A new criterion for bifurcation of homoclinic solutions for parameterized ordinary differential equations

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Nicolaus Copernicus University in Torun
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Plan of my talk

1. Introduction to the homoclinic problem for ordinary differential equations.
2. Mathematical methods how to study the above problem:
   - The nonlinear Nemytskii operator,
   - The concept of parity (definition and the main properties),
   - Stable and unstable manifolds vs the homoclinic trajectories,
   - The Evans function.
3. A new criterion of the existence of bifurcation points.
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Differential equations and homoclinic solutions

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda) \\ \lim\limits_{t \to \pm\infty} x_\lambda(t) = 0. \end{cases} \tag{HP}$$

(a) $f : \mathbb{R} \times \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ is continuous and $\Lambda \subset \mathbb{R}^k$ is a compact subset.
(b) $f(t, 0, \lambda) = 0$ for all $t \in \mathbb{R}$ and $\lambda \in \Lambda$.

Remark

1. *for each $\lambda \in \Lambda$, the function $x(t) \equiv 0$ is a solution of (HP) – a trivial homoclinic solution.*
2. *Question: For what parameters $\lambda \in \Lambda$ is there a non-trivial function $x(t) \not\equiv 0$ that solves Problem (HP)?*
3. *We are looking for the solutions in the space:*

$$C_0^1(\mathbb{R}, \mathbb{R}^d) := \{x : \mathbb{R} \to \mathbb{R}^d \mid x \text{ is of class } C^1 \text{ and } x, \dot{x} \in C_0(\mathbb{R}, \mathbb{R}^d)\},$$
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Example (Non-trivial homoclinic sol. for one-dimesnional space $\mathbb{R}$)

$$\begin{cases} \dot{x}(t) = \arctan(t) \cdot x(t) \\ \lim_{t \to \pm\infty} x(t) = 0 \end{cases}$$

- *Here* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is given by* $f(t, x) = \arctan(t) \cdot x$.
- *Each solution reads as follows* $x(t) = \sqrt{1 + t^2} e^{-t \cdot \arctan(t)} x_0$, $x_0 \in \mathbb{R}$.
- *Thus* $\lim_{t \to \pm\infty} x(t) = 0$.

Remark (Motivation)

Homoclinic Equations (HP) arises in many real phenomena, for instance, in physics as the study of traveling waves for parabolic reaction-diffusion equations or in biology/chemistry as the study of the Scott-Gray model for autocatalysis.

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How to study (HP)? - the first observation:

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda) \\ \lim\limits_{t \to \pm\infty} x_\lambda(t) = 0. \end{cases} \tag{HP}$$

Recall that $f(t, 0, \lambda) = 0$, $A(t, \lambda) := D_2 f(0, t, \lambda)$ exists and is continuous.

1. (HP) has a non-trivial solution $\iff W^s_\lambda(0) \cap W^u_\lambda(0) \neq \{0\}$, where

$$W^{s/u}_\lambda(0) := \{x_\lambda(0) \in \mathbb{R}^d \mid x_\lambda(\cdot) \text{ solves } (HP), \lim_{t \to +\infty/-\infty} x_\lambda(t) = 0\}.$$

2. We associate the problem (HP) with its linearization (LHP):

$$\begin{cases} \dot{x}(t) = A(t, \lambda) x(t) \\ \lim\limits_{t \to \pm\infty} x_\lambda(t) = 0. \end{cases} \tag{LHP}$$

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- Consider $F\colon C_0^1(\mathbb{R}, \mathbb{R}^d) \times \Lambda \to C_0(\mathbb{R}, \mathbb{R}^d)$ given by:
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Our strategy for studying the problem (HP)

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   we can define the Nemytskii operator $F\colon C_0^1(\mathbb{R}, \mathbb{R}^d) \times \Lambda \to C_0(\mathbb{R}, \mathbb{R}^d)$:

2. $F(x, \lambda)(t) := \dot{x}(t) - f(t, x(t), \lambda)$.
3. $F(0, \lambda) = 0$.
4. x is a solution of (HP) if and only if $F(x, \lambda) = 0$.
5. We are looking for the non-trivial zeros of F, i.e., $F(x, \lambda) = 0$ with $x \neq 0$ for some $\lambda \in \Lambda$.
6. For this purpose we can apply the bifurcation theory.
7. A bifurcation point $\lambda_0 \in \Lambda \Longrightarrow$ the existence of non-trivial solutions of F.

