Asymptotic decomposition of stochastic semigroups and its applications

Ryszard Rudnicki
Instytute of Mathematics
Polish Academy of Sciences

International Meetings on Differential Equations and Their Applications
Łódź 3.11.21

Outline:

1. Stochastic semigroups.
2. Asymptotic decomposition and corollaries.
3. Piecewise deterministic Markov processes.
4. Applications to a gene expression model, to immunology, and to a kinetic equation.

References:

1. K. Pichór, R.R., JMAA 2016.
2. K. Pichór, R.R., Stochastic and Dynamics 2017.
3. A. Tomski, R.R., J. Theor. Biol. 2015.
4. K. Pichór, R.R., Math Meth Appl Sci. 2020.
5. B. Lods, M. Mokhtar-Kharroubi, R.R., Ann. I.H.Poincaré AN 2020.
6. A. Bobrowski, R.R., Phil. Trans. R. Soc. A 2020.
7. R.R., M. Tyran Kamińska, *Piecewise deterministic processes in biological models*, Springer 2017.

Stochastic semigroups

(X, Σ, m) — σ-finite measure space.
$D = \{f \in L^1 : \ f \geq 0, \ \|f\| = 1\}$ – densities.
Stochastic operator (Markov operator):
$P : L^1 \to L^1$ linear, $P(D) \subset D$.

Stochastic semigroup : $\{P(t)\}_{t \geq 0}$,
$P(t)$ - stochastic operators,
$P(0) = Id$, $P(t + s) = P(t)P(s)$, $s, t \geq 0$,
(c) for each $f \in L^1$, the function $t \mapsto P(t)f$ is continuous.

Standard example - Fokker-Planck equation:

$(X_t)_{t\geq 0}$ diffusion process
f density of the distribution of X_0
$x \mapsto u(t,x)$ density of the distribution of X_t.

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{n} \frac{\partial (b^i(x)u)}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 (a^{ij}(x)u)}{\partial x^i\, \partial x^j}.$$

$P(t)f(x) = u(t,x)$ a stochastic semigroup on $L^1(\mathbb{R}^n)$.

Asymptotic stability

$f_* \in D$ – *invariant* if $P(t)f_* = f_*$ for $t \geq 0$.
$\{P(t)\}$ – *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for} \quad f \in D.$$

Sweeping (zero-type property)

$\{P(t)\}$ – *sweeping* with respect to a family of sets $\mathcal{F}$ if for $B \in \mathcal{F}$ and for $f \in D$

$$\lim_{t\to\infty} \int_B P(t)f(x)\, m(dx) = 0.$$

$\{P(t)\}$ – partially integral if there exist $t > 0$, $k(t,x,y) \geq 0$

$$\int_X \int_X k(t,x,y)\, m(dx)m(dy) > 0$$

$$P(t)f(y) \geq \int k(t,x,y)f(x)\, m(dx) \quad \text{for} \quad f \in D.$$

Theorem 1 *If a partially integral stochastic semigroup $\{P(t)\}_{t\geq 0}$ has a unique invariant density f_* and $f_* > 0$ then it is asymptotically stable.*

X separable metric space, $\Sigma = \mathcal{B}(X)$,
$\{P(t)\}_{t\geq 0}$ stochastic semigroup with the kernel part $k(t,x,y)$,
(K) for every $x_0 \in X$ there exist $r > 0$, $t > 0$, and a function $\eta \geq 0$ s.t. $\int \eta \, dm > 0$ and

$$k(t,x,y) \geq \eta(y) \mathbf{1}_{B(x_0,r)}(x).$$

Theorem 2 *If (K) holds then:*
there are a countable (possible empty) set I,
continuous positive functionals α_i, $i \in I$,
and invariant densities f_i^*, $i \in I$,
with pairwise disjoint supports A_i,
such that for every density f and every compact set F we have

$$\lim_{t\to\infty} \|\mathbf{1}_{A_i} P(t)f - \alpha_i(f) f_i^*\| = 0,$$

$$\lim_{t\to\infty} \int_{F\cap Y} P(t)f(x)\, m(dx) = 0,\ Y = X \setminus \bigcup_{i\in I} A_i.$$

Remark. Theorem 2 has a version for stochastic operators, but we replace the condition

$$\lim_{t\to\infty} \|\mathbf{1}_{A_i} P(t) f - \alpha_i(f) f_i^*\| = 0,$$

by asymptotic periodicity.

