Positive periodic solutions to the system of nonlinear delay differential equations

Robert Hakl[1] José Oyarce[2]

[1]Institute of Mathematics CAS, Czech Republic

[2]Universidad del Bío-Bío, Chile

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Outline

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Motivation

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Positive equilibrium $u_0 = 2\lambda - 1$ exists iff $\lambda > 1/2$

Critical value $\lambda^* = 1/2$ is the same for (E1)–(E4)

- There is no positive periodic solution to (E1)–(E4) provided $\lambda \leq \lambda^*$
- There exists a positive periodic solution to (E1)–(E4) provided $\lambda > \lambda^*$

Motivation

$$u'(t) = -5u(t) + \lambda \frac{10u(t-1)}{1+u(t-1)} \tag{E1}$$

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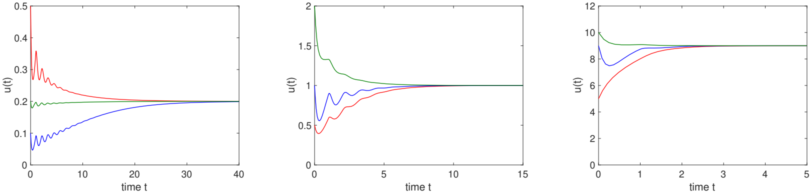

(a) $\lambda = 0.6$ (b) $\lambda = 1$ (c) $\lambda = 5$

Motivation

$$u'(t) = -5u(t) + \lambda \frac{10u(t)}{1 + u(\lfloor t \rfloor)} \tag{E2}$$

The equlibrium $u_0 = 2\lambda - 1$ is unique positive periodic solution to (E2) for every $\lambda \in (1/2, 5/6]$.

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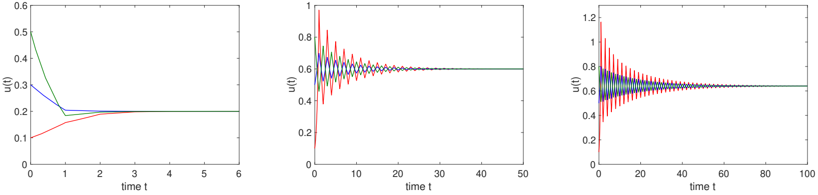

(a) $\lambda = 0.6$ (b) $\lambda = 0.8$ (c) $\lambda = 0.82$

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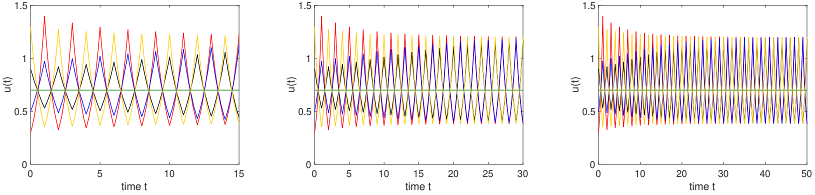

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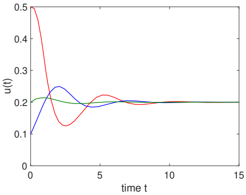

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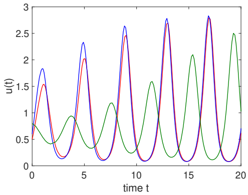
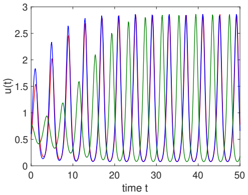


(b) $\lambda = 0.8$



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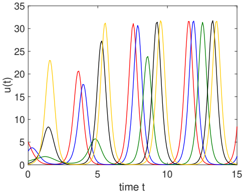

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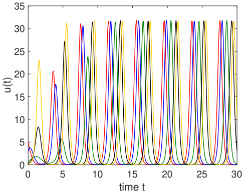
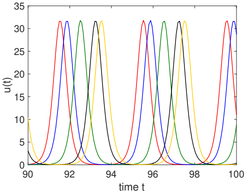


(b) $\lambda = 1$



(c) $\lambda = 1$

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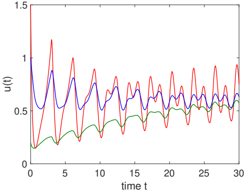

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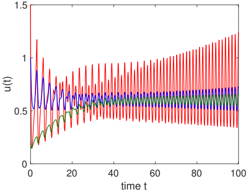

