Connected sets of periodic solutions of autonomous Hamiltonian systems

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The aim of my talk is to study connected sets of nonstationary periodic solutions of the autonomous Hamiltonian system

$$\dot{z}(t) = \lambda J H'(z(t)), \tag{HS}$$

where

1. $\Omega \subset \mathbb{R}^{2N}$ is open,
2. $H \in C^2(\mathrm{cl}(\Omega), \mathbb{R})$,
3. $H'^{-1}(0) \cap \Omega = H'^{-1}(0) \cap \mathrm{cl}(\Omega) = \{s_1, \ldots, s_k\}$.

Define sets $\mathcal{T}, \mathcal{N} \subset C_{2\pi}([0, 2\pi], \Omega) \times (0, +\infty)$ as follows

$$\mathcal{T} = \{s_1, \ldots, s_k\} \times (0, +\infty),$$

$$\mathcal{N} = \{(u(t), \lambda) : u(t) \text{ is a nonst. } 2\pi\text{-periodic solution of (HS)}\}.$$

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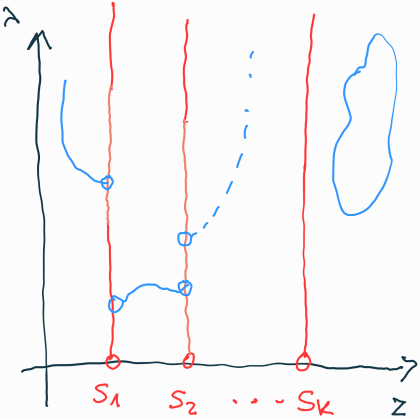

$\mathcal{T}$ – trivial solutions

$\mathcal{N}$ – nontrivial solutions
‖
nonstationary

Fix $(s_{k_0}, \lambda_0) \in \mathcal{T}$ and denote by $\mathcal{C}(s_{k_0}, \lambda_0)$ a closed connected component (= continuum) of $\mathrm{cl}(\mathcal{N})$ containing (s_{k_0}, λ_0).

Definition

A point $(s_{k_0}, \lambda_0) \in \mathcal{T}$ is said to be a global bifurcation point of 2π-periodic solutions of the system (HS) $\dot{z}(t) = \lambda J H'(z(t))$,

if either $\mathcal{C}(s_{k_0}, \lambda_0)$ is not compact in $C_{2\pi}([0, 2\pi], \Omega) \times (0, +\infty)$

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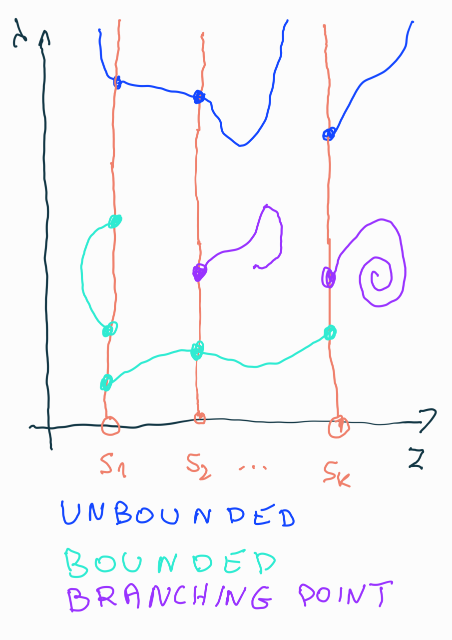
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Z
S1
S2 ...
SK
UNBOUNDED
BOUNDED
BRANCHING POINT

For $1 \leq k_0 \leq k$ define

$$\Lambda_{k_0} = \left\{ \frac{k}{\omega} \in (0, +\infty) : \pm i\omega \in \sigma(JH''(s_{k_0})), \omega > 0, k \in \mathbb{N} \right\}.$$

Theorem (Necessary condition)

If $(s_{k_0}, \lambda_0) \in \mathcal{T}$ is a global bifurcation point of 2π-periodic solutions of the system (HS) $\dot{z}(t) = \lambda JH'(z(t))$*, then $\lambda_0 \in \Lambda_{k_0}$.*

