p+1)} = \lambda^{2/(pq-p+1)}\|W_p\|_p^{2(1-p)/(pq-p+1)}. \tag{2.7}$$

Value of $\|W_p\|_p$. We apply the time map argument to (1.4). (cf. [13]). Since (1.4) is autonomous, we have

$$W_p(x) = W_p(1-x), \quad x \in [0, 1/2], \tag{2.8}$$

$$W_p'(x) > 0, \quad x \in [0, 1/2), \tag{2.9}$$

$$\xi := \|W_p\|_\infty = \max_{0\le x\le 1} W_p(x) = W_p(1/2). \tag{2.10}$$

Proof of Theorem 1.1; time map method

By (1.4), for $0 \le x \le 1$, we have

$$\{W_p''(x) + W_p(x)^p\}W_p'(x) = 0. \tag{2.11}$$

By this and (2.10), we have

$$\begin{aligned} &\frac{1}{2}W_p'(x)^2 + \frac{1}{p+1}W_p(x)^{p+1} = \text{constant} \\ &= \frac{1}{p+1}W_p(1/2)^{p+1} = \frac{1}{p+1}\xi^{p+1}. \end{aligned} \tag{2.12}$$

By this and (2.9), for $0 \le x \le 1/2$, we have

$$W_p'(x) = \sqrt{\frac{2}{p+1}(\xi^{p+1} - W_p(x)^{p+1})}. \tag{2.13}$$

By this, (2.8) and putting $\theta := W_p(x)$, we have

Proof of Theorem 1.1

$$\begin{aligned}
\|W_p\|_p^p &= 2\int_0^{1/2} W_p(x)^p dx \qquad (2.14)\\
&= 2\int_0^{1/2} W_p(x)^p \frac{W_p'(x)}{\sqrt{\frac{2}{p+1}(\xi^{p+1} - W_p(x)^{p+1})}} dx \\
&= \sqrt{2(p+1)} \int_0^{\xi} \frac{\theta^p}{\sqrt{\xi^{p+1} - \theta^{p+1}}} d\theta \qquad (\theta = \xi s) \\
&= \sqrt{2(p+1)} \xi^{(p+1)/2} \int_0^1 \frac{s^p}{\sqrt{1 - s^{p+1}}} ds \\
&= 2^{3/2}(p+1)^{-1/2}\xi^{(p+1)/2}.
\end{aligned}$$

Proof of Theorem 1.1

By this, (2.6), and (2.7), we have

$$u_\lambda(x) = \lambda^{1/(pq-p+1)}\{2^{3/2}(p+1)^{-1/2}\xi^{(p+1)/2}\}^{-q/(pq-p+1)}W_p(x). \quad (2.15)$$

By putting $x = 1/2$ in (2.15), we have

$$\alpha = \lambda^{1/(pq-p+1)}\{(2^{3/2}(p+1)^{-1/2}\xi^{(p+1)/2}\}^{-q/(pq-p+1)}\xi. \quad (2.16)$$

By (2.13), we have

$$\begin{aligned}
\frac{1}{2} &= \int_0^{1/2} 1dx = \int_0^{1/2} \frac{W_p'(x)}{\sqrt{\frac{2}{p+1}(\xi^{p+1} - W_p(x)^{p+1})}}dx \quad (2.17) \\
&= \sqrt{\frac{p+1}{2}}\int_0^{\xi} \frac{1}{\sqrt{\xi^{p+1} - \theta^{p+1}}}d\theta \qquad (\theta = \xi s) \\
&= \sqrt{\frac{p+1}{2}}\xi^{(1-p)/2}\int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}}ds = \sqrt{\frac{p+1}{2}}\xi^{(1-p)/2}L_p.
\end{aligned}$$