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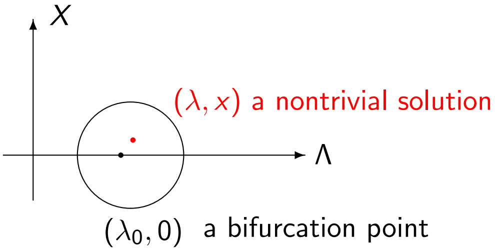
Bifurcation theory – when $\mathcal{B}(F) \neq \emptyset$?

Definition

A point $\lambda_0 \in \Lambda$ is called a bifurcation point *provided in any open neighborhood $\mathcal{O}$ of $(0, \lambda_0)$ there is a point $(x, \lambda) \in \mathcal{O}$ such that x is a nontrivial solution of $F(x, \lambda) = 0$, where $F : X \times \Lambda \to Y$. Then we write $\lambda_0 \in \mathcal{B}(F)$.*

Remark

Recall that $F(0, \lambda) = 0$ for all $\lambda \in \Lambda$.



Historical beginning of bifurcation theory

1. The definition of **a bifurcation point** can be traced to Poincaré in his studies of the equilibrium of rotating fluid masses:

   H. Poincaré, *Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation*, Acta Math. 7 (1885), 259-380.

2. The first fundamental sufficient condition for the existence of a bifurcation point was given by **Krasnoselskii**:

   M.A. Krasnoselskii, *On the problem of branch points* (Russian), Doklady Akad. Nauk SSSR (N.S.) 79 (1951), 389-392.

3. The **Rabinowitz global bifurcation theorem** was obtained in 1971:

   P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Functional Anal. 7 (1971), 487-513.

Bifurcation theory

Theorem (Krasnoselskii Local Bifurcation Theorem)

1. *Let* $0 \in \mathcal{U}$ *be an open subset of a Banach space* X*,*
2. *Let* $F : \mathcal{U} \times (a, b) \to X$ *be given by* $F(x, \lambda) = x - \lambda A x - G(x, \lambda)$*,*
3. A *is a linear compact operator,*
4. $G(\cdot, \lambda) : \overline{\mathcal{U}} \to X$ *is a compact (nonlinear) operator and*

$$\lim_{x \to 0} \frac{G(x, \lambda)}{\|x\|} = 0 \text{ for all } \lambda \in (a, b).$$

If $\lambda_0 \in (a, b)$ *is such that* $1/\lambda_0$ *is an eigenvalue of* A *and*

$$m_a\left(\frac{1}{\lambda_0}\right) = \dim \bigcup_{k=1}^{\infty} (I - \lambda_0 A)^k \in 2\mathbb{N} + 1 \ \ (\text{algebraic multiplicity}),$$

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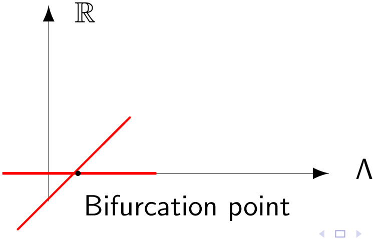
Bifurcation for $\Lambda = (-2, 2)$

Example

1. *Let* $F\colon \mathbb{R} \times (-2,2) \to \mathbb{R}$ *be given by* $F(x,\lambda) = x - \lambda x - x^2$.
2. $A : \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \times (-2,2) \to \mathbb{R}$ *are given by* $Ax = x$, $G(x,\lambda) = x^2$.
3. *Let* $\lambda_0 = 1$. *Then* $1/\lambda_0 \in \sigma(A)$ *and* $m_a(1/\lambda_0) = 1$ *is odd.*
4. *Thus* $\lambda_0 = 1$ *is a bifurcation point of* F.

Remark

$F(x,\lambda) = 0 \iff (x,\lambda) = (0,\lambda)$ *or* $(x,\lambda) = (1-\lambda,\lambda)$, *for all* $\lambda \in (0,2)$.



