On determining the homological Conley index of Poincaré maps

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Łódź 2023

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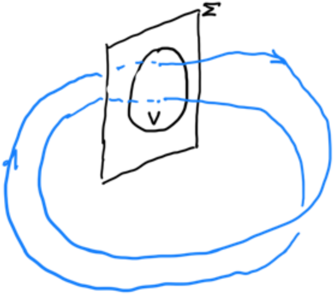

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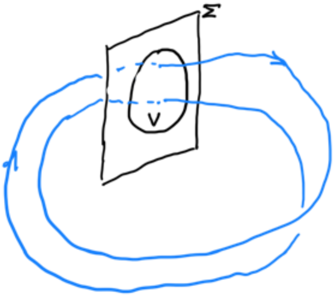

$\Pi \colon V \to \Sigma$ is the Poincaré map associated to Σ; its domain V is open in Σ.

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$f\colon V \to X$ is a homeomorphism onto $f(V)$ open in X.

$A \subset V$.
The *invariant part of A* is defined as

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A compact $S \subset V$ is called isolated invariant if there exists a neighborhood U of S such that

$$S = \mathrm{Inv}(U)$$

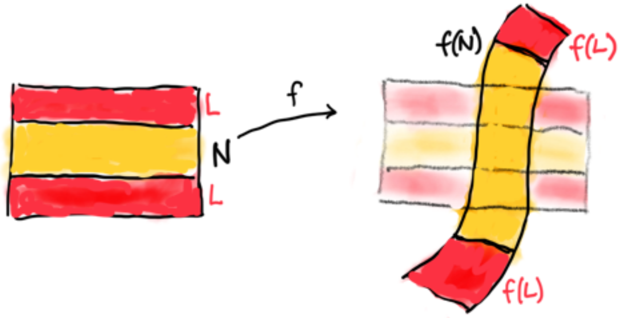
(i.e. S is the maximal invariant set in U).

(N, L) is a pair of compact subsets of V.

Definition

(N, L) is an *index pair* (for f) if

1. $\mathrm{Inv}(\mathrm{cl}(N \setminus L)) \subset \mathrm{int}(N \setminus L)$,
2. $f(L) \cap N \subset L$,
3. $\mathrm{cl}(f(N) \setminus N) \cap N \subset L$.



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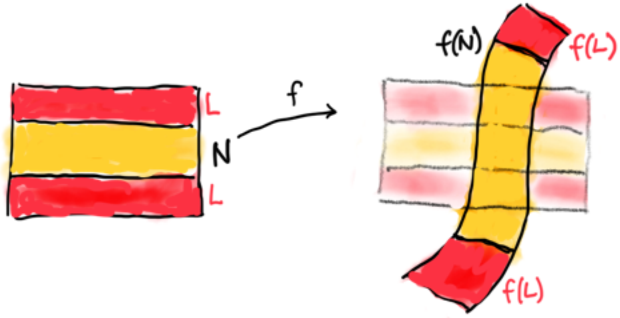

$$f_{(N,L)}\colon N/L \to N/L, \quad f_{(N,L)}[x] := \begin{cases} f(x), & \text{if } x, f(x) \in N \setminus L, \\ *, & \text{otherwise,} \end{cases}$$

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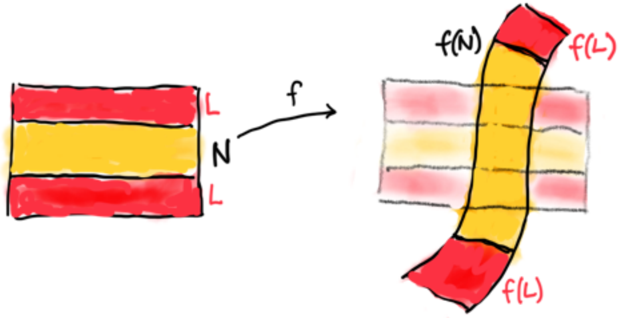

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Remark

If (N, L) is an index pair then $f_{(N,L)}$ is continuous.

(N, L) is called an *index pair for* (S, f) if $S = \mathrm{Inv}(\mathrm{cl}(N \setminus L))$.

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Definition (M. Mrozek, 1990)

The *Conley index of* (S, f) is the conjugacy class of the automorphism $RH(f_{(N,L)})$ for an index pair (N, L) for (S, f).

Here $RH(f_{(N,L)})$ is the *Leray reduction* of the endomorphism

$$H(f_{(N,L)})\colon H(N/L, *) \to H(N/L, *),$$

where H denotes the singular homology functor with coefficients in a field $\mathbb{F}$.

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Remarks

1. *If X is metrizable locally compact then there exists at least one index pair for (S, f).*
2. *The Conley index does not depend on the choice of an index pair.*
3. *The nozero part of the spectrum of $H(f_{(N,L)})$ is an invariant of S.*
4. *If the Lefschetz number of $H(f_{(N,L)})$ is nonzero then f has a fixed point in $N \setminus L$.*

E is a vector space over $\mathbb{F}$,
$\alpha \colon E \to E$ is a linear endomorphism.