For $\lambda_0 \in \Lambda_{k_0}$ choose $\mu > 0$ such that $[\lambda_0 - \mu, \lambda_0 + \mu] \cap \Lambda_{k_0} = \{\lambda_0\}$ and define a symmetric $(4N \times 4N)$-matrix

$$T_j((\lambda_0 \pm \mu)H''(s_{k_0})) = \begin{pmatrix} -\frac{\lambda_0 \pm \mu}{j} H''(s_{k_0}) & -J \\ J & -\frac{\lambda_0 \pm \mu}{j} H''(s_{k_0}) \end{pmatrix},$$

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Define bifurcation index $\mathcal{BIF}(s_{k_0}, \lambda_0) \in \bigoplus_{j=1}^{\infty} \mathbb{Z}$ as follows

$$\mathcal{BIF}(s_{k_0}, \lambda_0) = (\gamma_1(s_{k_0}, \lambda_0), \dots, \gamma_j(s_{k_0}, \lambda_0), \dots) \in \bigoplus_{j=1}^{\infty} \mathbb{Z},$$

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where $\mathrm{i_B}(s_{k_0}, H') = \deg_{\mathrm{B}}(H', B_{\varepsilon}^{2N}(s_{k_0}), 0)$.

If $\det H''(s_{k_0}) \neq 0$, then $\mathrm{i_B}(s_{k_0}, H') = \mathrm{sign}\ \det H''(s_{k_0})$

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$$= \mathrm{i_B}(s_{k_0}, H') \cdot \left(\mathrm{m}^- \left(T_j((\lambda_0 + \mu) H''(s_{k_0})) \right) - \mathrm{m}^- \left(T_j((\lambda_0 - \mu) H''(s_{k_0})) \right) \right),$$

where $\mathrm{i_B}(s_{k_0}, H') = \deg_{\mathrm{B}}(H', B^{2N}_{\varepsilon}(s_{k_0}), 0)$.

If $\det H''(s_{k_0}) \neq 0$, then $\mathrm{i_B}(s_{k_0}, H') = \mathrm{sign} \ \det H''(s_{k_0})$

Define bifurcation index $\mathcal{BIF}(s_{k_0}, \lambda_0) \in \bigoplus_{j=1}^{\infty} \mathbb{Z}$ as follows

$$\mathcal{BIF}(s_{k_0}, \lambda_0) = (\gamma_1(s_{k_0}, \lambda_0), \dots, \gamma_j(s_{k_0}, \lambda_0), \dots) \in \bigoplus_{j=1}^{\infty} \mathbb{Z},$$

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Theorem (Global bifurcation theorem for autonomous Hamiltonian systems)

Fix $(s_{k_0}, \lambda_0) \in \mathcal{T}$ such that $\mathcal{BIF}(s_{k_0}, \lambda_0) \neq \Theta$. Then (s_{k_0}, λ_0) is a global bifurcation point of nonstationary 2π-periodic solutions of the system (HS). *Moreover,*

1. *either the continuum $\mathcal{C}(s_{k_0}, \lambda_0)$ is not compact in $C_{2\pi}([0, 2\pi], \Omega) \times (0, +\infty)$,*
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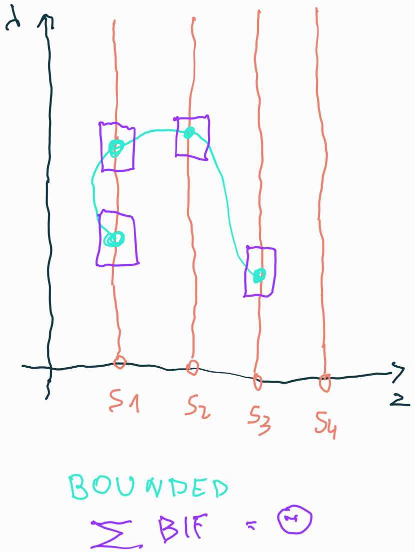
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S^1-orbits of solutions of the equation

$$\nabla_z \Phi(z, \lambda) = 0 \tag{E}$$

are in one-to-one correspondence with 2π-periodic solutions of the Hamiltonian system

$$\dot{z}(t) = \lambda J H'(z(t)). \tag{HS}$$

We study S^1-orbits of solutions of the equation (E) using the degree for S^1-equivariant gradient maps.