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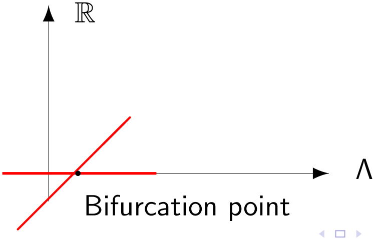

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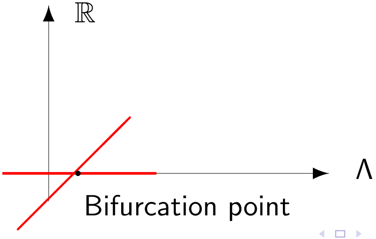

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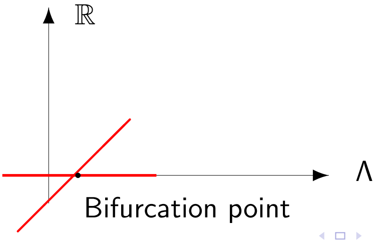

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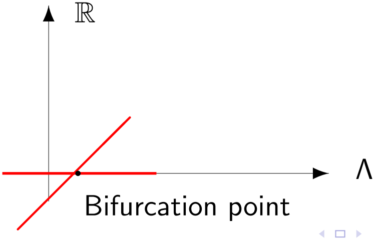

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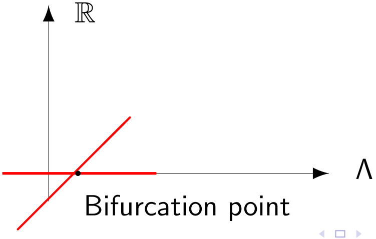
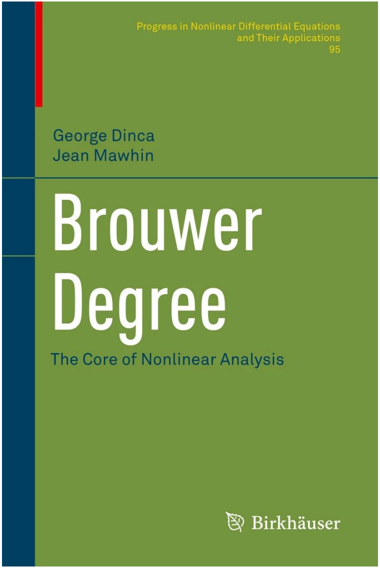
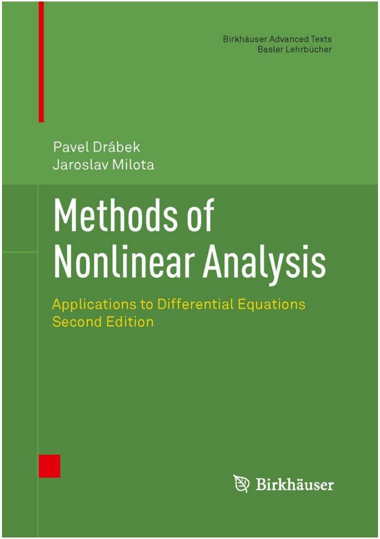


Book by J. Dinca and J. Mawhin



Book by P. Drabek and J. Milota



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2. Is it possible to apply the Krasnoselskii Local Bifurcation Theorem to the Nemytskii operator?
3. Answer: NO.
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2. It is not possible to apply the Krasnoselskii Local Bifurcation Theorem to the Nemytskii operator.
3. We try to apply the local bifurcation theorem for Fredholm maps.

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2. It is not possible to apply the Krasnoselskii Local Bifurcation Theorem to the Nemytskii operator.
3. We try to apply the local bifurcation theorem for Fredholm maps.

Fredholm maps

Definition

A Fredholm operator is a bounded linear operator $L : X \to Y$ between two Banach spaces with finite-dimensional $\ker(L)$ and $\operatorname{coker}(L) := Y/\operatorname{im}(L)$. The index of L (denoted by $\operatorname{ind}(L)$) is defined by

$$\operatorname{ind}(L) = \dim \ker(L) - \dim \operatorname{coker}(L).$$

The set of Fredholm operators of index zero is denoted by $\Phi_0(X, Y)$.

Definition

Let $\mathcal{U} \subset X$ be an open subset. We say that a function $F : \mathcal{U} \to Y$ of class C^1 is Fredholm of index zero if $DF(x) : X \to Y$ is a Fredholm operator of index 0 for all $x \in \mathcal{U}$, where X and Y are Banach spaces.