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First step: QE denotes the quotient of E by the *generalized kernel* of α, i.e.,

$$QE := E \Big/ \bigcup_k \ker \alpha^k$$

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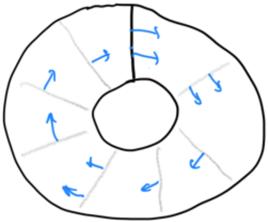
Second step: $\mathrm{gim}(Q\alpha)$ denotes the the *generalized image* of $Q\alpha$, i.e.,

$$\mathrm{gim}(Q\alpha) := \bigcap_k \operatorname{im} Q\alpha^k \subset QE.$$

Then the map $\mathrm{gim}(Q\alpha) \to \mathrm{gim}(Q\alpha)$ induced by $Q\alpha$ is an automorphism and it is denoted by $R\alpha$.

ϕ is the local dynamical system generated by v on an open $\Omega \subset \mathbb{R}^n$.

We assume that ϕ is *rotating*,



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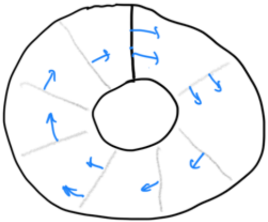
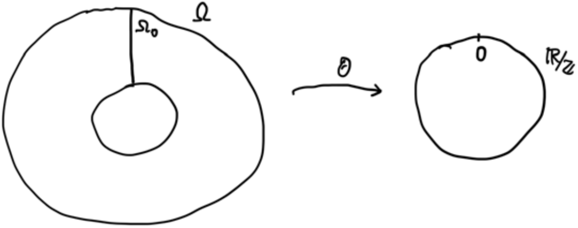
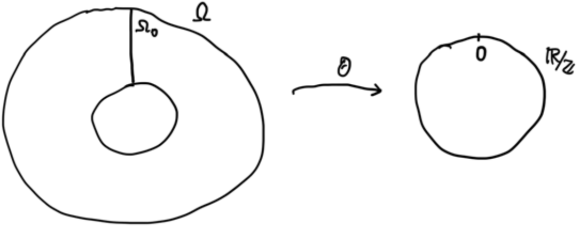
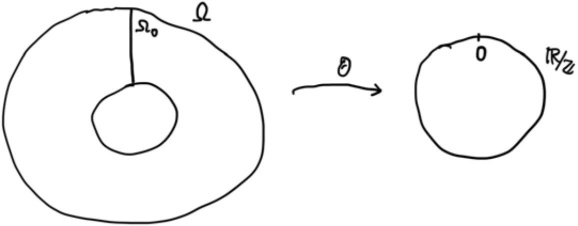


i.e. there is a smooth map $\theta \colon \Omega \to \mathbb{R}/\mathbb{Z}$ such that $\frac{d}{dt}\theta(\phi(x,t)) > 0$ for each $x \in \Omega$.





For $a \in \mathbb{R}$ set $\Omega_a := \theta^{-1}(a + \mathbb{Z})$.
For $Z \subset \Omega$ and $a \in \mathbb{R}$ set $Z_a := \Omega_a \cap Z$.



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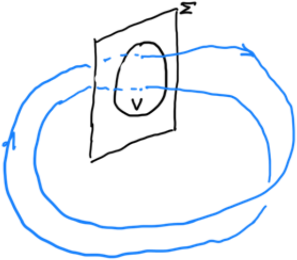
Ω_0 is a section for ϕ. We assume that $\Sigma := \Omega_0$, hence $\Pi\colon V \to \Omega_0$ is the Poincaré map.



$h > 0$, $\phi^h := \phi(\cdot, h)$ is the h-discretization of ϕ.

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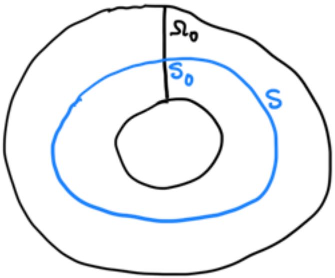
Lemma

If S is an isolated invariant set for ϕ^h then S_0 is an isolated invariant set for Π.

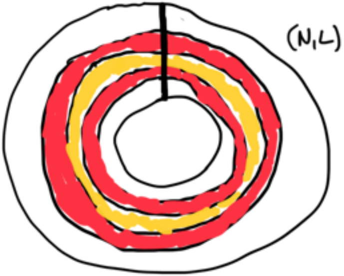


Theorem (R.S.)

