On some isoperimetric inequalities in the non-local setting

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Shape Optimization

Admissible shapes $\mathcal{O}_{\rm ad}$ - a family of admissible shapes.

- Bounded open sets contained in a design region $D \subset \mathbb{R}^d$. Additional constraints - volume, perimeter, number of holes, convexity etc.
- Class of polyhedra with given number of faces and given volume.

Shape functional

$$J : \mathcal{O}_{\rm ad} \to \mathbb{R}\,.$$

Shape Optimization Problem

$$\inf\{J(\Omega) : \Omega \in \mathcal{O}_{\rm ad}\}.$$

- Classical isoperimetric inequality Of all planar domains of given perimeter determining that which has the maximum area.

$$\sup\left\{\mathrm{Area}(\Omega) : \Omega \subset \mathbb{R}^2, \mathrm{Per}(\Omega) = L\right\}.$$

- Faber-Krahn-Rayleigh inequality Of all planar vibrating drums of given area determining that which has the lowest principal frequency.

$$\inf\left\{\lambda_1(\Omega) : \Omega \subset \mathbb{R}^2, \mathrm{Area}(\Omega) = a\right\}.$$

Classical Results

Isoperimetric inequality

- plane $L^2 \geq 4\pi A$.
- space $S^3 \geq 36\pi V^2$.
- higher dimensions $|\partial\Omega| \geq d\omega_d^{\frac{1}{d}}|\Omega|^{1-\frac{1}{d}}$; ω_d volume of the unit ball.

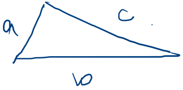
Pólya-Szegő inequality ($1 \leq p < \infty$)

$$\int_\Omega |\nabla u|^p \, dx \geq \int_{\Omega^\star} |\nabla u^\star|^p \, dx \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

Consequences

Faber-Krahn inequality, Sobolev inequality etc.

Isoperimetric inequality for triangles



$$2s = a + b + c = L.$$

$$A^2 = s\left[(s-a)(s-b)(s-c)\right]$$

$$\le s\left(\frac{s-a+s-b+s-c}{3}\right)^{\frac{1}{3}}$$

A.M–G.M

$$= s\left(\frac{s}{3}\right)^{1/3}$$

equality only if $s-a = s-b = s-c$

• $|\partial\Omega| \geq |\partial\Omega^*|$ when $|\Omega| = |\Omega^*|$.

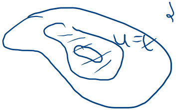
Fleming-Rischel

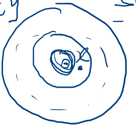
$$\int_\Omega |\nabla u| = \int_0^M \int_{\{u=t\}} dS\, dt$$

$$\geq \int_0^M P(\{u=t\}^*)\, dt$$

$$\geq \int_0^M \int_{\{u^*=t\}} dS\, dt = \int_{\Omega^*} |\nabla u^*|$$

$p=1$





Schwarz symm

$\Omega \Rightarrow \Omega^* = B$, $|B| = |\Omega|$

$u : \Omega \to \mathbb{R}$

$u^*(x) = \sup\{t : x \in \{u > t\}^*\}$

Faber-Krahn inequality

$$\lambda_1(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} ; u \in H^1_0(\Omega) \right\}$$

- $u \in H^1_0(\Omega)$, $u > 0$, $u \in C^\infty$
- $$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \geq \frac{\int_{\Omega^*} |\nabla u^*|^2}{\int_{\Omega^*} |u^*|^2} \geq \lambda_1(\Omega^*)$$

The Problem Setting - nonlocal functionals

Prinicipal Dirichlet eigenvalue for the fractional p-Laplacian

$\Omega \subseteq \mathbb{R}^d$ - bounded domain, $d > 1$; $0 < s < d$

$$\lambda_{1,p}^{s}(\Omega) = \inf\left\{ \frac{\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}}dxdy}{\displaystyle\int_\Omega |u(x)|^p dx} : u \in C_0^\infty(\Omega) \quad \text{y} \quad u \not\equiv 0 \right\},$$