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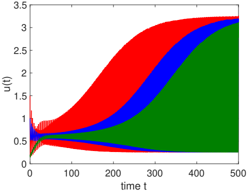

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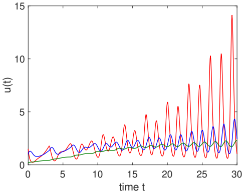

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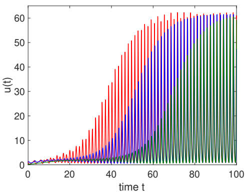

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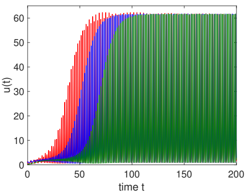

(c) $\lambda = 1.5$

Statement of the Problem

$$u_i'(t) = -d_i(t)u_i(t) - H_i(t)u_i(t-\sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)u_j(t-\nu_{ij}(t))$$
$$+ \lambda_i \sum_{k=1}^{N} P_{ik}(t)u_i(t-\tau_{ik}(t))f_{ik}(u_i(t-\mu_{ik}(t))) \quad (i=1,\dots,n), \quad (1)$$

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(i) $d_i,\ H_i,\ a_{ij},\ P_{ik} : \mathbb{R} \to \mathbb{R}_0^+$ $\dots\omega$-periodic locally integrable,

$$\int_0^\omega \Big[d_i(s) + H_i(s)\Big]ds > 0 \qquad \int_0^\omega \sum_{k=1}^{N} P_{ik}(s)ds > 0,$$

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(iv) $\lambda_i \geq 0$.

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Introduce a directed graph (V, A), $V = \{x_1, \dots, x_n\}$, oriented edge (arc) $[x_i, x_j] \in A$ iff

$$\int_0^{\omega} a_{ij}(s)ds > 0.$$

We assume that (V, A) is a strongly connected digraph, i.e., if for every $x, y \in V$ $(x \neq y)$ there exists an oriented path from x to y.

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$$\lambda_i = r\Lambda_i(\theta), \qquad r > 0, \qquad \theta_m \in [0, \pi/2]$$

$$\Lambda_1(\theta) = \cos\theta_1, \quad \Lambda_i(\theta) = \cos\theta_i \prod_{m=1}^{i-1} \sin\theta_m \quad (i = 2, \dots, n-1), \quad \Lambda_n(\theta) = \prod_{m=1}^{n-1} \sin\theta_m,$$

Main Results

$$u_i'(t) = -d_i(t)u_i(t) - H_i(t)u_i(t-\sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)u_j(t-\nu_{ij}(t))$$
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Theorem 1

Let there exist $\gamma_i \in AC\Big(\mathbb{R};\mathbb{R}_+\Big)$ $(i=1,\dots,n)$ and $\beta = \Big(\beta_i\Big)_{i=1}^{n} \in AC_\omega\Big(\mathbb{R}_+^n\Big)$ such that

$$\gamma_i'(t) \le -d_i(t)\gamma_i(t) - H_i(t)\gamma_i(t-\sigma_i(t)) \quad \text{for a. e. } t \in \mathbb{R},$$

$$\beta_i'(t) \ge -d_i(t)\beta_i(t) - H_i(t)\beta_i(t-\sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\beta_j(t-\nu_{ij}(t)) \quad \text{for a. e. } t \in \mathbb{R},$$

and β is not a solution. Then, for every $\theta \in [0,\pi/2]^{n-1}$ there exists a threshold $r^* > 0$ such that

i) there is a positive ω-periodic solution to (1) provided $r > r^*$;

ii) there is no nontrivial ω-periodic solution to (1) provided $r \le r^*$.

Sketch of Proof

$$u_i(t) = r\sum_{j=1}^{n}\int_{t-\omega}^{t} g_{ij}(t,s)\Lambda_j(\theta)\sum_{k=1}^{N} P_{jk}(s)u_j(s-\tau_{jk}(s))f_{jk}(u_j(s-\mu_{jk}(s)))ds$$

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$\mathcal{R}(\theta) = \{r \geq 0 :$ the system (1) has a positive ω-periodic solution$\}$

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$$r_1 \in \mathcal{R}(\theta), \quad r_2 > r_1 \quad \Longrightarrow \quad r_2 \in \mathcal{R}(\theta)$$

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r^* is continuous with respect to θ