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Examples

1. Any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is Fredholm of index $\mathrm{ind}(T) = n - m$.
2. $L\colon C^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ given by
$$(Lx)(t) = \dot{x}(t)$$
is Fredholm of index 1.
3. $L\colon C_0^1(\mathbb{R},\mathbb{R}^d) \to C_0(\mathbb{R},\mathbb{R}^d)$ given by
$$(Lx)(t) = \dot{x}(t) - Ax(t),$$
is Fredholm of index 0, where $A \in \mathbb{R}^{d\times d}$ is a hyperbolic matrix, i.e., $\sigma(A) \cap i\mathbb{R} = \emptyset$.

Remark

The third example suggests that the Fredholm theory should be appropriate for the problem (HP) and the Nemytskii operator $F_\lambda\colon C_0^1(\mathbb{R},\mathbb{R}^d) \to C_0(\mathbb{R},\mathbb{R}^d)$ given by
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Bifurcation theorem for Fredholm maps and $\Lambda = [a, b]$

Theorem (Fitzpatrick, Pejsachowicz, Rabier)

1. *Let* $F : \mathcal{U} \times [a, b] \to Y$ *be continuous and* $\mathcal{U} \subset X$ *be an open subset,*
2. $F(0, \lambda) = 0$ *for all* $\lambda \in [a, b]$,
3. $(x, \lambda) \mapsto D_x F(x, \lambda)$ *exists as continuous function,*
4. $D_x F(x, \lambda) \in \Phi_0(X, Y)$,
5. $D_x F(0, a)$ *and* $D_x F(0, b)$ *are invertible.*

a. *Then for the map* $[a, b] \ni \lambda \xmapsto{LF} D_x F(0, \lambda)$ *one can define the parity* $\sigma(LF, [a, b]) \in \{-1, 1\}$.

b. *If* $\sigma(LF, [a, b]) = -1$*, then the set of bifurcation points for* F *is nonempty.*

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Bartsch, Benevieri, Furi, Kryszewski, Mawhin and others studied bifurcation problems for Fredholm maps.

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Theorem (Special case for the finite-dimensional spaces)

1. *Let* $F : \mathbb{R}^d \times [a, b] \to \mathbb{R}^d$ *be of class* C^1,
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Let $LF : [a, b] \to \mathcal{L}(\mathbb{R}^d)$ *be continuous with* $LF(a), LF(b) \in GL(\mathbb{R}^d)$. *The parity of* LF *on* $[a, b]$, *which is denoted by* $\sigma(LF, [a, b])$, *is defined by*

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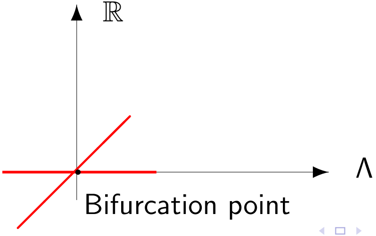
Example for $\Lambda = [-1, 1]$

Example

1. *Let* $F\colon \mathbb{R} \times [-1,1] \to \mathbb{R}$ *be given by* $F(x,\lambda) = \lambda x - x^2$.
2. $LF(\lambda) := D_x F(0,\lambda) = \lambda \implies LF(\lambda) \neq 0$ *for* $\lambda \neq 0$,
3. $\operatorname{sgn}\det LF(-1) \cdot \operatorname{sgn}\det LF(1) = -1 \cdot 1 = -1$.
4. *Then* $\mathcal{B}(F) \neq \emptyset$. *But* $LF(\lambda) \neq 0$ *for all* $\lambda \neq 0$. *Thus* $\mathcal{B}(F) = \{0\}$.

Remark

$F(x,\lambda) = 0 \iff (x,\lambda) = (0,\lambda)$ *or* $(x,\lambda) = (\lambda,\lambda)$, *for all* $\lambda \in [-1,1]$.



Parity for $L : [a, b] \to \mathcal{L}(\mathbb{R}^d)$

Remark

Given metric spaces X and Y with $X_0 \subset X$ and $Y_0 \subset Y$, the symbol $f : (X, X_0) \to (Y, Y_0)$ denotes a continuous function $f : X \to Y$ with $f(X_0) \subset Y_0$.

Definition

Let $L : ([a, b], \partial[a, b]) \to (\mathcal{L}(\mathbb{R}^d), GL(\mathbb{R}^d))$ be continuous. The parity of L on $[a, b]$, which is denoted by $\sigma(L, [a, b])$, is defined by

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Remark

Parity is one of the simplest topological invariants that allows us to study the existence of zeros for operators in infinite-dimensional spaces.