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Theorem

Classification of the critical points of the Hamiltonian H is as follows:

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If $\omega = 1$ and $ab \neq 0$, then

1. $H'^{-1}(0) = \{E_1\} = \{0, 0, 0, 0\}$,
2. $\det H''(E_1) = 0$,
3. $\mathrm{i_B}(E_1, H') = \begin{cases} -2, & \text{if } ab < 0 \\ 0, & \text{if } ab > 0 \end{cases}$,
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From now on we assume that $\omega = 1, ab < 0$.

Fix $\left(E_1, \frac{k}{2}\right) \in \{E_1\} \times \Lambda_1 \subset \mathcal{T}$.
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$$\mathcal{BIF}\left(E_1, \tfrac{k}{2}\right) = \left(\gamma_1\left(E_1, \tfrac{k}{2}\right), \ldots, \gamma_j\left(E_1, \tfrac{k}{2}\right), \ldots\right),$$

we obtain

$$\gamma_j\left(E_1, \tfrac{k}{2}\right) =$$

$$\mathrm{i_B}(E_1, H') \cdot \left(\mathrm{m}^-\left(T_j((\tfrac{k}{2} + \mu)H''(E_1))\right) - \mathrm{m}^-\left(T_j((\tfrac{k}{2} - \mu)H''(E_1))\right)\right) =$$

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Corollary

From the above formula it follows that for any choice $k_1, \ldots, k_r \in \mathbb{N}$ *the following formula holds true*

$$\mathcal{BIF}\left(E_1, \frac{k_1}{2}\right) + \ldots + \mathcal{BIF}\left(E_1, \frac{k_r}{2}\right) \neq \Theta$$

Theorem

Assume that $\omega = 1, ab < 0$. *Then for any* $k \in \mathbb{N}$ *the continuum* $\mathcal{C}\left(E_1, \frac{k}{2}\right)$ *is unbounded in* $C_{2\pi}([0, 2\pi], \mathbb{R}^4) \times (0, +\infty)$.

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Open question (Strong version)

Are there any continua of periodic solutions of the system (HS) emanating from the equilibrium $E_1 = (0,0,0,0)$, when $\omega = 1$ and $ab > 0$?

Open question (Weak version)

Is it true that $\left(E_1, \frac{k}{2}\right) \in \mathrm{cl}(\mathcal{N})$? i.e. is there a sequence $\{(u_r, \lambda_r)\} \subset \mathcal{N}$ converging to $\left(E_1, \frac{k}{2}\right)$ in L_∞-norm, when $\omega = 1$ and $ab > 0$?

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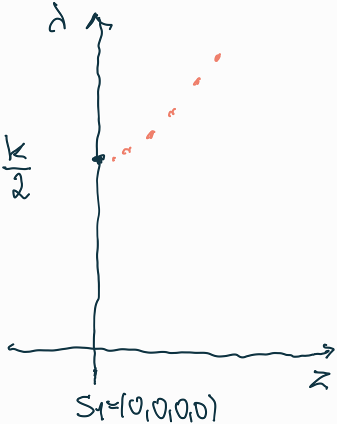

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A. Maciejewski and S. R., Global bifurcations of periodic solutions of the Restricted Three Body Problem, Celestial Mechanics and Dynamical Astronomy **88(3)** (2004), 293-324,

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3. The Planar Restricted Three Body Problem

In this section, we consider the planar restricted three body problem and, using Theorem 2.1, study the global behaviour of continua of non-stationary periodic solutions, which emanate from the stationary ones. Our aim is to classify the admissible continua of non-stationary periodic solutions of the restricted three body problem. The equations of motion of the restricted three body problem, in the rotating coordinates, can be written in the following form

$$\dot{x} = J\nabla H_\mu(x), \tag{3.1}$$

where $\mu \in (0, 1/2]$ and the Hamiltonian H_μ is given by the formula

$$H_\mu(x_1, x_2, x_3, x_4) = \frac{1}{2}(x_3^2 + x_4^2) + x_2x_3 - x_1x_4 - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2}} - \frac{1-\mu}{\sqrt{(x_1 + \mu)^2 + x_2^2}}. \tag{3.2}$$

Remark 3.1. System (3.1) has the following properties:

(1) $\mathcal{S}(H_\mu) = \{(1-\mu, 0), (-\mu, 0)\} \times \mathbb{R}^2$,
(2) $H_\mu \in C^\infty(\mathbb{R}^4 \setminus \mathcal{S}(H_\mu), \mathbb{R})$,
(3) $J\nabla H_\mu(\mathcal{R}x) = -\mathcal{R}J\nabla H_\mu(x)$, where

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

(4) $\nabla H_\mu^{-1}(0) = \{E_1^\mu, E_2^\mu, E_3^\mu, L_4^\mu, L_5^\mu\}$,

(5) $\det \nabla^2 H_\mu(E_l^\mu) < 0$, for $l = 1, 2, 3$, and $\det \nabla^2 H_\mu(L_l^\mu) > 0$, for $l = 4, 5$,
(6) $\mathcal{R}E_l^\mu = E_l^\mu$ for $l = 1, 2, 3$, $\mathcal{R}L_4^\mu = L_5^\mu$.