Principal eigenvalue for the Riesz potential operator

$\Omega \subseteq \mathbb{R}^d$ - bounded domain, $d > 1$; $0 < \alpha < d$. For $u \in L^2(\Omega)$

$$(I_\alpha u)(x) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \int_\Omega \frac{u(y)}{|x-y|^{d-\alpha}} dy \qquad \text{a.e. } x \in \Omega$$

$$\lambda_1(\Omega) = \max\left\{ \int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{d-\alpha}}dxdy : u \in L^2(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}.$$

Connection between the Riesz potential operator y fractional Laplacian

- in $\mathbb{R}^d$ The Riesz potential operator is the inverse of the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$.
- in bounded domain The Riesz potential operator is the inverse of the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ together with a non-local boundary condition.

Main Results

- Theorem 1 The equilateral triangle has the least first eigenvalue for the fractional Dirichlet p-Laplacian among all triangles of given area. The square has the least first eigenvalue for the fractional Dirichlet p-Laplacian among all quadrilaterals of given area. Moreover, the equilateral triangle and the square are the unique minimizers in the above problems.
- Theorem 2 The maximum of $\lambda_1(\Omega)$ among all triangles (open) of given area is obtained when Ω is an equilateral triangle and only when Ω is an equilateral triangle. Similarly, the maximum of $\lambda_1(\Omega)$ among all quadrilaterals (open) of given area is obtained when Ω is a square and only when Ω is a square.

Key Principles

Riesz's inequality under Steiner symmetrization Let f, g, and h be non-negative Borel measurable functions that vanish at infinity on $\mathbb{R}^d$, and let $f^\star, g^\star$, and $h^\star$ be their respective Steiner symmetrizations with respect to a given hyperplane H taken, as above, as $\{x \in \mathbb{R}^d | x_d = 0\}$. Then, for $I(f,g,h) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(x-y)h(y)\, dxdy$, we have

$$I(f,g,h) \leq I(f^\star, g^\star, h^\star).$$

Continuity of the eigenvalue with respect to the convergence in the Hausdorff complementary distance Let B be a fixed compact set in $\mathbb{R}^d$ and Ω_n be a family of nonempty convex open subsets of B which converges, for the complementary Hausdorff distance, to a nonempty convex open set Ω. Then, $\lambda_1(\Omega) = \lim_{n \to \infty} \lambda_1(\Omega_n)$.

Properties of the eigenvalue

Properties of $\lambda^s_{1,p}$

1. (translation invariance) $\lambda^s_{1,p}(\Omega) = \lambda^s_{1,p}(\Omega + x)$ for all $x \in \mathbb{R}^n$.
2. (invariance under orthonormal transformations) $\lambda^s_{1,p}(\Omega) = \lambda^s_{1,p}(T(\Omega))$ for every orthonormal transformation T.
3. (homothety law) $\lambda^s_{1,p}(t\Omega) = t^{-sp}\lambda^s_{1,p}(\Omega)$ for $t > 0$.
4. (domain monotonicity) If $A \subset B$ are open sets, then $\lambda^s_{1,p}(B) \leq \lambda^s_{1,p}(A)$.

Properties of λ_1

1. (translation invariance) $\lambda_1(\Omega) = \lambda_1(\Omega + x)$ for all $x \in \mathbb{R}^d$.
2. (invariance under orthonormal transformations) $\lambda_1(\Omega) = \lambda_1(T(\Omega))$ for every orthonormal transformation $T \in O(n)$.
3. (homothety law) $\lambda_1(k\Omega) = k^\alpha \lambda_1(\Omega)$ for $k > 0$.
4. (domain monotonicity) Given bounded domains A and B in $\mathbb{R}^d$, if $A \subset B$, then $\lambda_1(A) \leq \lambda_1(B)$.