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Properties of the parity for $L : [a, b] \to \mathcal{L}(\mathbb{R}^d)$

Lemma (Normalization property)

If $T_0 \colon [a, b] \to GL(\mathbb{R}^d)$, *then* $\sigma(T_0, [a, b]) = 1$.

Theorem (Homotopy property)

Let $L, S \colon ([a, b], \partial[a, b]) \to (\mathcal{L}(\mathbb{R}^d), GL(\mathbb{R}^d))$ *be continuous. Then the following are equivalent:*

(1) $\sigma(L, [a, b]) = \sigma(S, [a, b])$.

(2) *There exists a homotopy* $H(t, \lambda)$ *with the following properties:*

1. $H \colon ([0, 1] \times [a, b], [0, 1] \times \{a, b\}) \to (\mathcal{L}(\mathbb{R}^d), GL(\mathbb{R}^d))$.
2. $H(0, \lambda) = L(\lambda)$ *and* $H(1, \lambda) = S(\lambda)$.

Then we write $L \simeq_H S$.

Corollary

$\sigma(L, [a, b]) = 1 \iff L \simeq_H T_0$.

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Properties of the parity for $L : [a, b] \to \mathcal{L}(\mathbb{R}^d)$

Theorem

Further properties:

- *(Additivity property) If* $c \in I = [a, b]$ *and* $L(c) \in GL(\mathbb{R}^d)$, *then*

$$\sigma(L, [a, b]) = \sigma(L, [a, c]) \cdot \sigma(L, [c, d]).$$

- *(Product property) If* $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ *and* $L = L_1 \times L_2$, *then*

$$\sigma(L, [a, b]) = \sigma(L_1, [a, b]) \cdot \sigma(L_2, [a, b])$$

  where

  1. $L(\lambda) = L_1(\lambda) \times L_2(\lambda)$ *for* $\lambda \in [a, b]$,
  2. $L_i \colon ([a, b], \partial[a, b]) \to (\mathcal{L}(\mathbb{R}^{d_i}), GL(\mathbb{R}^{d_i})$ *for* $i = 1, 2$.

The construction of parity for Fredholm maps of index zero

Question

How to define the concept of parity for $L : [a, b] \to \mathcal{L}(X, Y)$, where X and Y are infinite-dimensional Banach spaces?

1. Let $X \subset Y$ be two infinite-dimensional Banach spaces.
2. Let $L : [a, b] \to \Phi_0(X, Y)$ be a continuous family of Fredholm operators of index 0, i.e., $L(\lambda) \in \Phi_0(X, Y)$.
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2. Let $L : [a, b] \to \Phi_0(X, Y)$ be a continuous family of Fredholm operators of index 0, i.e., $L(\lambda) \in \Phi_0(X, Y)$.
3. Then there exists a finite-dimensional subspace $V \subset Y$ such that $\mathrm{im}(L(\lambda)) + V = Y$ for all $\lambda \in [a, b]$.
4. It may happen that $0 \in \mathrm{im}(L(\lambda)) \cap V \neq \{0\}$ for some $\lambda \in [a, b]$.
5. For any $\lambda \in [a, b]$ we take $W_\lambda := L(\lambda)^{-1}(V) \subset X$.
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The above definition does not depend on the choice of V*.*

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Bifurcation theorem for Fredholm maps and (HP)

Theorem (Fitzpatrick, Pejsachowicz, Rabier)

1. *Let* $F : \mathcal{U} \times [a,b] \to Y$ *be continuous and* $\mathcal{U} \subset X$ *be an open subset,*
2. $F(0,\lambda) = 0$ *for all* $\lambda \in [a,b]$,
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- a. *Then for the map* $[a,b] \ni \lambda \xmapsto{LF} D_x F(0,\lambda)$ *one can define the parity* $\sigma(LF,[a,b]) \in \{-1,1\}$.
- b. *If* $\sigma(LF,[a,b]) = -1$, *then the set of bifurcation points for* F *is nonempty.*

We want to apply the above theorem to the following problem:

$$\begin{cases} \dot{x}(t) = f(t,x(t),\lambda) \\ \lim\limits_{t\to\pm\infty} x(t) = 0. \end{cases} \qquad \text{(HP)}$$

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Let $F\colon C_0^1(\mathbb{R},\mathbb{R}^d)\times[a,b]\to C_0(\mathbb{R},\mathbb{R}^d)$ be the Nemytski operator.

