Stability of Solutions Differential Equations with Delays and Advances

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Differential equations

Tumor growth cancer model

Consider the system

$$\begin{cases} x' = 1 + a_1 x(1-x) - k_1 xy - k_2 x \\ y' = a_2 yz - a_3 y - k_3 xy \\ z' = a_4 z(1-z) - a_5 yz - a_6 z - k_4 xz \end{cases}$$

density of tumor cells

density of hunting predator cells

density of resulting cells

- a_1 is the growth rate of tumor cells,
- a_2 represents the conversion rate of the resulting cells to hunting predator cells,
- a_3 is the specific loss rates of hunting predator cells,
- a_4 represents the growth rate of resting cells,
- a_5 is the conversion rate of resting cells to hunting predator cells,
- a_6 is the specific loss rates of the resting cells,
- k_1 is the rate of killing of tumor cells by hunting cells,
- k_2 is the specific loss rates of tumor cells,
- k_3 represents the rate of killing of hunting predator cells by tumor cells,
- k_4 represents rate of killing of resting cells by tumor cells.

Tumor growth cancer model

The equilibrium points of the system

$$\begin{cases} x' = 1 + a_1 x(1-x) - k_1 xy - k_2 x \\ y' = a_2 yz - a_3 y - k_3 xy \\ z' = a_4 z(1-z) - a_5 yz - a_6 z - k_4 xz \end{cases}$$

are:

- $E_0(0,0,0)$
- $E_1(x_1,0,0)$ where $x_1 = \frac{1}{2}\left[\left(1-\frac{k_2}{a_1}\right) + \sqrt{\left(1-\frac{k_2}{a_1}\right)^2 + \frac{4}{a_1}}\right]$ $\quad (a_1 > k_2)$
- $E_2(x_2,0,z_2)$ where $x_2 = \frac{1}{2}\left[\left(1-\frac{k_2}{a_1}\right) + \sqrt{\left(1-\frac{k_2}{a_1}\right)^2 + \frac{4}{a_1}}\right]$ and $z_2 = 1 - \frac{a_6}{a_4} - \frac{k_4}{a_4}x_2$

  ($a_1 > k_2$ and $a_4 > a_6 + k_4 x_2$)
- $E_3(x_3,y_3,z_3)$ where $y_3 = \frac{1+a_1x_3(1-x_3)-k_2x_3}{k_1x_3}$ and $z_3 = \frac{a_3+k_3x_3}{a_2}$ $\quad (a_1 > k_2)$

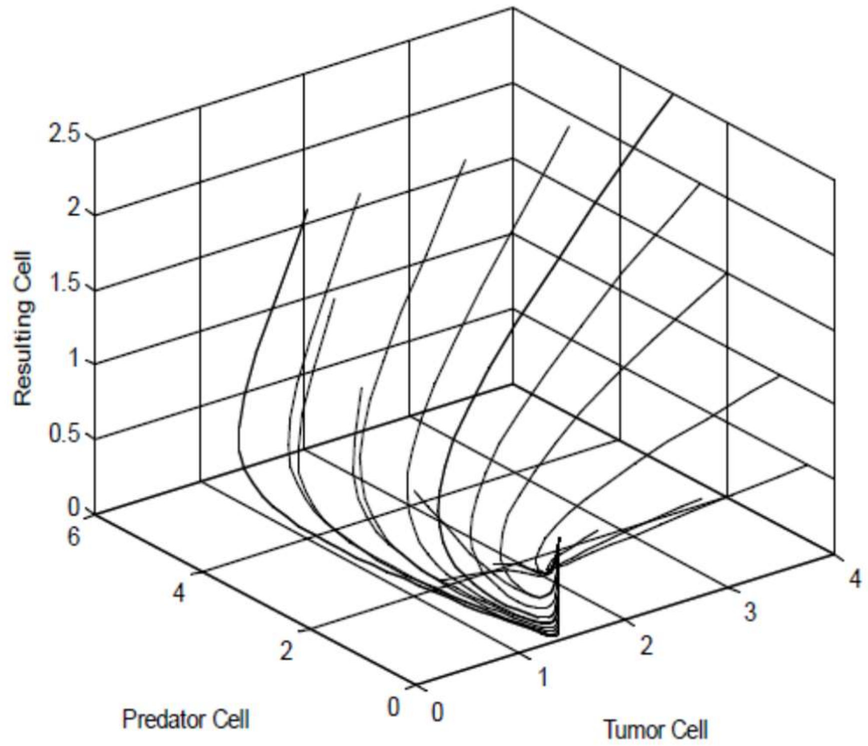
Tumor growth cancer model

The equilibrium point $E_3(x_3, y_3, z_3)$ is globaly asymptotically stable

Example

- $a_1 = 0{,}6 = a_4$
- $a_2 = 0{,}99$
- $a_3 = 0{,}1$
- $a_5 = 0{,}06$
- $a_6 = 0{,}118$
- $k_1 = 0{,}9$
- $k_2 = 0{,}5$
- $k_3 = 0{,}854$
- $k_4 = 0{,}02$