If (N, L) *is an index pair for* (S, ϕ^h), N_0 *and* L_0 *are ANRs and* $F_a\colon (N_0, L_0) \to (N_a, L_a)$ *for* $a \in [0, 1]$ *be a family of maps such that*

$$F\colon (N_0, L_0) \times [0, 1] \to (N, L), \quad F(x, a) := F_a(x)$$

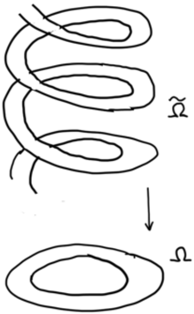
is a continuous. If $F_0 = \mathrm{id}_{N_0}$, $\phi^t(F_a(N_0)) \subset N$, *and* $\phi^t(F_a(L_0)) \subset L$ *for all* $a \in [0, 1]$ *and* $t \in [0, h]$ *then* $CH(S_0, \Pi)$ *is equal to the conjugacy class of* $RH(F_1)$.



$$\widetilde{\Omega} := \{(x,a) \in \Omega \times \mathbb{R} \colon x \in \Omega_a\}.$$

$$\zeta \colon \widetilde{\Omega} \ni (x,a) \to x \in \Omega, \quad \widetilde{\theta} \colon \widetilde{\Omega} \ni (x,a) \to a \in \mathbb{R}$$

$$\begin{array}{ccc} \widetilde{\Omega} & \xrightarrow{\ \widetilde{\theta}\ } & \mathbb{R} \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle (\cdot)+\mathbb{Z}} \\ \Omega & \xrightarrow{\ \theta\ } & \mathbb{R}/\mathbb{Z} \end{array}$$



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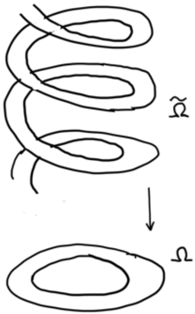


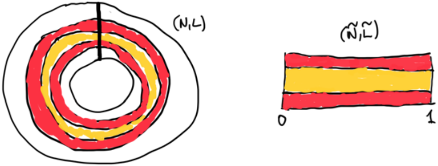
ϕ induces a local dynamical system $\widetilde{\phi}$ on $\widetilde{\Omega}$.

$$\widetilde{Z} := \{(x, a) \in \widetilde{\Omega} \colon x \in Z_a\}.$$

For an interval $J \subset \mathbb{R}$ set

$$\widetilde{Z}_J := \{(x, a) \in \widetilde{Z} \colon a \in J\}.$$

$h > 0$, (N, L) is an index pair for ϕ^h.



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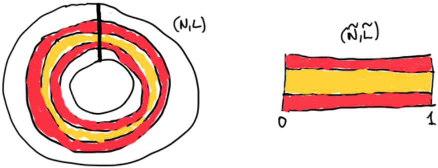

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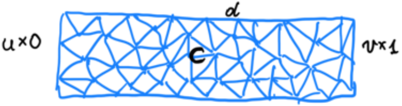
Definition

(u, v) is called a *pair of contiguous cycles over* $[0, 1]$ if there exist chains $c \in S(\widetilde{N}_{[0,1]})$ and $d \in S(\widetilde{L}_{[0,1]})$ such that

$$u \times 0 - v \times 1 = \partial c + d,$$



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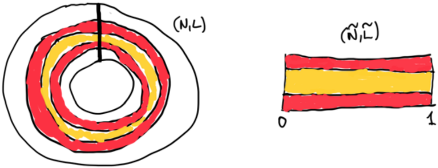

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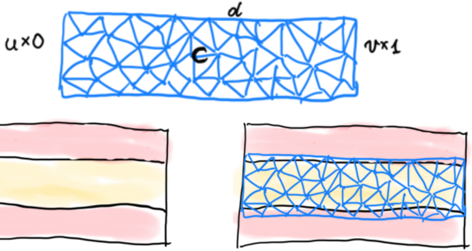

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If, moreover, $\widetilde{\phi}(|c|, [0, h]) \subset \widetilde{N}$ and $\widetilde{\phi}(|d|, [0, h]) \subset \widetilde{L}$ the pair of contiguous cycles (u, v) is called *h-movable*.

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Theorem (R.S.)

Let $h > 0$ and let (N, L) be an index pair for an isolated invariant set S with respect to ϕ^h. If N_0 and L_0 are ANRs, $n = \dim H(N_0, L_0)$, $A = [a_{ij}]$ is a graded $(n \times n)$-matrix over $\mathbb{F}$, and $(u_j, \sum_{i=1}^{n} a_{ij} u_i)$ for $j = 1, \ldots, n$ is an h-movable pair of contiguous cycles over $[0, 1]$ such that $\{[u_j] \colon j = 1, \ldots, n\}$ is a basis of $H(N_0, L_0)$ then $CH(S_0, \Pi)$ is equal to the conjugacy class of the Leray reduction RA.

M. Mrozek, R. Srzednicki, F. Weilandt, A topological approach to the algorithmic computation of the Conley index for Poincaré maps, SIAM J. Appl. Dyn. Syst. 14 (2015), 1348-1386.

R. Srzednicki, On determining the homological Conley index of Poincaré maps in autonomous systems, Topol. Methods Nonlinear Anal. 60 (2022), 5-32.