Steiner symmetrization of sets and functions with respect to the hyperplane $x_d = 0$

Steiner symmetrization of sets

$\Omega \subset \mathbb{R}^d$ - measurable set; for $x' \in \mathbb{R}^{d-1}$, we have the 1-d section of Ω en x':

$$\Omega(x') := \{x_n \in \mathbb{R} : (x', x_d) \in \Omega\},$$

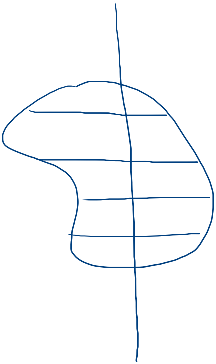
The set

$$\Omega^\star := \{x = (x', x_d) : -\frac{1}{2}|\Omega(x')| < x_d < \frac{1}{2}|\Omega(x')|, \}.$$

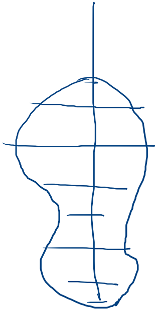
is the Steiner symmetrization of Ω with respect to the hyperplane $x_d = 0$.

Steiner symmetrization of functions Let f be a non-negative measurable function defined on Ω, which vanishes on $\partial\Omega$. The Steiner symmetrization of f is the function $f^\star$ defined on $\Omega^\star$ by

$$f^\star(x) = \sup\{c : x \in \{y \in \Omega : f(y) \geq c\}^\star\}$$



Ω



Ω^s (Steiner symmetrization)

Basic properties for Steiner symmetrization

A convex body is a compact convex set. For a convex body A in $\mathbb{R}^n$, the inradius $r(A)$ is the supremum of the radii of balls contained in A and the circumradius $R(A)$ is the infimum of the radii of balls containing A. The Steiner symmetrization of sets has the following properties. Let A, B be convex bodies. Then,

1. $A^\star \subseteq B^\star$ if $A \subseteq B$.
2. $r(A) \leq r(A^\star)$.
3. $R(A^\star) \leq R(A)$.
4. $V(A) = V(A^\star)$ where $V(A)$ denotes the volume of A.

The Steiner symmetrization of functions has the following properties.

1. The definitions of $A^\star$ y $f^\star$ are consistent, i.e.,

$$\chi_{A^\star} = (\chi_A)^\star \text{ and } \{x : f(x) \geq t\}^\star = \{x : f^\star(x) \geq t\}.$$

2. Let f be a nonnegative measurable function defined on Ω vanishing on $\partial\Omega$. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a measurable function. Then,

$$\int_\Omega F(f(x))dx = \int_{\Omega^\star} F(f^\star(x))dx.$$

3. Riesz's inequality.

Minkowski sums and differences

Minkowski sum Given $X, Y \subset \mathbb{R}^d$ is defined by

$$X \oplus Y := \bigcup_{y \in Y} (X + y).$$

Minkowski difference

$$X \ominus Y := \bigcap_{y \in Y} (X - y).$$

Inner parallel body at distance ϵ of K is the subset $K \ominus B(0, \epsilon)$.

Some properties

- Let $X, Z \subset \mathbb{R}^d$ with $X \subset Z$ and let $\epsilon > 0$. Then

$$X \ominus B(0, \epsilon) \subseteq Z \backslash ((Z \backslash X) \oplus B(0, \epsilon)).$$

- If $B(0, r) \subset K \subset B(0, R)$ and $0 < \epsilon < \frac{r^2}{4R}$, then

$$\left(1 - 4\frac{R\epsilon}{r^2}\right) K \subset K \ominus B(0, \epsilon) \subset K.$$

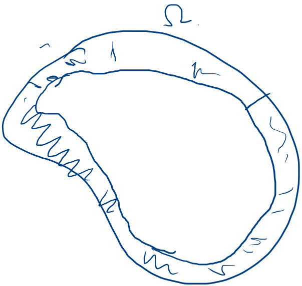
Hausdorff complementary distance

Let K and C be two non-empty compact sets in $\mathbb{R}^d$. Then their Hausdorff distance is defined as