Theorem 2 *If (K) holds then:*
there are a countable (possible empty) set I,
continuous positive functionals α_i, $i \in I$,
and invariant densities f_i^*, $i \in I$,
with pairwise disjoint supports A_i,
such that for every density f and every compact set F we have

$$\lim_{t\to\infty} \|\mathbf{1}_{A_i} P(t)f - \alpha_i(f) f_i^*\| = 0,$$

$$\lim_{t\to\infty} \int_{F\cap Y} P(t)f(x)\, m(dx) = 0, \ Y = X \setminus \bigcup_{i\in I} A_i.$$

Corollary 1 *Assume (K) and that* $\{P(t)\}_{t\geq 0}$ *has no invariant density. Then* $\{P(t)\}_{t\geq 0}$ *is sweeping with respect to compact sets.*

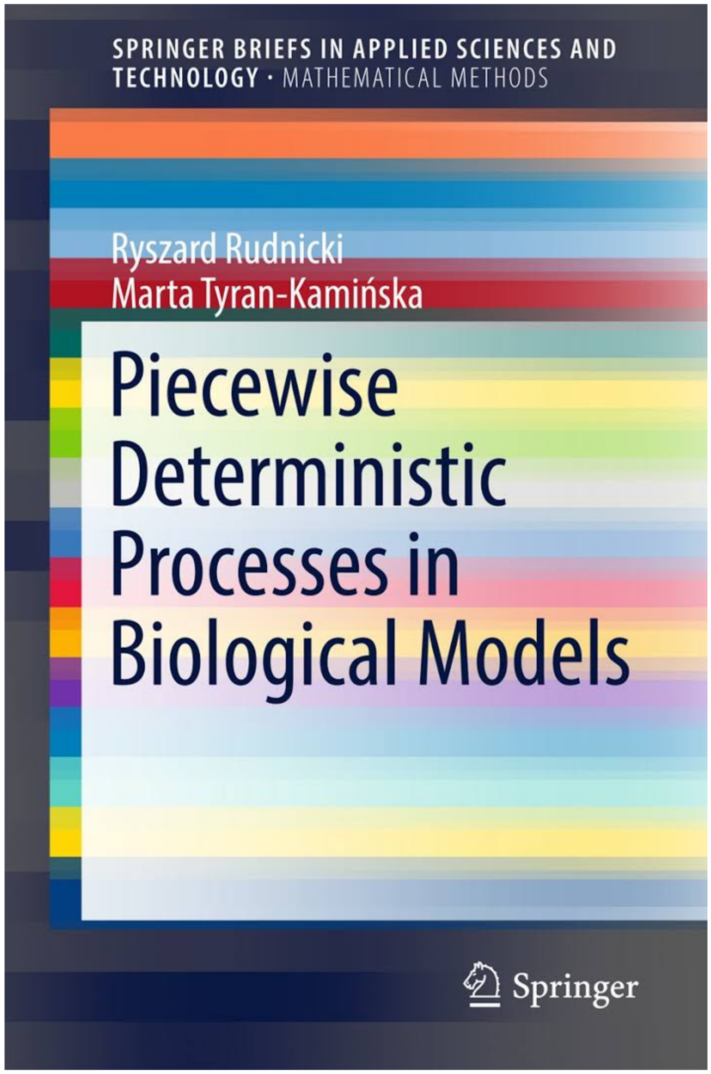
Corollary 2 *Assume (K), and* $\int_0^\infty P(t)f\,dt > 0$ *a.e. for* $f \in D$. *Then* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable or sweeping from compact sets.*

Corollary 3 *Let* X *be a compact space. Assume (K), and that* $\int_0^\infty P(t)f\,dt > 0$ *a.e. for* $f \in D$. *Then* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable.*

Corollary 4 *If (K) holds and there exists a point* x_0 *such that for each* $\varepsilon > 0$ *and each density* f *we have*

$$\int_{B(x_0,\varepsilon)} P(t)f\,dt > 0 \quad \textit{for some } t \geq 0. \qquad (1)$$