$E_3(1.3213, 0.5656, 0.1186)$



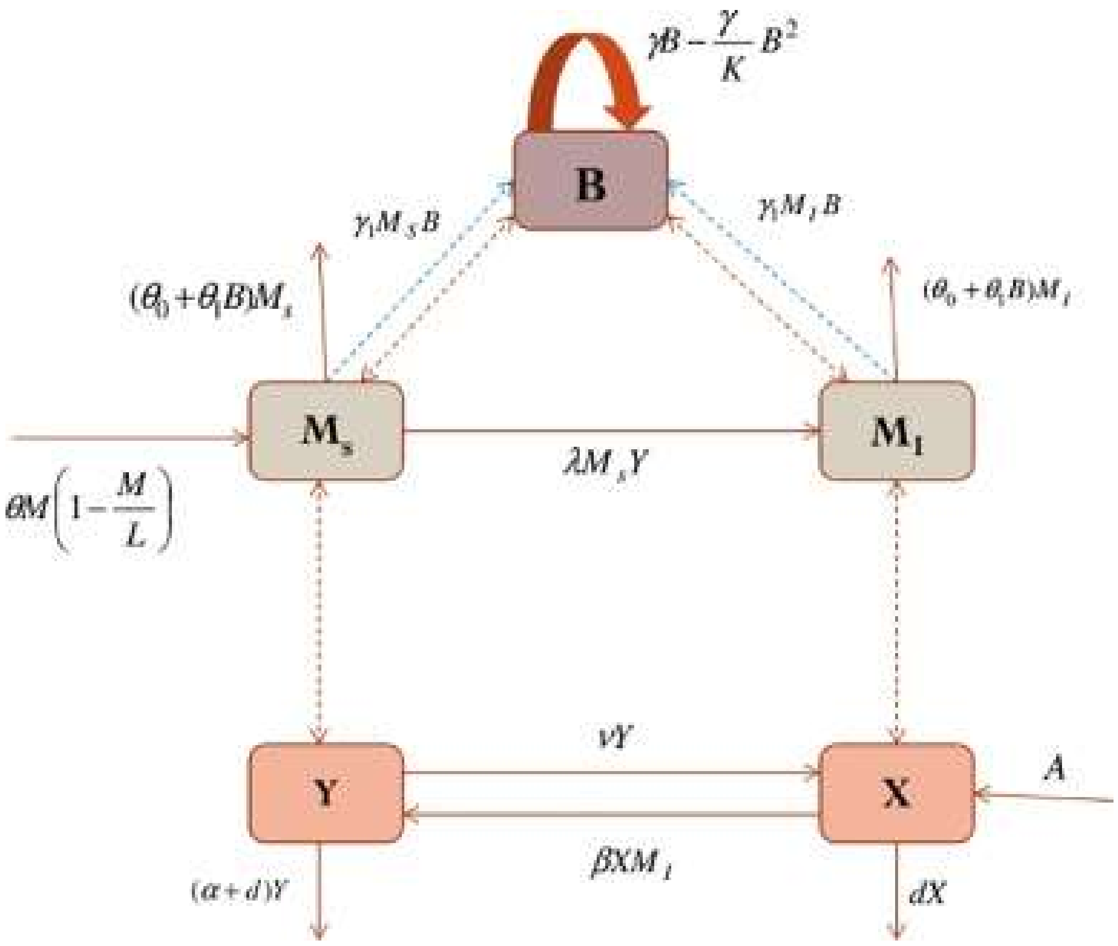
Biolarvicide vs Malaria Model

Consider the system

$$\begin{cases} x' = a - \beta x m_I - dx + vy \\ y' = \beta x m_I - (v - \alpha - d)y \\ m'_S = \theta M\left(1 - \dfrac{m_S + m_I}{L}\right) - (\theta_0 + \theta_1 B)m_S - \lambda m_s y \\ m'_I = \lambda m_I y - (\theta_0 + \theta_1 B)m_I \\ B' = \gamma B\left(1 - \dfrac{B}{k}\right) + \gamma_1(m_S + m_I)B \end{cases}$$

- x represent the susceptible humans
- y represent the infected humans
- m_S represent the susceptible mosquitoes
- m_I represent the infected mosquitoes
- B represent the biolarvicide population

Biolarvicide vs Malaria Model



The direction of each solid line represents movement of population along that line within the same species.

Example: $\beta x m_I$ is a removal from x population and an addition to y population.

The bi-directional dotted lines between boxes indicates a mass-action interaction.

The single directional dotted line indicates increase of bacteria population.

Biolarvicide vs Malaria Model

The equilibrium points of the system

$$\begin{cases} x' = a - \beta x m_I - dx + vy \\ y' = \beta x m_I - (v - \alpha - d)y \\ m'_S = \theta M\left(1 - \dfrac{m_S + m_I}{L}\right) - (\theta_0 + \theta_1 B)m_S - \lambda m_s y \\ m'_I = \lambda m_I y - (\theta_0 + \theta_1 B)m_I \\ B' = \gamma B\left(1 - \dfrac{B}{k}\right) + \gamma_1(m_S + m_I)B \end{cases}$$

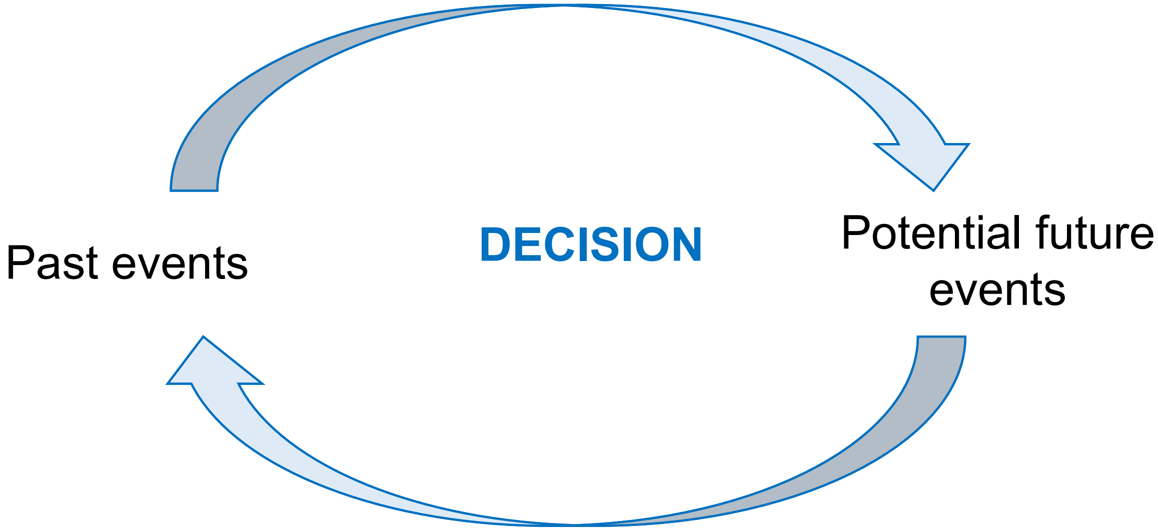
- $E_0\left(\frac{a}{d}, 0, 0, 0, 0\right)$ ⇐ Disease free, Unstable
- $E_1\left(\frac{a}{d}, 0, 0, \frac{L(\theta-\theta_0)}{\theta}, 0\right)$ ⇐ Disease free, Unstable
- $E_2\left(\frac{a}{d}, 0, 0, 0, K\right)$ ⇐ Disease free, Unstable if $\frac{(\theta-\theta_0)}{\theta_1} > k$
- $E_3\left(x^*, y^*, m_S^*, m_I^*, 0\right)$ ⇐ Endemic, Unstable
- $E_4\left(\frac{a}{d}, 0, m_S^*, 0, B^*\right)$ ⇐ Disease free, Stable under conditions
- $E_5\left(x^*, y^*, m_S^*, m_I^*, B^*\right)$ ⇐ Endemic, Stable under conditions

