Explicit and implicit control problems. A fixed point approach

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Łódź, 2023

General problem of control theory

Reconstitution of a (differential/integral) system, as a matter of fact some of its parameters viewed as control variables, from certain properties of solution.

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Basic reference: **JM Coron**: *Control and Nonlinearity*, AMS 2007

RP: On some applications of the controllability principle for fixed point equations, *Results in Applied Mathematics* 13 (2022), 100236.

A Hofman, RP: On some control problems for Kolmogorov type systems, *Mathematical Modelling and Control* 2022, 2(3).

LG Parajdi, F Patrulescu, RP, IS Haplea: Two numerical methods for solving a nonlinear system of integral equations of mixed Volterra-Fredholm type arising from a control problem related to leukemia, *Journal of Applied Analysis and Computation* doi: 10.11948/20220197.

Outline of my talk

1. **Some examples of control problems**

- explicit
- implicit

2. **General control problem of fixed point equations**

3. **Techniques of solving control problems**

- basic fixed point principles (Banach,Schauder,Leray-Schauder...)
  An example: control of Lotka-Volterra system

- vector fixed point techniques (Perov ...)
  An example: control of a Kolmogorov sistem

- approximation and numerical techniques:
  Lower and upper solution method

Examples of explicit control problems

1) A control problem on the Lotka-Volterra prey-predator system

$$\begin{cases} x' = ax\left(1 - \lambda by\right) & \text{(control of the attack rate } b\text{)} \\ y' = -cy\left(1 - dx\right) & \\ & \\ x\left(0\right) = x_0, \ y\left(0\right) = y_0, & x\left(T\right) = x_T \end{cases}$$

2) Control by vaccination of the SIR epidemiologic model

$$\begin{cases} S' = -aSI - \lambda & \text{(control of the infection rate)} \\ I' = aSI - bI & \\ R' = bI & \\ & \\ S(0) = S_0, \ I(0) = I_0, & R(T) = pN \end{cases}$$

Examples of implicit control problems

1) Nonlocal problems

$$\begin{cases} u' = f(t, u), & t \in [0, T] \\ g(u) = 0 \end{cases}$$

Integration leads to fixed point equation

$$\begin{cases} u(t) = \lambda + \int_0^t f(s, u(s))ds & (t \in [0, T]) \\ \\ g(u) = 0 & \text{(controllability condition)} \end{cases}$$

where $\lambda = u(0)$

2) Radial solutions of Neumann BVP in annulus

$$\begin{cases} -\text{div}\ \left(\phi\left(\left|\nabla u\right|\right)\nabla u\right) + u = f\left(\left|x\right|, u\right) & \text{in } \Omega : R_0 < |x| < R_1 \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\phi : (-a, a) \to \mathbb{R} \text{ - increasing homeomorphism}$$

Radial solution: $u(x) = v(|x|)$

$$\begin{cases} -\left(r^{n-1}\phi\left(v'\right)\right)' + r^{n-1}v = r^{n-1}f\left(r, v\right) & \text{in } \left(R_0, R\right) \\ v'\left(R_0\right) = v'(R) = 0. \end{cases}$$

Double integration leads to fixed point equation

$$v\left(r\right) = \lambda + \int_{R_0}^{r} \phi^{-1}\left(s^{1-n}\int_s^R \tau^{n-1}\left(f(\tau, v) - v\right)d\tau\right)ds \quad \left(r \in \left[R_0, R\right]\right)$$

$$\int_{R_0}^{R} r^{n-1}\left(f\left(r, v\right) - v\right)dr = 0 \quad \text{(controllability condition)}$$

where $\lambda = v\left(R_0\right)$

3) The nonlinear stationary Stokes system

$$\begin{cases} -\Delta u + \nabla p = \Phi(u) & \text{in } \Omega \\ \text{div } u = 0 \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

Physically: $u =$ fluid velocity
$p =$ pressure due to the medium porosity
$\Phi(u) =$ external reaction force