Proof of Theorem 1.1

By this, we have

$$\xi = (2(p+1))^{1/(p-1)} L_p^{2/(p-1)}. \tag{2.18}$$

Thus, using also (2.15) and (2.16), we obtain (1.5) and (1.6). □

(ii) Let $p = 1$. Then by (1.1), we have

$$-u_\lambda''(x) = \frac{\lambda}{\|u_\lambda\|_1^q} u_\lambda(x) = \pi^2 u_\lambda(x). \tag{2.19}$$

Since $u_\lambda(x) = \alpha \sin \pi x$ $(\alpha = \|u_\lambda\|_\infty)$, and

$$\|u_\lambda\|_1 = \int_0^1 \alpha \sin \pi x dx = \frac{2\alpha}{\pi}. \tag{2.20}$$

By this and (2.19), we have

$$\lambda = \pi^2 \|u_\lambda\|_1^q = 2^q \pi^{2-q} \alpha^q. \tag{2.21}$$

This implies (1.8). Thus the proof of Theorem 1.1 is complete. □

$\|W_p\|_p$ and $\|W_p\|_\infty$

By (2.14) and (2.18), for $p > 1$, we obtain

$$\|W_p\|_p = 2^{(2p-1)/(p(p-1))}(p+1)^{1/(p(p-1))}L_p^{(p+1)/(p(p-1))}. \qquad (2.22)$$

This implies (1.9). By (2.18), we obtain (1.10).

The Idea of the Proof of Theorems 1.2 and 1.3: $b > 0, p > 1$

In this section, let $p > 1$. First, we consider (1.1) under the condition (1.2). The approach to find the solutions of (1.1) is a variant of (2.3)–(2.4). Namely, we seek the solutions of (1.1) of the form

$$u_\lambda(x) := t\|w_\lambda\|_p^{-1} w_\lambda(x) = t\|W_p\|_p^{-1} W_p(x) \tag{3.1}$$

for some $t > 0$. To do this, let $M(s) := (s + b)^q$. If we have solutions of (1.1) of the form (3.1), then since $\|u_\lambda\|_p = t$ by (3.1), we have

$$\begin{aligned} -M(\|u_\lambda\|_p^p) u_\lambda''(x) &= -M(t^p) \frac{t}{\|w_\lambda\|_p} w_\lambda''(x) \\ &= M(t^p) \frac{t}{\|w_\lambda\|_p} \lambda w_\lambda(x)^p \\ &= \underline{M(t^p) t^{1-p} \|w_\lambda\|_p^{p-1}} \lambda u_\lambda(x)^p. \end{aligned} \tag{3.2}$$

The Idea of the Proof of Theorems 1.2 and 1.3: $b > 0, p > 1$

By this, we look for t satisfying

$$M(t^p)t^{1-p}\|w_\lambda\|_p^{p-1} = 1. \tag{3.3}$$

Namely, we solve the equation

$$g(t) := (t^p + b)^q - \|w_\lambda\|_p^{1-p}t^{p-1} = 0. \tag{3.4}$$

By this, we have

$$\begin{aligned} g'(t) &= pq(t^p + b)^{q-1}t^{p-1} - (p-1)\|w_\lambda\|_p^{1-p}t^{p-2} \qquad (3.5) \\ &= t^{p-2}\{pq(t^p + b)^{q-1}t - (p-1)\|w_\lambda\|_p^{1-p}\} =: t^{p-2}\tilde{g}(t). \end{aligned}$$

The Idea of the Proof of Theorems 1.2 and 1.3: $b > 0, p > 1$

By direct calculation, we see that $\tilde{g}(t)$ is strictly increasing for $t > 0$. Further, by (3.5), for $0 < t \ll 1$, we have $g'(t) < 0$. Therefore, we see that there exists a unique $t_0 > 0$ such that $g'(t) < 0$ for $0 < t_0 < t$, $g'(t_0) = 0$ and $g'(t) > 0$ for $t > t_0$. By using (3.4) and (3.5), we find that

$$g(t_0) \;=\; \frac{\|w_\lambda\|_p^{1-p}}{pqt_0}\left\{-(pq-p+1)t_0^p + b(p-1)\right\}. \qquad (3.6)$$

If $g(t_0) < 0$, then there exists exactly t_1, t_2 with $0 < t_1 < t_0 < t_2$ such that $g(t_1) = g(t_2) = 0$. If $g(t_0) > 0$, then (3.3) has no solutions. This idea will be also used in the next sections.