Difference and differential equations with delays and advances

Introduction

Mixed differential equations (or equations with mixed arguments) occur in many problems:

- Economy,
- Biology
- Physics
- Engineering
- …



However, this class of equations has been much less studied than other classes of functional differential equations.

Introduction

Why is this kind of equations a challenge?

It is well known that the solutions of these types of equations cannot be obtained in closed-form.

It is not quite clear how to formulate an initial value problem for such equations and the existence and uniqueness of solutions becomes a complicated issue.

To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equations on the half line.

Introduction

Example 1

Let the initial value problem with $t_0 = 0$

$$x'(t) - x(h(t)) = 0, \qquad t \geq 0,$$

$$x(0) = 0$$

$$h(t) = \begin{cases} 1 & 0 \leq t < 1 \\ -t^2 + 4t - 2 & 1 \leq t < 2 \\ t - |\sin(t-2)| & t \geq 2 \end{cases}$$

advanced on $[0, 2)$ and delayed on $(2, \infty)$

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ \dfrac{1}{2-t} & 1 \leq t < 2 \end{cases}$$

is a solution of this initial value problem on $[0, 2)$, which is unbounded on $[0, 2)$ and cannot be extended to $[2, \infty)$

Introduction

Example 2

Let $\alpha > 0$

$$x'(t) + 2\alpha x(t) - \alpha\, x(t+1) = 0, \qquad t \geq 0,$$

$$x(0) = 1$$

has both an infinitely growing and a decaying solutions $e^{\lambda t}$ on $[0, \infty)$, with λ positive and negative, respectively.

For $\alpha = 0.25$ $\Longrightarrow$ $x(t) = e^{-0.31812t}$

$\Longrightarrow$ $x(t) = e^{2.4773t}$

Remark: Note that for the delayed argument $h(t) \leq t$ and $0 \leq \alpha < 1$, any solution of the equation $x'(t) + 2\alpha x(t) - \alpha\, x\big(h(t)\big) = 0$ tends to zero as $t \to +\infty$.

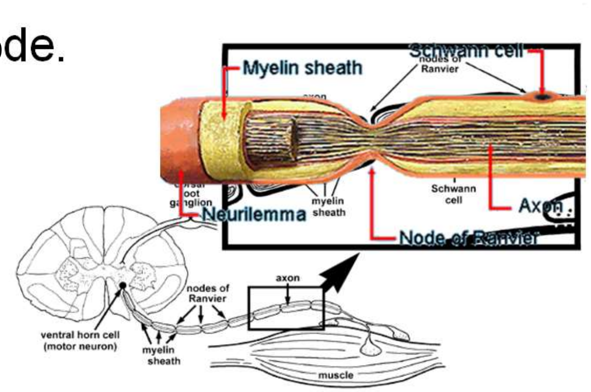
Introduction

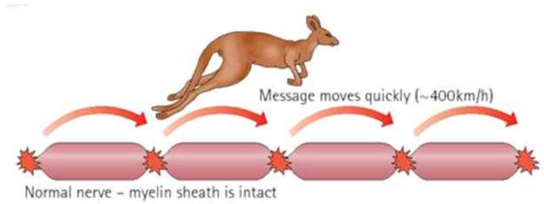
H. Chi, J. Bell, and B. Hassard, ***Numerical solution of a nonlinear advance-delay-differential Equation from nerve conduction theory***, J. Math. Biol. 24 (1986), 583-601.

The equation

$$RC\, v'(t) = F\big(v(t)\big) + v(t-\tau) + v(t+\tau)$$

where $t \in \mathbb{R}$, $v(-\infty) = 0$ and $v(+\infty) = 1$, represents a model conduction in a myelinated nerve axon in which the myelin completely insulates the membrane, so that the potential change jumps from node to node.





Introduction

In the equation

$$RCv'(t) = F(v(t)) + v(t-\tau) + v(t+\tau)$$

we have:

- $v(t)$ represents the transmembrane potential at a node;
- the internodal delay τ, represents the reciprocal of the speed of the potential wave as itpropagates down the axon. This constant r is unknown a priori and must be found simultaneously with $v(t)$.
- The constants R and C represent axoplasmic nodal resistivity and nodal capacity, respectively.
- F includes the model current-voltage relation.