By $\mathcal{S}(r,\theta)$ we denote the set of all positive ω-periodic solutions to (1) for corresponding r and θ. Further, we put

$$\mathcal{S}(r) = \bigcup_{\theta\in[0,\pi/2]^{n-1}} \mathcal{S}(r,\theta).$$

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Theorem 2

Let all the assumptions of Theorem 1 be fulfilled. Then, all positive ω-periodic solutions to (1) uniformly tends to infinity as r tends to infinity, i.e.,

$$\lim_{r\to+\infty} \inf \Big\{ \min\{u_i(t) : t \in \mathbb{R}, i \in \{1,\dots,n\}\} : u \in \mathcal{S}(r) \Big\} = +\infty.$$

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Theorem 3

Let all the assumptions of Theorem 1 be fulfilled, and let $\theta_0 = [0,\pi/2]^{n-1}$ be a fixed vector. Then

$$\lim_{\substack{(r,\theta)\to(r^*(\theta_0),\theta_0) \\ r>r^*(\theta)}} \sup \Big\{ \|u\|_{C_\omega} : u \in \mathcal{S}(r,\theta) \Big\} = 0.$$

Theorem 4

Let all the assumptions of Theorem 1 be fulfilled. Let, moreover, $r > 0$, and let $\theta \in [0, \pi/2]^{n-1}$ be a fixed vector. Then, the system of linear differential equations

$$u_i'(t) = -d_i(t)u_i(t) - H_i(t)u_i(t - \sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)u_j(t - \nu_{ij}(t)) + r\Lambda_i(\theta)\sum_{k=1}^{N} P_{ik}(t)f_{ik}(0)u_i(t - \tau_{ik}(t)) \quad (i = 1, \dots, n) \tag{2}$$

has a positive ω-periodic solution u^* iff $r = r^*(\theta)$. Moreover, the set of ω-periodic solutions to (2) is one-dimensional, generated by u^*.

Dependence on Parameter

Theorem 4

Let all the assumptions of Theorem 1 be fulfilled. Let, moreover, $r > 0$, and let $\theta \in [0, \pi/2]^{n-1}$ be a fixed vector. Then, the system of linear differential equations

$$u_i'(t) = -d_i(t)u_i(t) - H_i(t)u_i(t - \sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)u_j(t - \nu_{ij}(t))$$
$$+ r\Lambda_i(\theta) \sum_{k=1}^{N} P_{ik}(t) f_{ik}(0) u_i(t - \tau_{ik}(t)) \quad (i = 1, \dots, n) \qquad (2)$$

has a positive ω-periodic solution u^* iff $r = r^*(\theta)$. Moreover, the set of ω-periodic solutions to (2) is one-dimensional, generated by u^*.

We have

$$\frac{u^*}{r^*(\theta)} = A(u^*; \theta).$$

Therefore,

$$r^*(\theta) = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{\|A^n(\cdot; \theta)\|}}.$$

Theorem 5

Let all the assumptions of Theorem 1 be fulfilled. Let, moreover, $\theta \in [0, \pi/2]^{n-1}$ be a fixed vector, and let there exist $x, y \in AC_\omega\left(\mathbb{R}_+^n\right)$ and $r_1 > 0$, $r_2 > 0$ such that

$$x_i'(t) \geq -d_i(t)x_i(t) - H_i(t)x_i(t-\sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)x_j(t-\nu_{ij}(t)) + r_1\Lambda_i(\theta)\sum_{k=1}^{N} P_{ik}(t)f_{ik}(0)x_i(t-\tau_{ik}(t)) \quad (i = 1, \dots, n),$$

$$y_i'(t) \leq -d_i(t)y_i(t) - H_i(t)y_i(t-\sigma_i(t)) + \sum_{j=1}^{n} a_{ij}(t)y_j(t-\nu_{ij}(t)) + r_2\Lambda_i(\theta)\sum_{k=1}^{N} P_{ik}(t)f_{ik}(0)y_i(t-\tau_{ik}(t)) \quad (i = 1, \dots, n).$$

Then,

$$r_1 \leq r^*(\theta) \leq r_2.$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)\,\texttt{exp}(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-3)\,\texttt{exp}(-u_2(t-4))$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-2)$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-2)$$