Then there is at most one invariant density for this semigroup.
In particular, if X *is compact then the stochastic semigroup is asymptotically stable.*

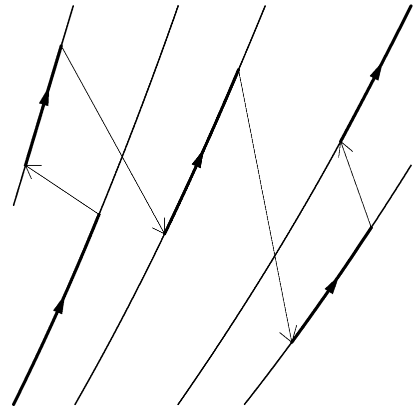

SPRINGER BRIEFS IN APPLIED SCIENCES AND TECHNOLOGY · MATHEMATICAL METHODS
Ryszard Rudnicki
Marta Tyran-Kamińska
Piecewise Deterministic Processes in Biological Models
Springer

Piecewise deterministic Markov processes

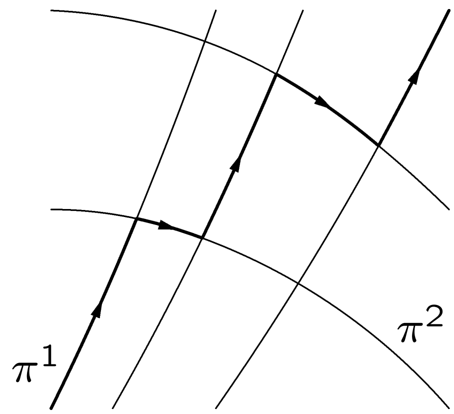
Davis (1984): "PDMPs is a general family of stochastic models covering virtually all non-difussion applications."

A continuous time (homogeneous) Markov process $X(t)$ is a PDMP if there is an increasing sequence of random times (t_n), called jumps, such that sample paths of $X(t)$ are defined in a deterministic way in each interval (t_n, t_{n+1}).

Two types of jumps: the process can jump to a new point or can change the dynamics which defines its trajectories.



Dynamical system with random jumps



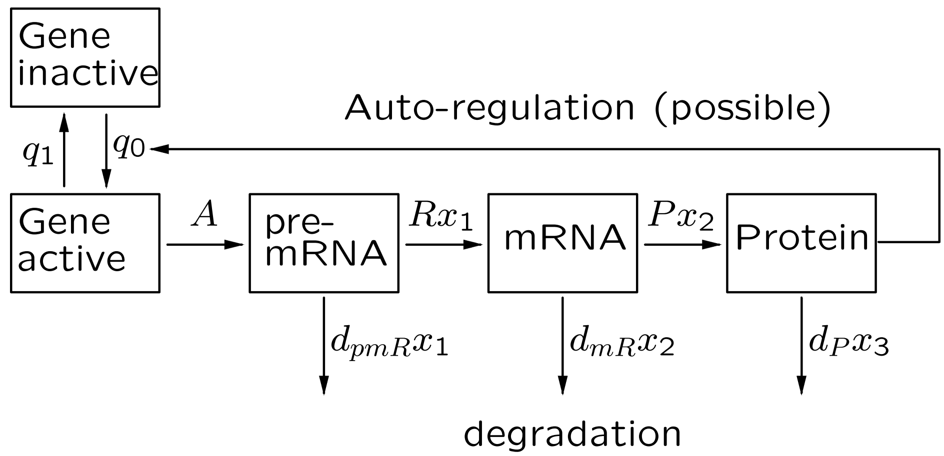
Process with switching dynamics


APPLICATION

Examples:

1. Pure jump processes: Markov chains, kangaroo movement.
2. Velocity jump processes: dispersal of cells and insects, stochastic billiards.
3. Semiflows with jumps: cell cycle models, immune systems.
4. Processes with switching dynamics: stochastic gene expression.
5. Mixture of 3 and 4: neural activity model, production of subtilin.
6. Individual-based models (agent-based m.): structured population models, coagulation-fragmentation process.