Introduction

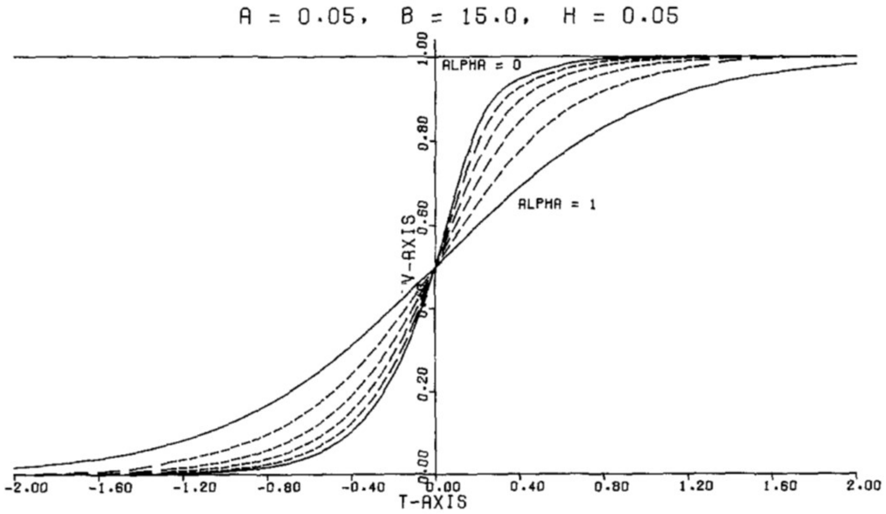
Using the Ohm law and the Taylor expansion around 0 the equation

$$RCv'(t) = F(v(t)) + v(t - \tau) + v(t + \tau)$$

will be transformed at

$$v'(t) = a_1 v(t) + a_2 v^2(t) + a_3 v^3(t) + v(t - \tau) - 2v(t) + v(t + \tau) + O(v^4)$$

Using numerical methods we obtain the solutions



This means the rise time of the membrane potential is faster for lower threshold potential.

Introduction

The linear autonomous mixed type differential equation

$$x'(t) = \sum_{i=1}^{p} a_j x(t - r_i) + \sum_{j=1}^{q} b_j x(t + \tau_j)$$

where a_i and b_j are nonzero real numbers and each r_i and τ_j are positive real numbers can arise in the study of traveling waves in regions with non-local interactions initiated in:

- J. Mallet-Paret, The Fredholm alternative for functional differential equations of mixed type, J. Dyn. Diff. Eq. 11 (1999) 1-47.
- J. Mallet-Paret, The global structure of traveling waves in spatially discrete dynamical systems, J. Dyn. Diff. Eq. 11 (1999) 49-127.

Introduction

H. d'Albis, E. Augeraud-Véron, H.J. Hupkes, **Stability and determinacy conditions for mixed -type functional differential equations**, J. Math. Econom. 53 (2014) 119-129

deals with the linear mixed-type functional differential equation

$$x'(t) = \int_{-a}^{b} x(t+\theta)\,d\mu(\theta)\,,$$

where $\mu(\theta)$ is real valued function of bounded variation on $[-a, b]$.

They also obtained the necessary conditions for the existence, uniqueness, and stability of a solution to mixed type functional equations.

Stability of solutions in differential equations with delays and advances

Introduction

Consider the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big) d\nu(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big) d\eta(\theta)$$

where:

- $x(t) \in \mathbb{R}$,
- $r_1(\theta)$ and $r_2(\theta)$ are real nonnegative continuous function on $[-1,0]$
- $\nu(\theta)$ and $\eta(\theta)$ are real valued function of bounded variation on $[-1,0]$

Introduction

We define

- $\|r_1\| = \max\{r_1(\theta) \colon \ -1 \le \theta \le 0\}$
- $\|r_2\| = \max\{r_2(\theta) \colon \ -1 \le \theta \le 0\}$

We specify an **initial condition** of the form

$$x(t) = \phi(t) \,, \qquad -\|r_1\| \le t \le \|r_2\|$$

where the initial function ϕ is a given continuous real-valued function on the interval $[-\|r_1\| \,, \|r_2\| \,]$

satisfying the "**consistency condition**"

$$\phi'(0) = \int_{-1}^{0} \phi\big(-r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} \phi\big(r_2(\theta)\big) d\eta(\theta) \ .$$

Introduction

By a **solution** of

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big)dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big)d\eta(\theta)$$

we mean a continuous function $x : [-\|r_1\|, +\infty) \to \mathbb{R}$, which is differentiable on $[0, +\infty)$ and satisfies, the equation for every $t \geq 0$.

If a solution of

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big)dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big)d\eta(\theta)$$

is searched in the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$, the **characteristic equation** will be

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)}dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)}d\eta(\theta) \ .$$

Introduction

The solution of

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big) d\eta(\theta)$$

is said to be **stable** if for every $\varepsilon > 0$, there exists a number $\ell = \ell(\varepsilon) > 0$ such that, for any initial

function ϕ with $\|\phi\| = \max_{-\|r_1\| \leq t \leq \|r_2\|} |\phi(t)| < \ell$ the solution satisfies

$$|x(t)| < \varepsilon\,, \quad \text{for all } \; t \in [-\|r_1\|\;, \infty).$$

Otherwise, the solution is said to be **unstable**.

The solution is called **asymptotically stable** if it is stable in the above sense and in addition there exists a number $\ell_0 > 0$ such that, for any initial function ϕ with $\|\phi\| < \ell_0$, the solution satisfies

$$\lim_{t \to \infty} x(t) = 0.$$

The Asymptotic Result

Theorem 1 Let λ_0 be a real root of the characteristic equation with the property

$$\mu(\lambda_0) = \int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)}\, dV(v)(\theta) + \int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)}\, dV(\eta)(\theta) < 1$$

and set

$$\beta(\lambda_0) = \int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)} dv(\theta) - \int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)} d\eta(\theta)$$

Then, for every $\phi \in C([-\|r_1\|, \|r_2\|], \mathbb{R})$, the solution of

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big)dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big)d\eta(\theta)$$

satisfies

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}.$$

$$L(\lambda_0;\phi) = \phi(0) + \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)}\left(\int_{-r_1(\theta)}^{0} e^{-\lambda_0 s}\phi(s)ds\right)dv(\theta) - \int_{-1}^{0} e^{\lambda_0 r_2(\theta)}\left(\int_{0}^{r_2(\theta)} e^{-\lambda_0 s}\phi(s)ds\right)d\eta(\theta)$$