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Whether F is continuous on $C_0^1(\mathbb{R},\mathbb{R}^d)\times[a,b]$ and differentiable with respect to the first variable (in a continuous way)?

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Whether F is the nonlinear Fredholm operator?

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$LF : [a,b]\to\Phi_0(C_0^1(\mathbb{R},\mathbb{R}^d),C_0(\mathbb{R},\mathbb{R}^d))$ is given by $LF(\lambda)=D_xF(0,\lambda)$. When $\sigma(LF,[a,b])=-1$?

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The answer to Q1 (Pötzsche, Rabier, Stuart, Skiba):

Let $F\colon C_0^1(\mathbb{R},\mathbb{R}^d)\times[a,b]\to C_0(\mathbb{R},\mathbb{R}^d)$ be the Nemytski operator induced by

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If a continuous function $f\colon \mathbb{R}\times\mathbb{R}^d\times[a,b]\to\mathbb{R}^d$ satisfies the conditions:

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Preparations to give the answer to Q2.

Given a differential system:

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda) \\ \lim\limits_{t \to \pm\infty} x(t) = 0, \end{cases} \tag{HP}$$

one can associate the linearization of (HP) at $x \equiv 0$ given by

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Definition

Consider a linear ODE

$$\dot{x}(t) = A(t)x(t) \tag{L}$$

with a continuous function $A : \mathbb{R} \to L(\mathbb{R}^d)$ *and an associated evolution matrix* $U : \mathbb{R}^2 \to L(\mathbb{R}^d)$. *Suppose that* $J \subset \mathbb{R}$ *is an unbounded interval.*

1. *An invariant projector is a function* $P : J \to L(\mathbb{R}^d)$ *of projections* $P(t) \in L(\mathbb{R}^d)$ *such that*

$$U(t,s)P(s) = P(t)U(t,s) \text{ for all } t,s \in J.$$

2. *A linear ODE* (L) *has an exponential dichotomy on* J *with a projector* P *if there exist constants* $K \geq 1$ *and* $\alpha > 0$ *such that*

$$|U(t.s)P(s)| \leq Ke^{-\alpha(t-s)}, \ \ |U(s,t)(I_d - P(t))| \leq Ke^{\alpha(s-t)}$$

for all $s \leq t$ *with* $t,s \in J$.

Preparations to give the answer to Q2: Example of an ED

Consider a linear ODE

$$\dot{x}(t) = Ax(t) \tag{L}$$

where A is an hyperbolic matrix, i.e., $\sigma(A) \cap i\mathbb{R} = \emptyset$.

1. An associated evolution matrix is given by $U(t,s) = e^{(t-s)A}$.
2. $\mathbb{R}^d = E^s \oplus E^u$, where

$$E^s = \bigoplus_{\{\lambda_i \in \sigma(A) | Re \lambda_i < 0\}} \mathrm{Eig}(\lambda_i) \text{ and } E^u = \bigoplus_{\{\lambda_j \in \sigma(A) | Re \lambda_j > 0\}} \mathrm{Eig}(\lambda_j).$$

3. Let $P : \mathbb{R}^d \to \mathbb{R}^d$ be a projection with $\mathrm{Im} P = E^s$ and $\mathrm{Ker} P = E^u$.

One can show that

1. $e^{(t-s)A} P = P e^{(t-s)A}$ for all $t, s \in \mathbb{R}$.
2. there exist constants $K \geq 1$ and $\alpha > 0$ such that

$$|e^{(t-s)A} P| \leq K e^{-\alpha(t-s)}, \quad |e^{(s-t)A}(I_d - P)| \leq K e^{\alpha(s-t)}$$

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for all $s \leq t$ with $t, s \in \mathbb{R}$.

A recommended book about an exponential dichotomy

Remark

Significant contributions to the study of an ED are made by Perron, Massera, Schaeffer, Coppel, Sacker, Sell, Palmer, Pötzsche and others.