In the following lemma we describe the normal form of matrices of the linearization of system (3.1) at the Euler stationary solutions $E_1^\mu, E_2^\mu, E_3^\mu$.

If M is a symmetric matrix, then $\sigma(M)$ denotes the spectrum of M.

LEMMA 3.1. *Let us fix $\mu \in (0, 1/2]$ and $l \in \{1, 2, 3\}$. Then there is $S_l^\mu \in \mathrm{Sp}(\mathbb{R}^4)$ such that*

$$J S_l^{\mu T} \nabla^2 H_\mu(E_l^\mu) S_l^\mu = \begin{bmatrix} a_l^\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & b_l^\mu \\ 0 & 0 & -a_l^\mu & 0 \\ 0 & -b_l^\mu & 0 & 0 \end{bmatrix},$$

where $a_l^\mu, b_l^\mu > 0$. Moreover, for any $j \in \mathbb{N}$

$$\sigma(Q(j, \lambda \nabla^2 H_\mu(E_l^\mu))) = \{\pm\sqrt{j^2 + \lambda^2 (a_l^\mu)^2}, -\lambda b_l^\mu \pm j\},$$

where the multiplicity of any eigenvalue equals 2.

The normal form of matrices of the linearization of system (3.1) at the Lagrange stationary solutions L_4^μ, L_5^μ depends only on μ. We describe them in the next lemma. Let $\mu_R = (1/2)(1 - \sqrt{69}/9)$ be the Routh critical mass ratio.

LEMMA 3.2. *Let us fix $l \in \{4, 5\}$. Then*

(1) *if $\mu \in (0, \mu_R)$, then there is $S_l^\mu \in \mathrm{Sp}(\mathbb{R}^4)$ such that*

$$J S_l^{\mu T} \nabla^2 H_\mu(L_l^\mu) S_l^\mu = \begin{bmatrix} 0 & 0 & \omega_l^\mu & 0 \\ 0 & 0 & 0 & -\omega_2^\mu \\ -\omega_l^\mu & 0 & 0 & 0 \\ 0 & \omega_2^\mu & 0 & 0 \end{bmatrix},$$

where $\omega_l^\mu > \sqrt{2}/2 > \omega_2^\mu > 0$. Moreover, for any $j \in \mathbb{N}$

$$\sigma(Q(j, \lambda \nabla^2 H_\mu(L_l^\mu)) = \{-\lambda \omega_l^\mu \pm j, \omega_2^\mu \pm j\},$$

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(2) *if $\mu = \mu_R$, then there is $S_l \in \mathrm{Sp}(\mathbb{R}^4)$ such that*

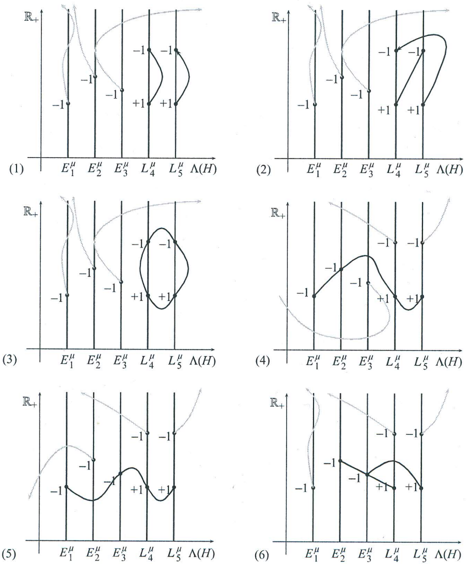

Figure 3. The bifurcations of periodic solutions in the planar restricted three body problem on 'level j'. The black lines denote the admissible continua. The subsequent figures correspond to six possibilities given in Theorem 3.1.