The Idea of the Proof of Theorems 1.2 and 1.3: $b > 0, p > 1$

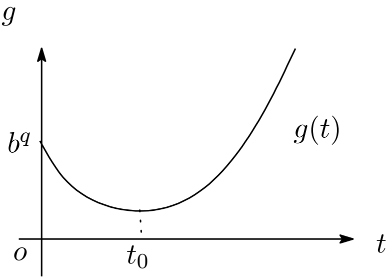

Fig. 3-1. The graph of $g(t)$

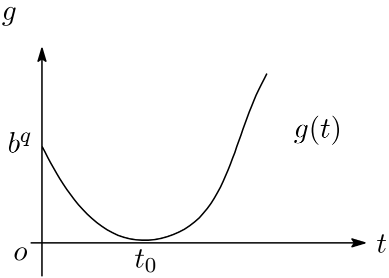

Fig. 3-2. The graph of $g(t)$

The Idea of the Proof of Theorems 1.2 and 1.3: $b > 0, p > 1$

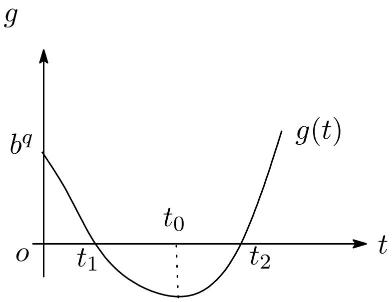

Fig. 3-3. The graph of $g(t)$

The Idea of the Proof of Theorems 1.2 and 1.3: $b > 0, p > 1$

Proof of Theorem 1.2 is more simple, since t_0 is obtained explicitly. Since $p = 2, q = 1$, by (3.4), we have

$$g(t) = t^2 + b - \lambda \|W_2\|_2^{-1} t = 0. \tag{3.7}$$

Then

$$t_{1,\lambda} = \frac{\lambda \|W_2\|_2^{-1} - \sqrt{\lambda^2 \|W_2\|_2^{-2} - 4b}}{2}, \tag{3.8}$$

$$t_{2,\lambda} = \frac{\lambda \|W_2\|_2^{-1} + \sqrt{\lambda^2 \|W_2\|_2^{-2} - 4b}}{2}. \tag{3.9}$$

By these and (3.1), we obtain (i), (ii) and (iii). Thus, the proof is complete. □

Proof of Theorems 1.3: $b > 0, p > 1, q = 1$

In this section, let $p > 1, q = 1$. Since $\|w_\lambda\|_p = \lambda^{-1/(p-1)}\|W_p\|_p$ by (2.2), we have from (3.5) that

$$g'(t) = t^{p-2}\{pt - \lambda\|W_p\|^{1-p}(p-1)\}. \tag{4.1}$$

We put

$$t_0 := \frac{p-1}{p}\lambda\|W_p\|_p^{1-p}. \tag{4.2}$$

Then $g'(t_0) = 0$. By this and (3.4), we have

$$\begin{aligned} g(t_0) &= \left(\frac{p-1}{p}\lambda\|W_p\|_p^{1-p}\right)^p + b - \lambda\|W_p\|_p^{1-p}\left(\frac{p-1}{p}\lambda\|W_p\|_p^{1-p}\right)^{p-1} \\ &= -\frac{1}{p-1}\left(\frac{p-1}{p}\lambda\|W_p\|_p^{1-p}\right)^p + b. \end{aligned} \tag{4.3}$$