The Asymptotic Result

Proof (a draft)

$$\mu(\lambda_0) = \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)}\, dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)}\, dV(\eta)(\theta) < 1 \Longrightarrow 1 + \beta(\lambda_0) > 0$$

Define $y(t) = e^{-\lambda_0 t} x(t)$, for $t \in [-\|r_1\|, \infty)$

$$\begin{cases} y'(t) + \lambda_0 y(t) = \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} y\big(t - r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} y\big(t + r_2(\theta)\big) d\eta(\theta) \\ y(t) = e^{-\lambda_0 t} \phi(t), \quad \text{for } t \in [-\|r_1\|, \|r_2\|] \end{cases}$$

$$y(t) - y(0) + \lambda_0 \int_0^t y(s) ds = \int_0^t \left(\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} y\big(s - r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} y\big(s + r_2(\theta)\big) d\eta(\theta) \right) ds$$

Using the fact that λ_0 is a real root of the characteristic equation

$$y(t) = L(\lambda_0; \phi) - \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t - r_1(\theta)}^{t} y(s) ds \right) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t}^{t + r_2(\theta)} y(s) ds \right) d\eta(\theta)$$

The Asymptotic Result

Proof (continuation)

Next, we set for $t \geq -\|r_1\|$

$$z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}$$

$$\begin{cases} z(t) = -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t-r_1(\theta)}^{t} z(s)ds \right) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t}^{t+r_2(\theta)} z(s)ds \right) d\eta(\theta) \\ z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \end{cases}$$

what we have to prove is

$$\lim_{t\to\infty} z(t) = 0$$

The Asymptotic Result

Proof (continuation)

$$M(\lambda_0;\phi) = \max_{t\in[-\|r_1\|,\|r_2\|]} \left| e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \right|$$

We claim that

$$|z(t)| < M(\lambda_0;\phi) + \epsilon \quad \text{for every} \quad t \geq -\|r_1\| .$$

Otherwise there exists a point $t_0 > 0$ such that

$$M(\lambda_0;\phi) + \epsilon = |z(t_0)|$$

$$= \left| -\int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t_0-r_1(\theta)}^{t_0} z(s)ds \right) dv(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t_0}^{t_0+r_2(\theta)} z(s) \right. \right.$$

$$\leq [M(\lambda_0;\phi)+\epsilon] \left(\int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} dV(\eta)(\theta) \right.$$

$$= [M(\lambda_0;\phi)+\epsilon]\mu(\lambda_0) < [M(\lambda_0;\phi)+\epsilon]$$

Then we have

$$|z(t)| \leq M(\lambda_0;\phi)$$

$$\|r_1\| \leq t \leq 0 .$$

Contradiction!

The Asymptotic Result

Proof (continuation)

$$|z(t)| \leq \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{t-r_1(\theta)}^{t} |z(s)| ds \right) dV(v)(\theta) + \int_{-1}^{0} e^{\lambda_0 r_2(\theta)} \left(\int_{t}^{t+r_2(\theta)} |z(s)| ds \right) dV(\eta)(\theta)$$

$$\leq M(\lambda_0; \phi) \left(\int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)} dV(\eta)(\theta) \right)$$

$$\leq \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for every } t \geq 0$$

Taking into consideration the definition of $\mu(\lambda_0)$, it can be shown, by an easy induction, that z meets

$$|z(t)| \leq [\mu(\lambda_0)]^n M(\lambda_0; \phi)$$

$$\lim_{n\to\infty} [\mu(\lambda_0)]^n = 0 \quad \Rightarrow \quad \lim_{t\to\infty} z(t) = 0$$

Estimation of Solutions and Stability Criteria

Theorem 2 Let λ_0 be a real root of the characteristic equation and

$$\mu(\lambda_0) = \int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)}\, dV(v)(\theta) + \int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)}\, dV(\eta)(\theta) < 1$$

$$\beta(\lambda_0) = \int_{-1}^{0} r_1(\theta)e^{-\lambda_0 r_1(\theta)} dv(\theta) - \int_{-1}^{0} r_2(\theta)e^{\lambda_0 r_2(\theta)} d\eta(\theta)$$

Then the solution x of

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big)dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big)d\eta(\theta)$$

satisfies

$$|x(t)| \leq \left[\frac{\big(1+\mu(\lambda_0)\big)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0)\right] N(\lambda_0;\phi)e^{\lambda_0 t}$$

$$N(\lambda_0;\phi) = \max_{-\|r_1\|\leq t\leq \|r_2\|} \left|e^{-\lambda_0 t}\phi(t)\right|$$

Moreover, the solution is*:*

- stable if $\lambda_0 = 0$,
- asymptotically stable if $\lambda_0 < 0$
- ustable if $\lambda_0 > 0$

Estimation of Solutions and Stability Criteria

Proof (a draft)

Let

$$y(t) = e^{-\lambda_0 t} x(t)$$

$$z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}$$

$$M(\lambda_0;\phi) = \max_{t\in[-\|r_1\|,\|r_2\|]} \left| e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \right|$$

$$\Rightarrow \quad |y(t)| \le \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} + \mu(\lambda_0) M(\lambda_0;\phi)$$

$$\mu(\lambda_0) < 1$$

$$N(\lambda_0;\phi) = \max_{-\|r_1\|\le t\le\|r_2\|} \left| e^{-\lambda_0 t}\phi(t) \right|$$

$$\Rightarrow \quad |L(\lambda_0;\phi)| \le |\phi(0)| + \int_{-1}^{0} e^{-\lambda_0 r_1(\theta)} \left(\int_{-r_1(\theta)}^{0} \left| e^{-\lambda_0 s}\phi(s) \right| ds \right) dV(v)(\theta)$$

$$\le \big(1+\mu(\lambda_0)\big) N(\lambda_0;\phi)$$

Estimation of Solutions and Stability Criteria

Proof (continuation)