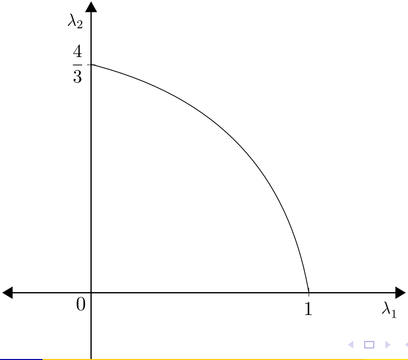
$$(2\lambda_1 - 3)(\lambda_2 - 2) - 2 = 0, \qquad 0 \le \lambda_1 < 3/2, \quad 0 \le \lambda_2 < 2$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-3)\exp(-u_2(t-4))$$

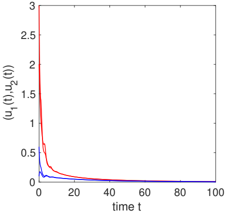
$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-2)$$

$$(2\lambda_1 - 3)(\lambda_2 - 2) - 2 = 0, \qquad 0 \le \lambda_1 < 3/2, \quad 0 \le \lambda_2 < 2$$

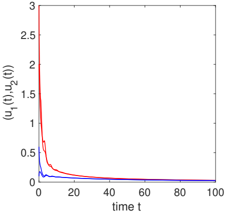


Examples

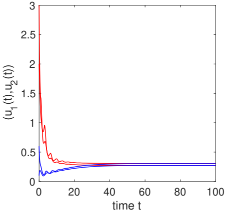
$$u_1'(t) = -3u_1(t) + 2u_2(t) + \lambda_1 2u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + \lambda_2 u_2(t-3)\exp(-u_2(t-4))$$



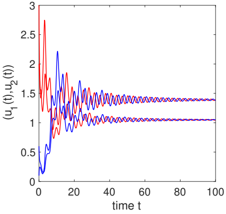
(a) $(\lambda_1, \lambda_2) = (0.5, 0.9)$



(b) $(\lambda_1, \lambda_2) = (0.5, 1)$



(c) $(\lambda_1, \lambda_2) = (0.5, 1.5)$



(d) $(\lambda_1, \lambda_2) = (0.5, 5)$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$

Examples

$$
\begin{aligned}
u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2)) \\
u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))
\end{aligned}
$$

$$
\begin{aligned}
u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1) \\
u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)
\end{aligned}
$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)2.1x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)1.05x_2(t-3)$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)2.1x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)1.05x_2(t-3)$$

$$x_1'(t) \geq -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)(2 + 0.1\cos t)x_1(t-1)$$
$$x_2'(t) \geq x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)(1 + 0.05\sin t)x_2(t-3)$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$

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$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)2.1x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)1.05x_2(t-3)$$

$$x_1'(t) \geq -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)(2 + 0.1\cos t)x_1(t-1)$$
$$x_2'(t) \geq x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)(1 + 0.05\sin t)x_2(t-3)$$

$$y_1'(t) = -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)1.9y_1(t-1)$$
$$y_2'(t) = y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)0.95y_2(t-3)$$

Examples

$$
\begin{aligned}
u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2)) \\
u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))
\end{aligned}
$$

$$
\begin{aligned}
u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1) \\
u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)
\end{aligned}
$$

$$
\begin{aligned}
x_1'(t) &= -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)2.1x_1(t-1) \\
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\end{aligned}
$$

$$
\begin{aligned}
x_1'(t) &\geq -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)(2 + 0.1\cos t)x_1(t-1) \\
x_2'(t) &\geq x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)(1 + 0.05\sin t)x_2(t-3)
\end{aligned}
$$

$$
\begin{aligned}
y_1'(t) &= -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)1.9y_1(t-1) \\
y_2'(t) &= y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)0.95y_2(t-3)
\end{aligned}
$$

$$
\begin{aligned}
y_1'(t) &\leq -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)(2 + 0.1\cos t)y_1(t-1) \\
y_2'(t) &\leq y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)(1 + 0.05\sin t)y_2(t-3)
\end{aligned}
$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$

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$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)2.1x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)1.05x_2(t-3)$$

$$x_1'(t) \geq -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)(2 + 0.1\cos t)x_1(t-1)$$
$$x_2'(t) \geq x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)(1 + 0.05\sin t)x_2(t-3)$$