Proof of Theorems 1.3: $b > 0, p > 1, q = 1$

We put

$$\lambda_0 := (b(p-1))^{1/p} \frac{p}{p-1} \|W_p\|_p^{p-1}. \tag{4.4}$$

By this, we see that (i)–(iii) are valid, since if λ_0 satisfies (4.4), then $g(t_0) = 0$ by (4.3). Further, $g(0) = b > 0$ and $g(t) > 0$ when $t \gg 1$. We now prove (1.17). We assume that $\lambda \gg 1$. Then there exists t_1, t_2 with $0 < t_1 < t_0 < t_2$ which satisfy $g(t_1) = g(t_2) = 0$. Since $t_0 \to \infty$ as $\lambda \to \infty$, we see that $t_2 \to \infty$ as $\lambda \to \infty$. Then by (3.4), we have

$$t_2 = \lambda \|W_p\|_p^{1-p} + R, \tag{4.5}$$

where R is the remainder term, and $R = o(\lambda)$. By (3.4), (4.5) and Taylor expansion, we have

Proof of Theorems 1.3: $b > 0, p > 1, q = 1$

$$
\begin{aligned}
&(\lambda\|W_p\|_p^{1-p})^p \left(1 + \frac{R}{\lambda\|W_p\|_p^{1-p}}\right)^p + b \\
&- (\lambda\|W_p\|_p^{1-p})^p \left(1 + \frac{R}{\lambda\|W_p\|_p^{1-p}}\right)^{p-1} \\
&= (\lambda\|W_p\|_p^{1-p})^p \left(1 + \frac{pR}{\lambda\|W_p\|_p^{1-p}}(1+o(1))\right) + b \\
&- (\lambda\|W_p\|_p^{1-p})^p \left(1 + \frac{R(p-1)}{\lambda\|W_p\|_p^{1-p}}(1+o(1))\right) = 0.
\end{aligned} \tag{4.6}
$$

This implies that

Proof of Theorems 1.3: $b > 0, p > 1, q = 1$

$$R = -b\left(\lambda\|W_p\|_p^{1-p}\right)^{1-p}(1+o(1)). \tag{4.7}$$

By this, (3.1) and (4.5), for $\lambda \gg 1$, we have

$$u_{2,\lambda}(x) = \left\{\lambda\|W_p\|_p^{1-p} - b\left(\lambda\|W_p\|_p^{1-p}\right)^{1-p}(1+o(1))\right\}\|W_p\|_p^{-1}W_p(x) \tag{4.8}$$

By putting $x = 1/2$ in (4.8), we have

$$\alpha = \left\{\lambda\|W_p\|_p^{1-p} - b\left(\lambda\|W_p\|_p^{1-p}\right)^{1-p}(1+o(1))\right\}\|W_p\|_p^{-1}\|W_p\|_\infty. \tag{4.9}$$

Proof of Theorems 1.3: $b > 0, p > 1, q = 1$

By this, we obtain

$$\lambda \;=\; \|W_p\|_p^p \|W_p\|_\infty^{-1} \alpha + b\|W_p\|_\infty^{p-1} \alpha^{1-p} + o(\alpha^{1-p}). \qquad (4.10)$$

By this and (4.8), we obtain (1.17). We next prove (1.16). To do this, we consider the asymptotic behavior of t_1 as $\lambda \to \infty$. If there exists a constant $C > 0$ such that $C < t_1 < C^{-1}$. Then by (3.3), for $\lambda \gg 1$, we have

$$g(t_1) = t_1^p + b - \lambda \|W_p\|_p^{1-p} t_1^{p-1} < 0. \qquad (4.11)$$

This is a contradiction, since $g(t_1) = 0$. If $t_1 \to \infty$ as $\lambda \to \infty$, then by (1.11), we see that $t_1 = (1 + o(1))\lambda \|W_p\|_p^{1-p}$ and by (4.2), we have $t_1 > t_0$ for $\lambda \gg 1$. This is a contradiction. Therefore, $t_1 \to 0$ as $\lambda \to \infty$. By (3.4), we have