On the other hand

$$M(\lambda_0;\phi) \le N(\lambda_0;\phi) + \frac{\big(1+\mu(\lambda_0)\big)N(\lambda_0;\phi)}{1+\beta(\lambda_0)} = \left(1+\frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)}\right)N(\lambda_0;\phi)$$

$$|y(t)| \le \left[\frac{\big(1+\mu(\lambda_0)\big)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0)\right]N(\lambda_0;\phi)$$

$$|x(t)| \le \left[\frac{\big(1+\mu(\lambda_0)\big)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0)\right]N(\lambda_0;\phi)e^{\lambda_0 t}$$

- stable if $\lambda_0 = 0$,
- asymptotically stable if $\lambda_0 < 0$
- ustable *if* $\lambda_0 > 0$

Some Important Lemmas

Lemma 1 Assume that

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r}\,, \qquad \int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}$$

and

$$\int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1$$

where $r = max\{\, \|r_1\|\,,\ \|r_2\|\,\}$.

Then, in the interval $\left(-\frac{1}{r}\,,\,\frac{1}{r}\right)$, the characteristic equation

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta)$$

has a unique root λ_0 , and this root satisfies the property

$$\mu(\lambda_0) = \int_{-1}^{0} r_1(\theta) e^{-\lambda_0 r_1(\theta)}\, dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\lambda_0 r_2(\theta)}\, dV(\eta)(\theta) < 1$$

Some Important Lemmas

Proof (a draft)

Define

$$F(\lambda) = \lambda - \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) - \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta) \qquad \text{for } \lambda \in \left[-\frac{1}{r}, \frac{1}{r}\right].$$

We have

$$F\left(-\frac{1}{r}\right) = -\frac{1}{r} - \int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) - \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) < 0$$

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r}$$

$$\int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}$$

$$F\left(\frac{1}{r}\right) = \frac{1}{r} - \int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) - \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) > 0$$

Some Important Lemmas

Proof (continuation)

$$F'(\lambda) = 1 + \int_{-1}^{0} r_1(\theta) e^{-\lambda r_1(\theta)} dv(\theta) - \int_{-1}^{0} r_2(\theta) e^{\lambda r_2(\theta)} d\eta(\theta)$$

$$\geq 1 - \left| \int_{-1}^{0} r_1(\theta) e^{-\lambda r_1(\theta)} dv(\theta) \right| - \left| \int_{-1}^{0} r_2(\theta) e^{\lambda r_2(\theta)} d\eta(\theta) \right|$$

$$\geq 1 - \int_{-1}^{0} r_1(\theta) e^{-\lambda r_1(\theta)} dV(v)(\theta) - \int_{-1}^{0} r_2(\theta) e^{\lambda r_2(\theta)} dV(\eta)(\theta)$$

$$> 1 - \int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) - \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \geq 0.$$

$$\int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1$$

Some Important Lemmas

Corollary Assume that

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r}, \qquad \int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}$$

and

$$\int_{-1}^{0} r_1(\theta) e^{\frac{r_1(\theta)}{r}} dV(v)(\theta) + \int_{-1}^{0} r_2(\theta) e^{\frac{r_2(\theta)}{r}} dV(\eta)(\theta) \leq 1$$

where $r = max\{ \|r_1\| , \|r_2\| \}$.

Then the solution of

$$x'(t) = \int_{-1}^{0} x\big(t - r_1(\theta)\big) dv(\theta) + \int_{-1}^{0} x\big(t + r_2(\theta)\big) d\eta(\theta)$$

is:

- asymptotically stable if $v(-1) + \eta(-1) > v(0) + \eta(0)$
- unstable if $v(-1) + \eta(-1) < v(0) + \eta(0)$.

Some Important Lemmas

Lemma 2 Suppose that v and η are decreasing on $[-1,0$. *Assume that*

$$\int_{-1}^{0} e^{\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{-\frac{r_2(\theta)}{r}} d\eta(\theta) > -\frac{1}{r}$$

Then,

i. in the interval $[0,+\infty)$, the characteristic equation

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta)$$

has no roots;

ii. in the interval $\left(-\frac{1}{r}, 0\right)$, the characteristic equation has a unique root;

iii. $\lambda = -\frac{1}{r}$ is not a root of the characteristic equation;

iv. in the interval $\left(-\infty, -\frac{1}{r}\right)$, the characteristic equation has a unique root.

Some Important Lemmas

Lemma 3 Suppose that v and η are increasing on $[-1,0]$. Assume that

$$\int_{-1}^{0} e^{-\frac{r_1(\theta)}{r}} dv(\theta) + \int_{-1}^{0} e^{\frac{r_2(\theta)}{r}} d\eta(\theta) < \frac{1}{r}$$

Then,

i. In the interval $(-\infty, 0]$, the characteristic equation

$$\lambda = \int_{-1}^{0} e^{-\lambda r_1(\theta)} dv(\theta) + \int_{-1}^{0} e^{\lambda r_2(\theta)} d\eta(\theta)$$

has no roots;

ii. in the interval $\left(0\,,\,\frac{1}{r}\right)$, the characteristic equation has a unique root;

iii. $\lambda = \frac{1}{r}$ is not a root of the characteristic equation;

iv. in the interval $\left(\frac{1}{r}\,,\infty\right)$, the characteristic equation has a unique root.