Denoting $\lambda := \nabla p$ the problem becomes

$$\begin{cases} u = (-\Delta)^{-1}(\Phi(u) - \lambda), & u \in H_0^1(\Omega; \mathbb{R}^n) \\ \text{div } u = 0 & \text{(controllability condition)} \end{cases}$$

$$\begin{cases} w = N(w, \lambda) \\ w \in W, \quad \lambda \in P, \quad (w, \lambda) \in D \end{cases} \tag{1}$$

w = state variable, λ = control variable, W = domain of the states, P = domain of controls, D = controllability domain/condition

$$\Sigma := \{(w, \lambda) \in W \times P : \ w = N(w, \lambda)\}$$

1. **Give expression of control variable λ in terms of state variable w:**

$$\lambda = \Lambda(w)$$

2. **Replace and get the fixed point equation without parameter:**

$$w = N(w, \Lambda(w))$$

3. **Find a fixed point w^***

$$(w^*, \Lambda(w^*)) \in \Sigma \cap D \quad \text{(i.e., a solution of the control problem)}$$

- **Existence by basic fixed point principles**

EXAMPLE: Control of Lotka-Volterra system

$$\begin{cases} x' = ax\left(1 - \lambda by\right) \\ y' = -cy\left(1 - dx\right) \\ x\left(0\right) = x_0,\ y\left(0\right) = y_0, \quad x\left(T\right) = x_T,\ x, y > 0. \end{cases}$$

Theorem

Let $x_0 < x_T < 2x_0$. *The system is controllable on a short time interval* $[0, T]$ *with*

$$\begin{aligned} T &< \min\left\{\frac{1}{4a}, \frac{1}{c}, \frac{2x_0 - x_T}{4ax_0}\right\}, \\ 1 - 2aT &> 2cT\left(1 - 2aT + dx_T\right). \end{aligned}$$

PROOF: Assume $b = 1$. Integrating gives

$$\begin{cases} x(t) = x_0 + a\int_0^t x(s)(1 - \lambda y(s))\,ds \\ y(t) = y_0 - c\int_0^t y(s)(1 - dx(s))\,ds. \end{cases}$$

Using the controllability condition $x(T) = x_T$ we obtain the necessary form of the control parameter

$$\lambda = \Lambda(x, y) := \frac{a\int_0^T x(s)\,ds - x_T + x_0}{a\int_0^T x(s)\,y(s)\,ds}.$$

Replacing gives

$$\begin{cases} x(t) = A(x, y) := x_0 + a\int_0^t x(s)(1 - \Lambda(x, y)\,y(s))\,ds \\ y(t) = B(x, y) := y_0 - c\int_0^t y(s)(1 - dx(s))\,ds. \end{cases}$$

We look for solution in the bounded set

$$D_0 := \left\{ (x, y) \in C\left([0, T] ; \mathbb{R}^2 \right) : \; m_x \leq x \leq M_x, \; m_y \leq y \leq M_y \right\}$$

where

$$m_x = \frac{2x_0(1-2aT)-x_T}{1-2aT}, \qquad M_x = \frac{x_T}{1-2aT},$$
$$m_y = y_0 \frac{1-2cT(1+dM_x)}{1-cT(1+dM_x)}, \quad M_y = \frac{y_0}{1-cT(1+dM_x)}.$$

Then the operator $N = (A, B)$ satisfies

$$N(D_0) \subset D_0$$

In addition, N is compact. Thus Schauder's fixed point theorem applies.

▶ Vector fixed point techniques

EXAMPLE: Control of a Kolmogorov system

$$\begin{cases} x'(t) = x(t)[f(x(t), y(t)) - \lambda_1] \\ y'(t) = y(t)[g(x(t), y(t)) - \lambda_2] \end{cases}$$

Controllability conditions: $x(T) = x_T, \quad y(T) = y_T$

Denote $C_1 := |\ln x_0| + |\ln(x_0/x_T)|, \quad C_2 := |\ln y_0| + |\ln(y_0/y_T)|$

Theorem

Let $f, g : [0, \rho]^2 \to \mathbb{R}$ *be bounded by a constant* M. *Assume that for all* $x, y, \bar{x}, \bar{y} \in [0, \rho]$:

$$\begin{aligned} |f(x,y) - f(\bar{x}, \bar{y})| &\leq a_{11}|x - \bar{x}| + a_{12}|y - \bar{y}| \\ |g(x,y) - g(\bar{x}, \bar{y})| &\leq a_{21}|x - \bar{x}| + a_{22}|y - \bar{y}| \end{aligned}$$