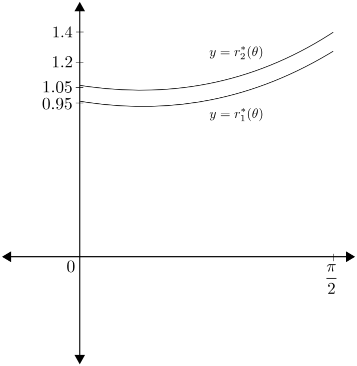
$$y_1'(t) = -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)1.9y_1(t-1)$$
$$y_2'(t) = y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)0.95y_2(t-3)$$

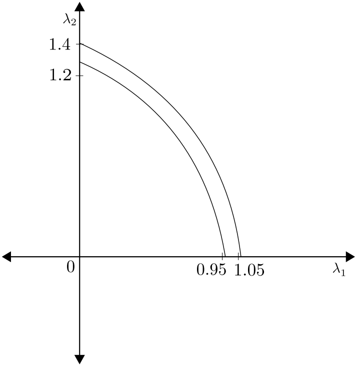
$$y_1'(t) \leq -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)(2 + 0.1\cos t)y_1(t-1)$$
$$y_2'(t) \leq y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)(1 + 0.05\sin t)y_2(t-3)$$

$$r_1(\theta) \leq r^*(\theta) \leq r_2(\theta)$$

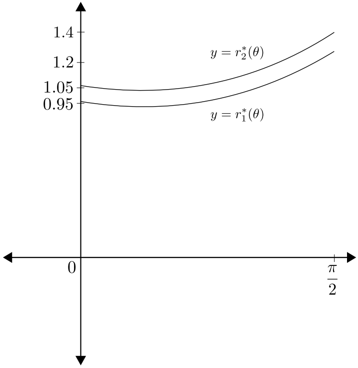
Examples

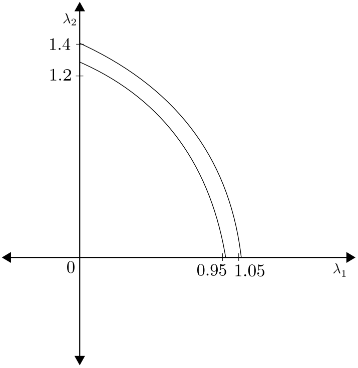
$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$





$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$

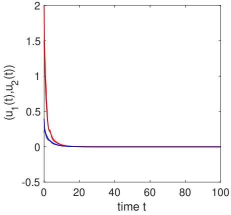




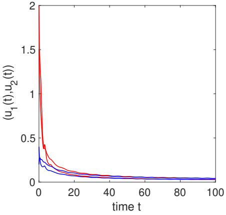
$$0.92371 \le r^*(\pi/6) \le 1.02095$$

Examples

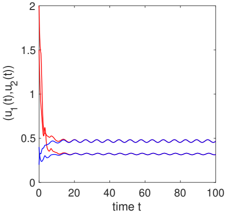
$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2 + 0.1\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1 + 0.05\sin t)u_2(t-3)\exp(-u_2(t-4))$$



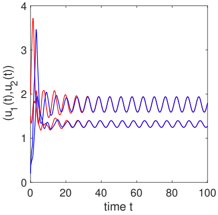
(a) $(r, \theta) = (0.5, \pi/6)$



(b) $(r, \theta) = (1, \pi/6)$



(c) $(r, \theta) = (1.5, \pi/6)$



(d) $(r, \theta) = (5, \pi/6)$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4))$$

Examples

$$\begin{aligned} u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2)) \\ u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4)) \end{aligned}$$

$$\begin{aligned} u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1) \\ u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3) \end{aligned}$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)3x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)2x_2(t-3)$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)3x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)2x_2(t-3)$$

$$y_1'(t) = -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)y_1(t-1)$$
$$y_2'(t) = y_1(t) - 2y_2(t)$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)$$

$$x_1'(t) = -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)3x_1(t-1)$$
$$x_2'(t) = x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)2x_2(t-3)$$

$$y_1'(t) = -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)y_1(t-1)$$
$$y_2'(t) = y_1(t) - 2y_2(t)$$

$$0.597 \le r^*(\pi/4) \le 2.828$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
$$u_2'(t) = u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4))$$

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)$$
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$$x_1(t) = 1 + a\sin t + b\cos t, \qquad x_2(t) = k(1 + c\sin t + d\cos t),$$
$$-0.7 \le a, b, c, d \le 0.7, \qquad 0.1 \le k \le 2$$