Gene expression model



x_1, x_2, x_3 — the number of pre-mRNA, mRNA, protein molecules,
d_{pmR}, d_{mR}, d_P — degradation coefficients,
A, Rx_1, Px_2 - velocities of transcription; conversion of pre-mRNA to mRNA; translation.

$$\begin{cases} x_1' = A\gamma(t) - (d_{pmR} + R)x_1 \\ x_2' = Rx_1 - d_{mR}x_2 \\ x_3' = Px_2 - d_P x_3, \end{cases} \qquad (2)$$

where $\gamma(t) = 1$ if a gene is active or
$\gamma(t) = 0$ if it is inactive.
We assume that the gene is activated with rate $q_0(\mathbf{x})$ and inactivated with rate $q_1(\mathbf{x})$.

$$x_1' = \gamma(t) - x_1; \; x_2' = \alpha(x_1 - x_2); \; x_3' = \beta(x_2 - x_3)$$

Processes with switching dynamics (PSD)

Markov process

$(x_1(t), x_2(t), x_3(t))$ is not a Markov process.

$$\xi_t = (x_1(t), x_2(t), x_3(t), \gamma(t)),\ t \geq 0.$$

The state-space

$$\mathsf{E} = \mathbb{R}^3_+ \times \{0, 1\}.$$

Partial density functions $f_i(x_1, x_2, x_3, t)$:

$$\Pr(\xi_t \in B \times \{i\}) = \iint_B f_i(x_1, x_2, x_3, t)\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3,$$

where B is a Borel subset of $\mathbb{R}^3_+$, $i = 0, 1$.

f – density of $\xi(0)$, $P(t)f$ – density of ξ_t.

$$x'(t) = b_i(x(t)), \tag{3}$$

$i = 1, \dots, N$ and at point $x \in G \subset \mathbb{R}^d$ it can jump from j to i state with intensity $q_{ij}(x)$. $\xi_t = (x(t), i(t))$, $t \geq 0$, is a PDMP.

$$\mathrm{prob}(\xi_t \in E \times \{i\}) = \int_E u(x,i,t)\,dx.$$

$$A_i f = -\sum_{k=1}^{d} \frac{\partial(b_i^k(x,i)f)}{\partial x_k}.$$

$$\frac{\partial u}{\partial t} = Mu + Au$$

where $Au = (A_1u_1, \dots, A_Nu_N)$, $u_i(x,t) = u(x,i,t)$ $M = [m_{ij}(x)]$, $m_{ij}(x) = q_{ij}(x)$ for $i \neq j$ and $m_{ii}(x) = -\sum_{k\neq i} q_{ki}(x)$.

Applications to PSD – how to check (K)?

The system is governed by k flows π_t^i and each flow π_t^i is the solution of a differential equation $x' = b^i(x)$ on $G \subset \mathbb{R}^d$. All transition intensities $q_{ij}(x)$ are continuous and positive in a neighbourhood of y_0.

(**Hörmander condition**) If vectors

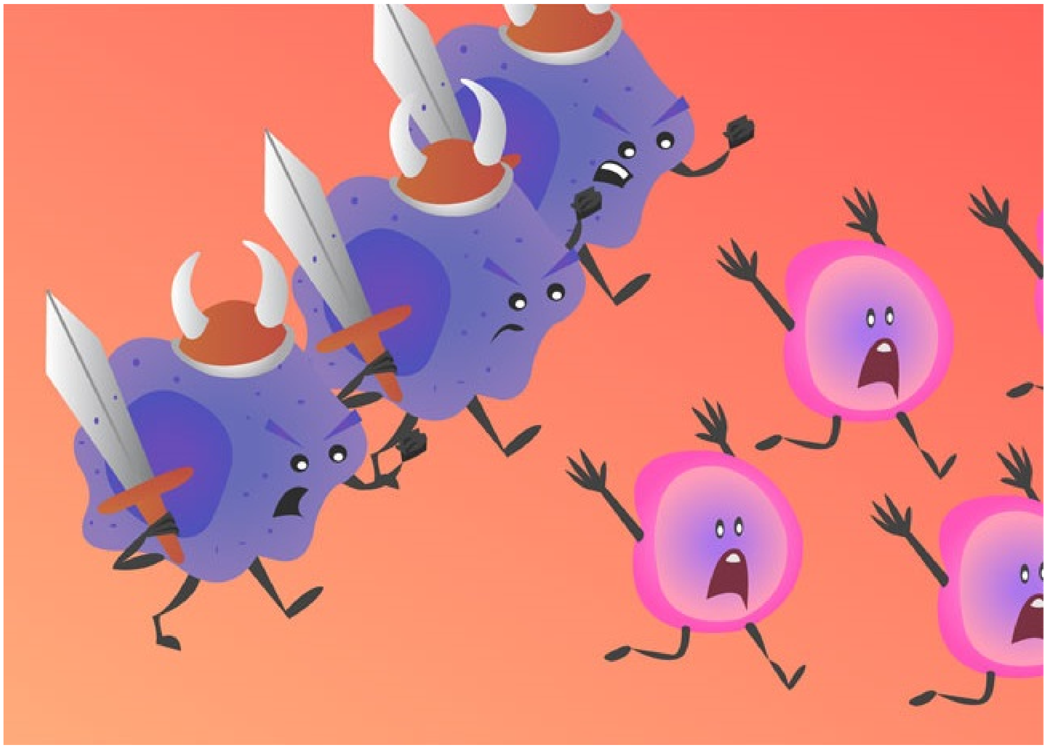
$$b^2(y_0) - b^1(y_0), \ldots, b^k(y_0) - b^1(y_0),$$
$$[b^i, b^j](y_0)_{1\le i,j\le k},\ [b^i, [b^j, b^l]](y_0)_{1\le i,j,l\le k}, \cdots$$