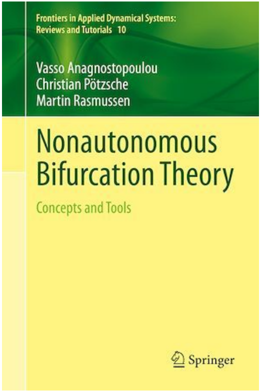

Figure: Anagnostopoulu, Pötzsche, Rasmusen, Nonautonomous bifurcation theory

The answer to Q2:

Question (Medium)

Whether $F\colon C_0^1(\mathbb{R},\mathbb{R}^d)\times[a,b]\to C_0(\mathbb{R},\mathbb{R}^d)$ *is Fredholm?*

Theorem (Palmer (Analysis+FA) or Pejsachowicz (Analysis+Top):)

Assume that

1. *F is admissible (i.e.,continuous and differentiable with respect to x),*
2. *the system*

$$\begin{cases} \dot{x}(t) = A(t,\lambda)x(t) \\ \lim_{t\to\pm\infty} x(t) = 0 \end{cases} \tag{LHP}$$

   has an ED on $\mathbb{R}^\pm$, where $A(t,\lambda) = D_x f(t,0,\lambda)$.

Then F is Fredholm. If there exists λ_0 such that (LHP) has only the trivial solution, then the Fredholm index $ind(F_\lambda) = 0$ for all $\lambda \in [a,b]$.

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The answer to Q3 – a new criterion for bifurcation

Question (Hard)

$LF : [a, b] \to \Phi_0(C_0^1(\mathbb{R}, \mathbb{R}^d), C_0(\mathbb{R}, \mathbb{R}^d))$ is given by $LF(\lambda) = D_x F(0, \lambda)$.
When $\sigma(LF, [a, b]) = -1$?

Theorem (C. Pötzsche and RS)

Assume that

1. *(LHP) has an ED on $\mathbb{R}^\pm$,*
2. *(LHP) only has a trivial solution for $\lambda = a$ and $\lambda = b$.*

Then the Evans function $E : [a, b] \to \mathbb{R}$ admits the following properties:

a. $\sigma(LF, [a, b]) = \operatorname{sgn} E(a) \cdot \operatorname{sgn} E(b)$,
b. *if* $\operatorname{sgn} E(a) \cdot \operatorname{sgn} E(b) = -1$, *then* $\mathcal{B}(F) \neq \emptyset$.

Remark

The construction of the Evans function E will be on the following slides.

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The construction of the Evans function E will be on the following slides.

The construction of the Evans function

Theorem (CP and RS)

We assume that a linear differential system

$$\begin{cases} \dot{x}(t) = A(t,\lambda)x(t) \\ \lim_{t\to\pm\infty} x(t) = 0. \end{cases} \tag{LHP}$$

has an ED on $\mathbb{R}^{\pm}$*. Then*

$$E^s := \bigcup_{\lambda\in[a,b]} \{\lambda\} \times E^s(\lambda), \quad E^u := \bigcup_{\lambda\in[a,b]} \{\lambda\} \times E^u(\lambda)$$

are the vector bundles over $\Lambda = [a,b]$*.*

Remark

$E^s(\lambda) := \{x(0) \in \mathbb{R}^d \mid \dot{x}(t) = A(t,\lambda)x(t)$ *and* $\lim_{t\to+\infty} x(t) = 0\}$,
$E^u(\lambda) := \{x(0) \in \mathbb{R}^d \mid \dot{x}(t) = A(t,\lambda)x(t)$ *and* $\lim_{t\to-\infty} x(t) = 0\}$.

The Evans function - a crucial step for bifurcation

Theorem

There exist continuous functions:

$$\gamma_1, ..., \gamma_p \colon [a, b] \to E^s \text{ and } \eta_1 ..., \eta_q \colon [a, b] \to E^u \text{ with } p + q = d$$

such that

1. $\gamma_1(\lambda), ..., \gamma_p(\lambda) \in E^s(\lambda) \subset \mathbb{R}^d$ *is a base of* $E^s(\lambda)$,
2. $\eta_1(\lambda), ..., \eta_q(\lambda) \in E^u(\lambda) \subset \mathbb{R}^d$ *is a base of* $E^u(\lambda)$.

Definition

A function $E \colon [a, b] \to \mathbb{R}$ *given by*

$$E(\lambda) := \det(\gamma_1(\lambda), ..., \gamma_p(\lambda), \eta_1(\lambda), ..., \eta_q(\lambda))$$

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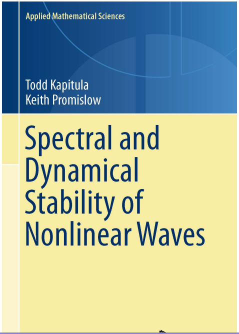
Historical notes about the Evans function

1. John Evans wrote his seminal papers in the 1970s in which the concept of the Evans function appeared.
2. Initially, it was used in the stability analysis of travelling waves in evolutionary PDEs.
3. Nowadays, the Evans function is used in many types of differential equations in various modified forms.
4. Todd Kapitula, Keith Promislow, Spectral and Dynamical Stability of Nonlinear Waves, Springer 2013.