Examples

Example 1

Consider the equation

$$x'(t) = \int_{-1}^{0} x\big(t-(-\theta)\big)d\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} x\big(t+(\theta+1)\big)d\eta\left(\frac{\theta}{4}+1\right)$$

$r_1(\theta) = -\theta$

$v(\theta) = -\frac{\theta}{4}$

$r_2(\theta) = \theta + 1$

$\eta(\theta) = \frac{\theta}{4} + 1$

Here the characteristic equations is

$$\lambda = \int_{-1}^{0} e^{-\lambda(-\theta)} d\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} e^{\lambda(\theta+1)} d\left(\frac{\theta}{4}+1\right)$$

$$= -\frac{1}{4}\int_{-1}^{0} e^{\lambda\theta} d\theta + \frac{1}{4}\int_{-1}^{0} e^{\lambda(\theta+1)} d\theta = \frac{1}{4\lambda}\left(e^{\lambda} + e^{-\lambda} - 2\right)$$

Examples

Example 1 (continuation)

So, $F_1(\lambda) = \lambda - \frac{1}{4\lambda}\left(e^{\lambda} + e^{-\lambda} - 2\right)$.

We have 3 roots: $\lambda = 0$ and $\lambda \cong \mp 4.36$

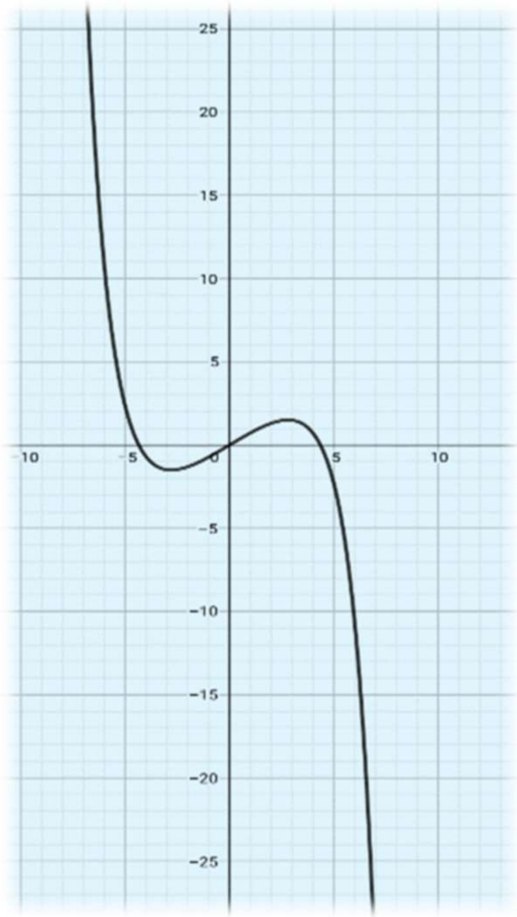
- $\lambda_0 = 0$.

  Since v is decreasing on $[-1,0]$ and η is increasing on $[-1,0]$, we get

$$\mu(\lambda_0) = \mu(0) = \int_{-1}^{0} (-\theta) dV\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} (\theta + 1) dV\left(\frac{\theta}{4} + 1\right)$$
$$\leq \left[\max_{-1\leq\theta\leq 0} (-\theta)\right] V\left(-\frac{\theta}{4}\right)(-1,0) + \left[\max_{-1\leq\theta\leq 0} (\theta + 1)\right] V\left(\frac{\theta}{4} + 1\right)(-1,0)$$
$$= 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = \frac{1}{2} < 1 .$$

  Therefore, for $\lambda_0 = 0$ the condition of the Theorem 2 is provided.

  So, the solution is stable.



Examples

Example 2

Consider the equation

$$x'(t) = \int_{-1}^{0} x\left(t - \left(\frac{\theta+1}{2}\right)\right) d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} x\left(t + \left(-\frac{\theta}{4}\right)\right) d\eta\left(-\frac{3}{2}\theta\right)$$

$r_1(\theta) = \frac{\theta+1}{2}$ $\quad v(\theta) = \frac{(\theta+1)^2}{4}$ $\quad r_2(\theta) = -\frac{\theta}{4}$ $\quad \eta(\theta) = -\frac{3}{2}\theta$

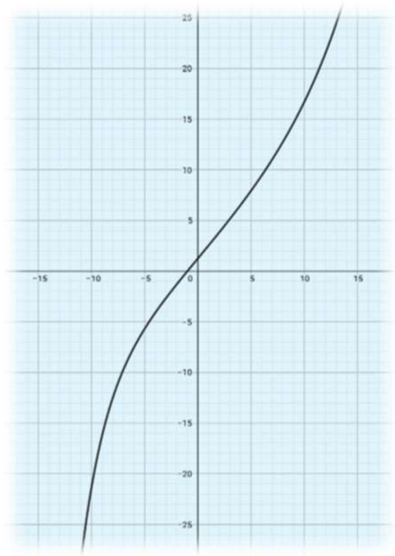
Here the characteristic equations is

$$\lambda = \int_{-1}^{0} e^{-\lambda\left(\frac{\theta+1}{2}\right)} d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{\lambda\left(-\frac{\theta}{4}\right)} d\left(-\frac{3}{2}\theta\right)$$

$$= \frac{1}{2}\int_{-1}^{0} \left[e^{-\lambda\left(\frac{\theta+1}{2}\right)}(\theta+1) - 3e^{-\frac{\lambda\theta}{4}}\right] d\theta = \frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1 - e^{-\frac{\lambda}{2}}\right) - e^{-\frac{\lambda}{2}} - 6\left(e^{\frac{\lambda}{4}} - 1\right)\right]$$

Examples

Example 2 (continuation)



So, $F_2(\lambda) = \lambda - \frac{1}{\lambda}\left[\frac{2}{\lambda}\left(1 - e^{-\frac{\lambda}{2}}\right) - e^{-\frac{\lambda}{2}} - 6\left(e^{\frac{\lambda}{4}} - 1\right)\right]$

The only one root of F_2 is $\lambda \cong -0.98$

Then, for $\lambda_0 = -0.98$ the condition of Theorem 2 is satisfied.