Then, if $0 < T \leq min\{\ln\rho - C_1, \ \ln\rho - C_2\}/(2M)$ *and the matrix* $\mathcal{M} := 2\rho T[a_{ij}]$ *converges to zero, the problem has a unique solution* $(x^*, y^*, \lambda_1^*, \lambda_2^*)$ *with* $\|x^*\|_\infty \leq \rho, \ \|y^*\|_\infty \leq \rho$.

PROOF: 1) Change of variables: $x = e^u$, $y = e^v$, $u_0 = \ln x_0$, $v_0 = \ln y_0$,

$$u(T) = u_T = \ln x_T, \quad v(T) = v_T = \ln y_T.$$

2) Substitution and integration gives

$$\begin{cases} u(t) = u_0 + \int_0^t f(e^{u(s)}, e^{v(s)})ds - \lambda_1 t \\ v(t) = v_0 + \int_0^t g(e^{u(s)}, e^{v(s)})ds - \lambda_2 t \end{cases}$$

3) Use the controllability condition to get the control expressions

$$\lambda_1 = \frac{1}{T}\left(u_0 - u_T + \int_0^T f(e^{u(s)}, e^{v(s)})ds\right)$$
$$\lambda_2 = \frac{1}{T}\left(v_0 - v_T + \int_0^T g(e^{u(s)}, e^{v(s)})ds\right)$$

4) Replace and get the fixed point system for the operator $N = (A, B)$ with:

$$A(u,v)(t) = u_0 - \frac{t}{T}(u_0 - u_T) - \frac{t}{T}\int_0^T f(e^u, e^v)ds + \int_0^t f(e^u, e^v)ds$$
$$B(u,v)(t) = v_0 - \frac{t}{T}(v_0 - v_T) - \frac{t}{T}\int_0^T g(e^u, e^v)ds + \int_0^t g(e^u, e^v)ds$$

5) Apply Perov's fixed point theorem in the set

$$D_R := \{(u,v) \in C([0,T]; \mathbb{R}^2) : \ ||u||_\infty \le R, \ ||v||_\infty \le R\}$$

where $R = \ln \rho$.

- $N(D_R \subset D_R$
- N is a Perov contraction:

$$\begin{bmatrix} ||A(u,v) - A(\bar{u}, \overline{v})||_\infty \\ ||B(u,v) - B(\bar{u}, \overline{v})||_\infty \end{bmatrix} \le \mathcal{M} \begin{bmatrix} ||u - \bar{u}||_\infty \\ ||v - \overline{v}||_\infty \end{bmatrix}$$

where the matrix $\mathcal{M}$ is assumed convergent to zero.

A second result based on Schauder's FPT:

Theorem

Let $f, g : \mathbb{R}^2_+ \to \mathbb{R}$ *be continuous and satisfy growth conditions*

$$\begin{aligned} |f(x,y)| &\leq a_{11} |\ln x| + a_{12} |\ln y| + b_1, \\ |g(x,y)| &\leq a_{21} |\ln x| + a_{22} |\ln y| + b_2, \end{aligned}$$

for all $x, y \in (0, \infty)$. *Then for each* $T > 0$ *for which the matrix* $\mathcal{M} = 2T[a_{ij}]$ *is convergent to zero, the control problem has at least one positive solution* $(x^*, y^*, \lambda_1^*, \lambda_2^*)$.