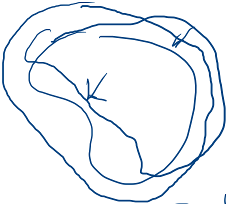
$$d^H(K, C) = \inf\{\epsilon \geq 0\,;\ K \subseteq C \oplus B(0, \epsilon) \text{ and } C \subseteq K \oplus B(0, \epsilon)\}$$

Let O_1, O_2 be two open subsets of a compact set B. Then the so called complementary Hausdorff distance is defined by:

$$d_H(O_1, O_2) = d^H(B \backslash O_1, B \backslash O_2).$$



$\Omega \ominus B(0,\varepsilon)$.



$\Omega_n \subseteq K$.

$\Omega_n \xrightarrow{d_H} \Omega$

$A_n \longrightarrow A$

$K \setminus \Omega_n \xrightarrow{d_H} K \setminus \Omega$

$\forall \varepsilon > 0$, $A_n \subseteq A \oplus B(0,\varepsilon)$

$n \geq$

$A \subseteq A_n \oplus B(0,\varepsilon)$

Continuity of the eigenvalue with respect to the convergence in the Hausdorff complementary distance

- Let $B(0,r) \subseteq \Omega \subset B(0,R)$. Since, $d_H(\Omega_n, \Omega) \to 0$, given $\epsilon > 0$

$$B \backslash \Omega_n \subset (B \backslash \Omega) \oplus B(0,\epsilon) \text{ for all } n \geq n_\epsilon .$$

From this we obtain

$$\Omega \ominus B(0,\epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon \text{ and so, for } 0 < \epsilon < \frac{r^2}{16R},$$

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega \subset \Omega \ominus B(0,\epsilon) \subset \Omega_n \text{ for all } n \geq n_\epsilon .$$

From this we get

$$\lambda_1(\Omega) \leq \varliminf_{n\to\infty} \lambda_1(\Omega_n).$$

- Similarly, starting from

$$B \backslash \Omega \subset (B \backslash \Omega_n) \oplus B(0,\epsilon) \text{ for all } n \geq n_\epsilon$$

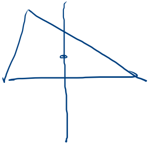
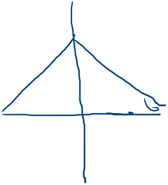
we can also obtain

$$\varlimsup_{n\to\infty} \lambda_1(\Omega_n) \leq \lambda_1(\Omega) .$$

Construction of the Steiner symmetrizations

Case of triangles Let $\triangle_1$ be an arbitrary open triangle of positive area a. We successively define the open triangles $\triangle_{n+1}$ by taking the Steiner symmetrization of $\triangle_n$ with respect to the perpendicular bisector of a side with respect to which there is no symmetry. It can be shown that that the sequence $\triangle_n$ converges with respect to the complementary Hausdorff distance to an open equilateral triangle $\triangle$.

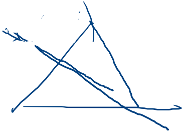
Case of quadrilaterals Apply Steiner symmetrization with respect to an axis which is perpendicular to a diagonal of the quadrilateral, for which the other two vertices aren't on the same side of this diagonal. The resulting object is a *convex* quadrilateral which is symmetric with respect to this axis. Next, we Steiner symmetrize with respect to a perpendicular axis and thereby get a rhombus. This is to be followed by a Steiner symmetrization with respect to an axis perpendicular to one of the sides to produce a rectangle. The rectangle is then Steiner symmetrized with respect to an axis perpendicular to a diagonal to get, again, a rhombus. By repeating the procedures for the rhombus and rectangle we end up with an infinite sequence of rhombi and rectangles which converge, ultimately, in the complementary Hausdorff distance, to a square.