span the space $\mathbb{R}^d$ then (K) holds for any point x which is connected with y_0.

Assume that G is a bounded set and there exist $x_0 \in G$ and $i_0 \in I$ such that starting from any state $(x, i) \in X$ we are able to go arbitrarily close to (x_0, i_0) by a cumulative flow and that the Hörmander's condition holds. Then the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable.

Corollary 5 *The semigroup generated by the gene expression model is asymptotically stable.*

Dynamics of antibody levels



The immune status x is the concentration of specific antibodies, which appear after infection with a pathogen and remain in serum, providing protection against future attacks of that same pathogen.

$x(t)$ is a stochastic process whose trajectories are decreasing functions $x(t)$ between subsequent infections:

$$x'(t) = g(x(t)), \quad g < 0, \tag{4}$$

If x is the concentration of antibodies at the moment of infection, then $Q(x) > x$ is the concentration of antibodies just after clearance of infection.
The moments of infections are independent of the state of the immune system and they are distributed according to a Poisson process $(N_t)_{t \geq 0}$ with rate $\Lambda > 0$.

$$\xi_{t_n} = Q(\xi_{t_n^-}), \quad \xi_t' = g(\xi_t) \text{ for } t \in [t_{n-1}, t_n),$$

The process $(\xi_t)_{t \geq 0}$ satisfies the stochastic differential equation

$$d\xi_t = g(\xi_t)\, dt + (Q(\xi_t) - \xi_t)\, dN_t.$$

If f is the density distribution of x before an infection then $P_Q f$ is the density distribution of x just after clearance of infection:

$$P_Q f(x) = \sum_{i \in I_x} f(\varphi_i(x))|\varphi_i'(x)|, \qquad (5)$$

where φ_i are the right-inverse functions of $Q\Big|_{(a_i,b_i)}$.

$$P_Q^* f(x) = f(Q(x)).$$

The density of ξ_t is given by $u(t)(x)$ and

$$u'(t) = \mathcal{A}u(t),$$

$$\mathcal{A}f(x) = -\frac{d}{dx}(g(x)f(x)) + \Lambda P_Q f(x) - \Lambda f(x).$$

$\mathcal{A}$ is a generator of a semigroup $\{U(t)\}_{t \geq 0}$.

Results:

Theorem 3 *The semigroup* $\{U(t)\}_{t\geq 0}$ *satisfies the Foguel alternative, i.e. it is asymptotically stable or for every* $f \in L^1[0,\infty)$ *and* $M > 0$

$$\lim_{t\to\infty} \int_0^M U(t)f(x)\,dx = 0.$$

The proof of (K) is based on the Dyson-Phillips expansion.