PROOF: Apply Schauder FPT in a set of the form

$$D = B_{R_1} \times B_{R_2},$$

where $B_{R_i} = \{w \in C([0, T]; \mathbb{R}_+) : \|w\|_\infty \leq R_i\} \ (i = 1, 2)$.
We need to prove that there are two positive numbers R_1 and R_2 such that the following invariance condition be satisfied:

$$\|u\|_\infty \leq R_1, \ \|v\|_\infty \leq R_2 \quad \Rightarrow \quad \|A(u, v)\|_\infty \leq R_1, \ \ \|B(u, v)\|_\infty \leq R_2$$

Use the growth conditions to get in the vector form:

$$\begin{bmatrix} \|A(u, v)\|_\infty \\ \|B(u, v)\|_\infty \end{bmatrix} \leq \mathcal{M} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Thus it suffices to have

$$\mathcal{M} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

equivalently:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \leq (I - \mathcal{M}) \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

If the matrix $\mathcal{M}$ is convergent to zero, then $(I - \mathcal{M})^{-1} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ and thus

$$(I - \mathcal{M})^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

This inequality allows the choice of the radii $R_1, R_2 > 0$ to guarantee the invariance condition. Thus Schauder's fixed point theorem can be applied in $B_{R_1} \times B_{R_2}$.

Lower and upper solution method

Assume a partition of the solution domain:

$$W \times P = \underline{D} \cup \overline{D}, \qquad \underline{D} \cap \overline{D} = D$$

Thus the condition of controllability can be targeted from the left or from the right.

Definition

By a lower (upper) solution of problem (1) we mean a pair $(w, \lambda) \in \Sigma \cap \underline{D}$ $(\Sigma \cap \overline{D})$.

Assume: W is a metric space, $P = conv\left\{\underline{\lambda}_0,\ \overline{\lambda}_0\right\}$ is a segment of a normed space and the sets $\underline{D}$, $\overline{D}$ are closed. In addition:

- (h1) The problem has a lower solution $\left(\underline{w}_0,\ \underline{\lambda}_0\right)$ and an upper solution $\left(\overline{w}_0,\ \overline{\lambda}_0\right)$
- (h2) For each $\sigma \in [0, 1]$, there is a unique $w =: S(\sigma) \in W$ with $(w, \lambda) \in \Sigma$ for $\lambda = \lambda(\sigma) := (1-\sigma)\underline{\lambda}_0 + \sigma\overline{\lambda}_0$
- (h3) The map $S : [0, 1] \to W$ is continuous

Bisection algorithm

Initialization: $\underline{\sigma}_0 = 0$ and $\overline{\sigma}_0 = 1$.
For $k \geq 1$ execute

Step 1. Calculate

$$\sigma := \frac{\underline{\sigma}_{k-1} + \overline{\sigma}_{k-1}}{2}$$

and find $S(\sigma)$.

Step 2. If $(S(\sigma), \lambda(\sigma)) \in D$, we are finished and $(S(\sigma), \lambda(\sigma))$ is a solution of the control problem; otherwise take

$$\begin{array}{ll}\text{(a)} & \underline{\sigma}_k := \sigma, \quad \overline{\sigma}_k := \overline{\sigma}_{k-1} \text{ if } (S(\sigma), \lambda(\sigma)) \in \underline{D} \setminus D; \\ \text{(b)} & \underline{\sigma}_k := \underline{\sigma}_{k-1}, \ \overline{\sigma}_k := \sigma \quad \text{if } (S(\sigma), \lambda(\sigma)) \in \overline{D} \setminus D,\end{array}$$

then set $k := k + 1$ and go to **Step 1**.

The algorithm leads either to a solution when it stops, or to two sequences $(\underline{\sigma}_k)$ and $(\overline{\sigma}_k)$ such that

(i) $0 \leq \underline{\sigma}_1 \leq \cdots \leq \underline{\sigma}_k \leq \underline{\sigma}_{k+1} \leq \cdots \leq 1$ and $(S(\underline{\sigma}_k), \lambda(\underline{\sigma}_k)) \in \underline{D}$

(ii) $0 \leq \cdots \leq \overline{\sigma}_{k+1} \leq \overline{\sigma}_k \leq \cdots \leq \overline{\sigma}_1 \leq 1$ and $(S(\overline{\sigma}_k), \lambda(\overline{\sigma}_k)) \in \overline{D}$

(iii) $0 \leq \overline{\sigma}_k - \underline{\sigma}_k = \frac{1}{2^k}$.