In fact, since v is increasing on $[-1,0]$ and η is decreasing on $[-1,0]$,

$$\mu(\lambda_0) = \mu(-0.98)$$

$$\leq \left[\max_{-1\leq\theta\leq0}\left(e^{0.98\left(\frac{\theta+1}{2}\right)}\left(\frac{\theta+1}{2}\right)\right)\right]V\left(\frac{(\theta+1)^2}{4}\right)(-1,0) + \left[\max_{-1\leq\theta\leq0}\left(e^{0.98\left(\frac{\theta}{4}\right)}\left(-\frac{\theta}{4}\right)\right)\right]V\left(-\frac{3}{2}\theta\right)(-1,0)$$

$$= \frac{e^{\frac{0.98}{2}}}{2}\cdot\frac{1}{4} + \frac{e^{-\frac{0.98}{4}}}{4}\cdot\frac{3}{2} \cong 0.5 < 1.$$

So, the solution is asymptotically stable.

Examples

Example 2 (continuation)

In this example, stability analysis can be performed using Corollary of Lemma 1 without using the characteristic equation. Indeed, we get

$$\int_{-1}^{0} e^{(\theta+1)} d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{\frac{\theta}{2}} d\left(-\frac{3}{2}\theta\right) = \frac{1}{2}\int_{-1}^{0}\left[(\theta+1)e^{\theta+1} - 3e^{\frac{\theta}{2}}\right] d\theta \cong -0.68 > -2\,,$$

$$\int_{-1}^{0} e^{-(\theta+1)} d\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} e^{-\frac{\theta}{2}} d\left(-\frac{3}{2}\theta\right) = \frac{1}{2}\int_{-1}^{0}\left[(\theta+1)e^{-(\theta+1)} - 3e^{-\frac{\theta}{2}}\right] d\theta \cong -1.68 < 2$$

$$\int_{-1}^{0} \left(\frac{\theta+1}{2}\right) e^{(\theta+1)} dV\left(\frac{(\theta+1)^2}{4}\right) + \int_{-1}^{0} \left(-\frac{\theta}{4}\right) e^{-\frac{\theta}{2}} dV\left(-\frac{3}{2}\theta\right) \le \frac{e}{2}\cdot\frac{1}{4} + \frac{\sqrt{e}}{4}\cdot\frac{3}{2} \cong 0.96 \le 1\,.$$

Thus, according to the Lemma 1, it states that a real root must pass in the interval $(-2\,,2)$.

Finally, from the Corollary we obtain

$$v(-1) + \eta(-1) = 0 + \frac{3}{2} > v(0) + \eta(0) = \frac{1}{4} + 0$$

and thus the solution is asymptotically stable.

Examples

Example 3

Consider the equation

$$x'(t) = \int_{-1}^{0} x\left(t - \left(-\frac{\theta}{2}\right)\right) d\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} x\left(t + \left(-\frac{\theta}{2}\right)\right) d\eta\left(-\frac{\theta}{4}\right)$$

$r_1(\theta) = -\frac{\theta}{2}$ $v(\theta) = -\frac{\theta}{4}$ $r_2(\theta) = -\frac{\theta}{2}$ $\eta(\theta) = -\frac{\theta}{4}$

Here the characteristic equations is

$$\lambda = \int_{-1}^{0} e^{-\lambda\left(-\frac{\theta}{2}\right)} d\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} e^{\lambda\left(-\frac{\theta}{2}\right)} d\left(-\frac{\theta}{4}\right)$$

$$= -\frac{1}{4}\int_{-1}^{0}\left(e^{\frac{\lambda\theta}{2}} + e^{-\frac{\lambda\theta}{2}}\right) d\theta$$

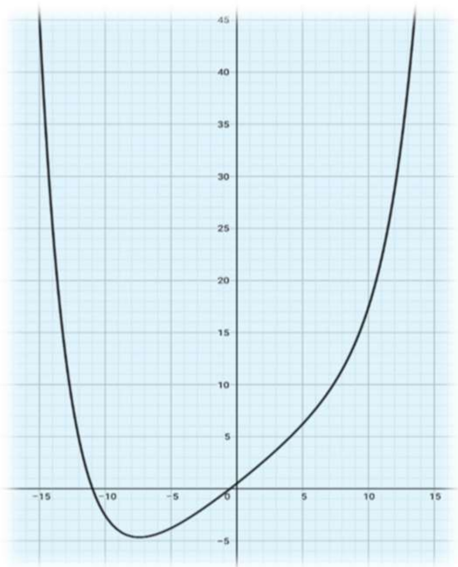
$$= -\frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right)$$

Examples

Example 3 (continuation)

So, $F_3(\lambda) = \lambda + \frac{1}{2\lambda}\left(e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}}\right)$.

The graph of the function F_3 shows that F_3 has two roots: $\lambda \cong -0.5$ and $\lambda \cong -11$.



Let $\lambda = -11$,

$$|\beta(-11)| = \left| \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{-\frac{11\theta}{2}} d\left(-\frac{\theta}{4}\right) - \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{\frac{11\theta}{2}} d\left(-\frac{\theta}{4}\right) \right| \cong 2.46 \leq \mu(-11)$$

So, Theorem 2, for $\lambda_0 = -11$ cannot be applied.

Let $\lambda = -0.5$,

$$\mu(\lambda_0) = \mu\left(-\frac{1}{2}\right) = \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{-\frac{\theta}{4}} dV\left(-\frac{\theta}{4}\right) + \int_{-1}^{0} \left(-\frac{\theta}{2}\right) e^{\frac{\theta}{4}} dV\left(-\frac{\theta}{4}\right)$$

$$\leq \frac{e^{\frac{1}{4}}}{2} \cdot \frac{1}{4} + \frac{e^{-\frac{1}{4}}}{2} \cdot \frac{1}{4} \cong 0.26 < 1$$

Then, for $\lambda_0 = -0.5$ the conditions of Theorem 2 are satisfied. So, the solution is asymptotically stable.

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Thank you very
much for your
attention!...