$$t_1^{p-1}(\lambda\|W_p\|_p^{1-p} - t_1) = b. \tag{4.12}$$

By this and Taylor expansion, we have

$$\begin{aligned} t_1 &= \left(\frac{b}{\lambda\|W_p\|_p^{1-p} - t_1}\right)^{1/(p-1)} \qquad (4.13) \\ &= \|W_p\|_p b^{1/(p-1)}\lambda^{-1/(p-1)}\left(\frac{1}{1 - t_1\lambda^{-1}\|W_p\|_p^{p-1}}\right)^{1/(p-1)} \\ &= \|W_p\|_p b^{1/(p-1)}\lambda^{-1/(p-1)}\left(1 + \frac{1}{p-1}\frac{t_1}{\lambda\|W_p\|_p^{1-p}}(1 + o(1))\right). \end{aligned}$$

By this and (3.1), we have

Proof of Theorems 1.3: $b > 0, p > 1, q = 1$

$$u_{1,\lambda}(x) = b^{1/(p-1)}\lambda^{-1/(p-1)} \times \left(1 + \frac{1}{p-1}\frac{b^{1/(p-1)}\|W_p\|_p^p}{\lambda^{p/(p-1)}}(1+o(1))\right) W_p(x). \quad (4.14)$$

By this, we obtain

$$\alpha = b^{1/(p-1)}\lambda^{-1/(p-1)}\left(1 + \frac{1}{p-1}\frac{b^{1/(p-1)}\|W_p\|_p^p}{\lambda^{p/(p-1)}}(1+o(1))\right)\xi. \quad (4.15)$$

By this, we have

$$\lambda_1 = b\|W_p\|_\infty^{p-1}\alpha^{-(p-1)}\left\{1 + b^{-1}\|W_p\|_p^p\|W_p\|_\infty^{-p}\alpha^p(1+o(1))\right\}. \quad (4.16)$$

By this and (4.14), we obtain (1.16). Thus the proof is complete. □

Proof of Theorems 1.4: $b > 0, p > 1, \lambda \gg 1$

In this section, we assume (1.2) and $p > 1, \lambda \gg 1$. We put $k := ((p-1)\|W_p\|_p^{1-p}/pq)^{1/(pq-p+1)}$. By (3.5), we have

$$t_0 = k\lambda^{1/(pq-p+1)}(1 + o(1)). \tag{5.1}$$

By this, (1.2) and (3.4), we see that $g(t_0) < 0$. Then there exists $0 < t_1 < t_0 < t_2$ such that $g(t_1) = g(t_2) = 0$. By (5.1), we see that $t_2 \to \infty$ as $\lambda \to \infty$. We first prove (1.19). We recall that $m := \|W_p\|_p^{(1-p)/(pq-p+1)}$. By (3.4), we have

$$t_2 = m\lambda^{1/(pq-p+1)} + r, \tag{5.2}$$

where r is the remainder term satisfying $r = o(\lambda^{1/(pq-p+1)})$. It is clear that (3.4) is equivalent to

$$(t_2^p + b)^q = \lambda\|W_p\|_p^{1-p} t_2^{p-1}. \tag{5.3}$$

By this, (5.2) and Taylor expansion, we have

$$\begin{aligned} \text{r.h.s. of (5.3)} &= \lambda \|W_p\|_p^{1-p} \left(m\lambda^{1/(pq-p+1)} + r \right)^{p-1} \qquad (5.4) \\ &= \lambda \|W_p\|_p^{1-p} m^{p-1} \lambda^{(p-1)/(pq-p+1)} \\ &\quad \times \left(1 + (p-1) \frac{r}{m\lambda^{1/(pq-p+1)}} (1 + o(1)) \right). \end{aligned}$$