Bifurcation for $\dot{x}(t) = f(t, x(t), \lambda)$, $\lim_{t\to\pm\infty} x(t) = 0$

Example

Consider the following problem:

$$\begin{cases} \dot{x}(t) = A(t,\lambda)x + g(x,\lambda) \\ \lim\limits_{t\to\pm\infty} x(t) = 0, \end{cases} \tag{HP}$$

where

1. $g\colon \mathbb{R}^2 \times [-1,1] \to \mathbb{R}^2$ is of class C^1 with $g(0,\lambda) = 0$, $D_x g(0,\lambda) = 0$,
2. $A(t,\lambda) := \begin{pmatrix} a(t) & 0 \\ \lambda^3 & -a(t) \end{pmatrix}$,
3. $a : \mathbb{R} \to \mathbb{R}$ is continuous with $a(t) \xrightarrow[t\to\pm\infty]{} \pm 1$.

Then (LHP) has the following form:

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Example (Continuation)

Linear ODE

$$\begin{cases} \dot{x}(t) = A(t,\lambda)x \\ \lim\limits_{t\to\pm\infty} x(t) = 0, \end{cases} \tag{LHP}$$

has the following stable and unstable manifolds:

- a) $E^s(\lambda) = \left\{\xi\left(1, \frac{-\lambda^3}{2}\right) \in \mathbb{R}^2 \mid \xi \in \mathbb{R}\right\}$,
- b) $E^u(\lambda) = \left\{\xi\left(1, \frac{\lambda^3}{2}\right) \in \mathbb{R}^2 \mid \xi \in \mathbb{R}\right\}$.

Then

- a) $\gamma_1 : [-1,1] \to \mathbb{R}^2$ is given by $\gamma_1(\lambda) = (1, \frac{-\lambda^3}{2}) \in E^s(\lambda)$,
- b) $\eta_1 : [-1,1] \to \mathbb{R}^2$ is given by $\eta_1(\lambda) = (1, \frac{\lambda^3}{2}) \in E^u(\lambda)$.

Bifurcation for $\dot{x}(t) = f(t, x(t), \lambda)$, $\lim_{t\to\pm\infty} x(t) = 0$

Example (Continuation)

1. The Evans function has the following form:

$$E(\lambda) = \det(\gamma_1(\lambda), \eta_1(\lambda)) = \det\begin{pmatrix} 1 & 1 \\ -\frac{1}{2}\lambda^3 & \frac{1}{2}\lambda^3 \end{pmatrix} = \lambda^3$$

2. Observe that $\operatorname{sgn} E(-1) \cdot \operatorname{sgn}\ \det E(1) = -1 \cdot 1 = -1$.
3. Consequently, $\mathcal{B}(F) \neq \emptyset$.
4. Since $E(\lambda) \neq 0$ for all $\lambda \neq 0$, it follows that $\mathcal{B}(F) = \{0\}$.

Remark

One can show that $E(\lambda) \neq 0$, then $\lambda \notin \mathcal{B}(F)$.

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Corollary

Since $\mathcal{B}(F) = \{0\}$*, one can conclude that for every neighbourhood* U_{λ^*} *of the point* $\lambda^* = 0$*, there is* $\lambda \in U_{\lambda^*}$ *and a non-trivial homoclinic solution* $x_\lambda(t)$ *to the equation:*

$$\begin{cases} \dot{x}(t) = A(t,\lambda)x + g(x,\lambda) \\ \lim\limits_{t\to\pm\infty} x(t) = 0, \end{cases} \tag{HP}$$

where $\lambda \in [-1, 1]$.

Concluding remarks

Remark

1. *Talk is based on the article: C. Pötzsche, R. Skiba, Evans function, parity and nonautonomous bifurcation of bounded entire solutions, manuscript 2024.*
2. *One can study a local bifurcation of bounded solutions for $\Lambda := [a, b]$, where $f(t, x, \lambda)$ is a Caratheodory function.*
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