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$$-0.7 \le a, b, c, d \le 0.7, \qquad 0.1 \le k \le 2$$

$$r_{11}(k,a,b,c,d) \overset{def}{=} \min\left\{ \frac{x_1'(t) + 3x_1(t) - 2x_2(t)}{\cos(\theta)(2+\cos t)x_1(t-1)} : t \in [0, 2\pi] \right\},$$
$$r_{12}(k,a,b,c,d) \overset{def}{=} \min\left\{ \frac{x_2'(t) + 2x_2(t) - x_1(t)}{\sin(\theta)(1+\sin t)x_1(t-3)} : t \in [0, 2\pi] \right\},$$
$$r_1(k,a,b,c,d) \overset{def}{=} \min\{r_{11}(k,a,b,c,d), r_{12}(k,a,b,c,d)\},$$
$$r_1(\theta) \overset{def}{=} \max\{r_1(k,a,b,c,d) : k,a,b,c,d\}$$

Examples

$$
\begin{aligned}
u_1'(t) &= -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2)) \\
u_2'(t) &= u_1(t) - 2u_2(t) + r\sin(\theta)(1+\sin t)u_2(t-3)\exp(-u_2(t-4))
\end{aligned}
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\end{aligned}
$$

$$
y_1(t) = 1 + a\sin t + b\cos t, \qquad y_2(t) = k(1 + c\sin t + d\cos t),
$$
$$
-0.7 \le a, b, c, d \le 0.7, \qquad 0.1 \le k \le 2
$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
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$$y_1(t) = 1 + a\sin t + b\cos t, \qquad y_2(t) = k(1 + c\sin t + d\cos t),$$
$$-0.7 \le a, b, c, d \le 0.7, \qquad 0.1 \le k \le 2$$

$$r_{21}(k,a,b,c,d) \stackrel{def}{=} \max\left\{ \frac{y_1'(t) + 3y_1(t) - 2y_2(t)}{\cos(\theta)(2+\cos t)y_1(t-1)} : t \in [0, 2\pi] \right\},$$
$$r_{22}(k,a,b,c,d) \stackrel{def}{=} \max\left\{ \frac{y_2'(t) + 2y_2(t) - y_1(t)}{\sin(\theta)(1+\sin t)y_1(t-3)} : t \in [0, 2\pi] \right\},$$
$$r_2(k,a,b,c,d) \stackrel{def}{=} \max\{r_{21}(k,a,b,c,d), r_{22}(k,a,b,c,d)\},$$
$$r_2(\theta) \stackrel{def}{=} \min\{r_2(k,a,b,c,d) : k, a, b, c, d\}$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
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$$x_1'(t) \geq -3x_1(t) + 2x_2(t) + r_1(\theta)\cos(\theta)(2+\cos t)x_1(t-1)$$
$$x_2'(t) \geq x_1(t) - 2x_2(t) + r_1(\theta)\sin(\theta)(1+\sin t)x_2(t-3)$$

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\end{aligned}
$$

$$
\begin{aligned}
y_1'(t) &\leq -3y_1(t) + 2y_2(t) + r_2(\theta)\cos(\theta)(2+\cos t)y_1(t-1) \\
y_2'(t) &\leq y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)(1+\sin t)y_2(t-3)
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$$

Examples

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$$y_2'(t) \leq y_1(t) - 2y_2(t) + r_2(\theta)\sin(\theta)(1+\sin t)y_2(t-3)$$

$$r_1(\theta) \leq r^*(\theta) \leq r_2(\theta)$$

Examples

$$u_1'(t) = -3u_1(t) + 2u_2(t) + r\cos(\theta)(2+\cos t)u_1(t-1)\exp(-u_1(t-2))$$
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For $\theta = \pi/4$ we can choose

$$x_1(t) = 1 + 0.24\sin t - 0.15\cos t, \qquad x_2(t) = 0.95(1 + 0.46\sin t + 0.26\cos t),$$
$$y_1(t) = 1 + 0.27\sin t - 0.06\cos t, \qquad y_2(t) = 0.66(1 + 0.44\sin t - 0.05\cos t),$$

Examples

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Then

$$0.712 \le r^*(\pi/4) \le 1.391$$

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Comparison with autonomous systems

$$0.597 \le r^*(\pi/4) \le 2.828$$

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Comparison with autonomous systems

$$0.597 \le r^*(\pi/4) \le 2.828$$

Other angles

$$0.779 \le r^*(\pi/3) \le 1.801$$
$$0.812 \le r^*(\pi/5) \le 1.432$$