By (5.2) and Taylor expansion, we have

$$
\begin{aligned}
\text{l.h.s. of (5.3)} &= t_2^{pq}\left(1+\frac{b}{t_2^p}\right)^q = t_2^{pq}\left(1+\frac{bq}{t_2^p}(1+o(1))\right) \\
&= \left(m\lambda^{1/(pq-p+1)}+r\right)^{pq} \\
&\quad\times\left\{1+bq(m\lambda^{1/(pq-p+1)}+r)^{-p}(1+o(1))\right\} \\
&= (m\lambda^{1/(pq-p+1)})^{pq}\left(1+\frac{r}{m\lambda^{1/(pq-p+1)}}(1+o(1))\right)^{pq} \\
&\quad\times\left\{1+bq(m\lambda^{1/(pq-p+1)})^{-p}(1+o(1))\right\}.
\end{aligned}
\tag{5.5}
$$

Proof of Theorems 1.4: $b > 0, p > 1, \lambda \gg 1$

By the definition of m, we see that the leading terms of (5.4) and (5.5) coinside each other. By (5.4) and (5.5), we have

$$pq\frac{r}{m\lambda^{1/(pq-p+1)}} + \frac{bq}{m^p\lambda^{p/(pq-p+1)}} = (p-1)\frac{r}{m\lambda^{1/(pq-p+1)}}. \tag{5.6}$$

This implies that

$$r = -\frac{bqm^{1-p}}{pq-p+1}\lambda^{(1-p)/(pq-p+1)}(1+o(1)). \tag{5.7}$$

By this and (5.2), for $\lambda \gg 1$, we have

$$t_2 = \left\{m\lambda^{1/(pq-p+1)} - \frac{bqm^{1-p}}{pq-p+1}\lambda^{(1-p)/(pq-p+1)}(1+o(1))\right\}. \tag{5.8}$$

By this and (3.1), we have

Proof of Theorems 1.4: $b > 0, p > 1, \lambda \gg 1$

$$u_{2,\lambda}(x) = \left\{ m\lambda^{1/(pq-p+1)} - \frac{bqm^{1-p}}{pq-p+1}\lambda^{(1-p)/(pq-p+1)}(1+o(1)) \right\} \|W_p\|_p^{-1} W_p(x). \tag{5.9}$$

This implies (1.19). We next show (1.18). We consider the asymptotic behavior of t_1 as $\lambda \to \infty$. By the same argument as that in Section 4, we find that $t_1 \to 0$ as $\lambda \to \infty$. By (5.3), we have

$$\lambda \|W_p\|_p^{1-p} t_1^{p-1} = b^q(1+o(1)). \tag{5.10}$$

This implies that

$$t_1 = b^{q/(p-1)} \|W_p\|_p \lambda^{-1/(p-1)}(1+\eta), \tag{5.11}$$

where η is the remainder term. Then by Taylor expansion and (5.11), we have

Proof of Theorems 1.4: $b > 0, p > 1, \lambda \gg 1$

By this, we have

$$\eta = \frac{q}{p-1} b^{-1} t_1^p = \frac{q}{p-1} b^{(pq-p+1)/(p-1)} \|W_p\|_p^p \lambda^{-p/(p-1)} (1 + o(1)). \quad (5.13)$$

By this, (3.1) and (5.11), we have

$$\begin{aligned} u_{1,\lambda}(x) &= b^{q/(p-1)} \lambda^{-1/(p-1)} \qquad (5.14) \\ &\quad \times \left\{ 1 + \frac{q}{p-1} b^{(pq-p+1)/(p-1)} \|W_p\|_p^p \lambda^{-p/(p-1)} (1 + o(1)) \right\} W_p(x). \end{aligned}$$

This implies (1.18). Thus the proof is complete. □

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Thank you very much

Thank You for Your Attention

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