Asymptotic stability — if we have V such that:

$$\limsup_{x\to\infty} [g(x)V'(x) + \Lambda V(Q(x)) - \Lambda V(x)] < 0.$$

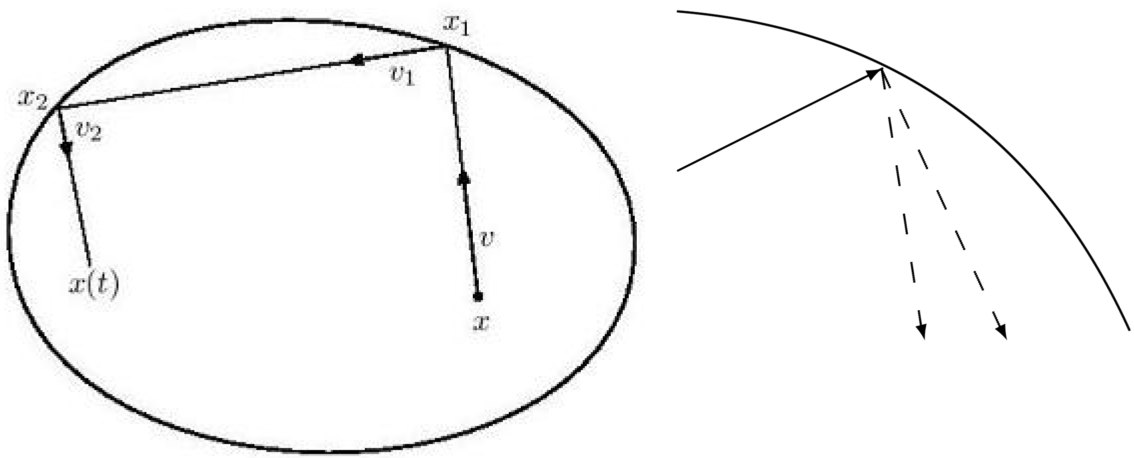
Example: if $\lim_{x\to\infty} g(x) = -\infty$ and $Q(x) \leq x + L$, then $V(x) = x$ is OK.

1) $a > \Lambda \log b$, $g(x) \leq -ax$ and $Q(x) \leq bx$ for a sufficiently large x, then $\{U(t)\}_{t\geq 0}$ is asymptotically stable.

2) If $a < \Lambda \log b$, $g(x) \geq -ax$ and $Q(x) \geq bx$ then the semigroup is sweeping from compact sets.

Sweeping can be interpreted as asymptotic permanent immunity of the population.

Collisionless kinetic equations



$$\partial_t u(t,x,v)+v\cdot\nabla_x u(t,x,v)=0,\ \ (x,v)\in\Omega\times V,\ t\geq 0$$

$$\psi_{|\Gamma_-}=\mathsf{H}(\psi_{|\Gamma_+}),$$

where

$$\Gamma_\pm=\{(x,v)\in\partial\Omega\times V;\ \pm v\cdot n(x)>0\}$$

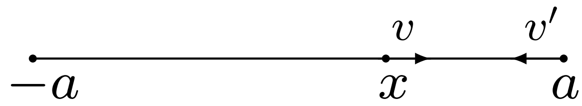
$n(x)$ – the outward unit normal at $x\in\partial\Omega$
H is a linear boundary operator relating the outgoing and incoming fluxes.

$f(x,v)=u(0,x,v)$, $P(t)f(x,v)=u(t,x,v)$ stochastic semigroup on $L^1(\Omega\times V)$.

Main result: Sufficient conditions for asymptotic stability of $\{P(t)\}_{t\geq 0}$.

Steps of the proof: Existence of invariant density, checking the semigroup is partially integral and irreducible.

One dimensional stochastic billiard



Stochastic process $\xi(t) = (x(t), v(t))$ with values in $X = [-a, a] \times ([-1, 0) \cup (0, 1])$.

$$\frac{\partial u}{\partial t}(t, x, v) + v\frac{\partial f}{\partial x}(t, x, v) = 0$$

f -invariant density $\Rightarrow$ f is a function of v and satisfies some integral equation.

Special case: velocity after hitting a boundary point is uniformly distributed.

Then the semigroup is sweeping from compact sets and, consequently, the distribution of velocity converges to δ_0.
It is interesting that in this case

$$f(t,x,v) \sim \frac{c}{|v|}(\log t)^{-1} \text{ when } t \to \infty$$

for $|v| \ge \varepsilon$ i $x \in [-a,a]$, where c is some constant.


Thank You!