Δ_1

$$\lambda_1(\Delta) \le \lambda_1(\Delta_n)$$

$$\lambda_1(\Delta) \le \lim \lambda_1(\Delta_n) = \lambda_1(\Delta^*)$$



$$\lambda_1(\Delta_n) = \int_{\Delta_n}\int_{\Delta_n} \frac{u_n(x)\,u_n(y)}{|x-y|^{d-\alpha}}\,dx\,dy$$

$$\le \int_{\Delta_{n+1}}\int_{\Delta_{n+1}} \frac{u_n^s(x)\,u_n^s(y)}{|x-y|^{d-\alpha}}\,dx\,dy \le \lambda_1(\Delta_{n+1})$$

Eigenvalue maximization for triangles of given area

Let f_n be an eigenfunction for $\lambda_1(\triangle_n)$. Let $f_n^\star$ be the corresponding Steiner symmetrization of $\widetilde{f_n}$ (the extension of f_n by zero outside $\triangle_n$) which is a Borel measurable function vanishing at infinity. Applying Riesz's inequality to the function $\widetilde{f_n}$ (taken twice) and with the function in the middle taken as the Riesz potential we obtain

$$\begin{aligned}
\lambda_1(\triangle_n) &= \int_{\triangle_n}\int_{\triangle_n} \frac{f_n(x)f_n(y)}{|x-y|^{2-\alpha}}\,dxdy \\
&\leq \int_{\triangle_{n+1}}\int_{\triangle_{n+1}} \frac{f_n^\star(x)f_n^\star(y)}{|x-y|^{2-\alpha}}\,dxdy \\
&\leq \max_{w\in L^2(\triangle_{n+1})}\left\{\int_{\triangle_{n+1}}\int_{\triangle_{n+1}} \frac{w(x)w(y)}{|x-y|^{2-\alpha}}\,dxdy : \|w\|_{L^2(\Delta_{n+1})} = 1\right\} \\
&= \lambda_1(\triangle_{n+1})
\end{aligned}$$

for each n and so $\lambda_1(\triangle_1) \leq \lambda_1(\triangle_n)$ for all n. So by the continuity property of the Riesz eigenvalue for the complementary Hausdorff convergence we get

$$\lambda_1(\triangle_1) \leq \lim_{n\to\infty} \lambda_1(\triangle_n) = \lambda_1(\triangle).$$

The equilateral triangle is the unique maximizer

Let Δ be an open triangle of given area which maximizes. If Δ is not already an equilateral triangle, then there is at least one axis m (perpendicular to one of its sides) such that Δ is not Steiner symmetric with respect to m. Let $\Delta^\star$ be the Steiner symmetrization of Δ respect to m. We take f to be a continuous positive eigenfunction of norm 1 (in the L^2 norm) associated to $\lambda_1(\Delta)$. Let $f^\star$ be the Steiner symmetrization of $\widetilde{f}$ (the extension of f by zero outside Δ) with respect to m. We note that $f^\star$ has to be supported on the closure of $\Delta^\star$. We note that $f^\star$ also has norm 1 in the L^2 norm. So, by Riesz's inequality, we obtain

$$\begin{aligned}\lambda_1(\Delta^\star) &\geq \int_{\Delta^\star}\int_{\Delta^\star} \frac{f^\star(x)f^\star(y)}{|x-y|^{2-\alpha}}dxdy \\ &\geq \int_{\Delta}\int_{\Delta} \frac{f(x)f(y)}{|x-y|^{2-\alpha}}dxdy \\ &= \lambda_1(\Delta)\,.\end{aligned}$$

So, $\lambda_1(\Delta^\star) = \lambda_1(\Delta)$ leading to the equality case in Riesz's inequality. Therefore, f is a translate of $f^\star$ up to a set of measure 0. Furthermore, $f^\star$ is a maximizer for $\lambda_1(\Delta^\star)$ and so, by Jentzsch's theorem, is continuous and positive on $\Delta^\star$. We, therefore, have

$$\Delta = \{x \in \Delta : f(x) > 0\} = \{x \in \mathbb{R}^d : f^\star(x-y) > 0\} = \Delta^\star + y\,.$$

Thank you for your kind attention!

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