Then the sequences $(\underline{\sigma}_k)$ and $(\overline{\sigma}_k)$ are convergent, have the same limit σ^* and $(S(\sigma^*), \lambda(\sigma^*)) \in \underline{D}$, $(S(\sigma^*), \lambda(\sigma^*)) \in \overline{D}$. Hence $(S(\sigma^*), \lambda(\sigma^*)) \in D$ that is $(S(\sigma^*), \lambda(\sigma^*))$ solves the control problem.

Theorem

Under assumptions (h1)-(h3), the Algorithm is convergent to a solution (w^*, λ^*) of the control problem.

APPLICATION: control of cell evolution after bone marrow transplantation

$$
\begin{cases}
x' = \frac{a}{1+b(x+y+z)} \cdot \frac{x+y}{x+y+\lambda_1 gz} x - cx \\
y' = \frac{\lambda_4 A}{1+B(x+y+z)} \cdot \frac{x+y}{x+y+\lambda_2 Gz} y - \lambda_5 Cy \\
z' = \frac{a}{1+b(x+y+z)} \cdot \frac{z}{z+\lambda_3 h(x+y)} z - cz
\end{cases}
$$

$x(t), y(t), z(t)$	=	normal, leukemic, donor cells
a, A	=	growth rates of normal/ leukemic cells
b, B	=	sensibility rates
c, C	=	cell death rates of normal/ leukemic cells
h, g, G	=	anti-graft/ anti-host/ anti-cancer effects

Therapies:
- immunosuppressive therapy (related to h)
- post-transplant consolidation chemotherapy (related to A, C)
- donor T-lymphocyte infusion (related to g, G)

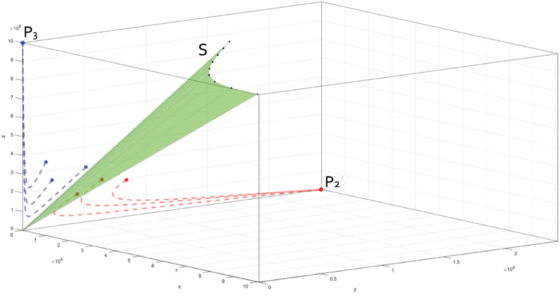

Figure 1: Border between the basins of attraction of the two asymptotically stable equilibria, "good" and "bad"

A good indicator of the location in the "bad" or "good" basin is:

$$\frac{z}{x} < \sqrt{\frac{h}{g}} \quad (\text{"bad" basin}) \qquad \frac{z}{x} > \sqrt{\frac{h}{g}} \quad (\text{"good" basin})$$

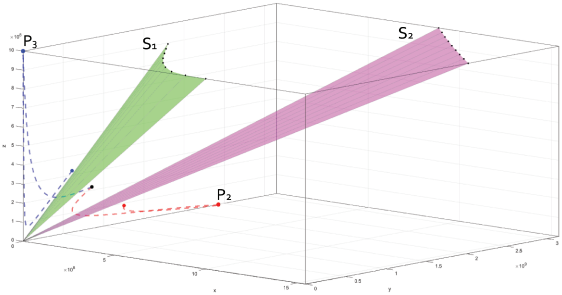

Figure 2: showing that the basin of attraction of the good equilibrium becomes larger by changing parameters

Assume that at transplant time $t = 0$ patient's condition $(x(0), y(0), z(0))$ is in the "bad" basin i.e. $\frac{z(0)}{x(0)} < \sqrt{\frac{h}{g}}$

The aim is that in a short time T, his condition is brought into the good "basin" i.e. $\frac{z(T)}{x(T)} > \sqrt{\frac{h}{g}}$

For a lower solution $\left(\underline{x}_0, \underline{y}_0, \underline{z}_0, \underline{\lambda}_0\right)$ we take the vector $\underline{\lambda}_0 = (1, 1, 1, 1, 1)$ which corresponds to the absence of any post-transplant therapy. Then

$$\frac{\underline{z}_0\,(T)}{\underline{x}_0\,(T)} < \sqrt{\frac{h}{g}}.$$

An upper solution $\left(\overline{x}_0, \overline{y}_0, \overline{z}_0, \overline{\lambda}_0\right)$ is chosen by checking several vectors λ to have

$$\frac{\overline{z}_0\,(T)}{\overline{x}_0\,(T)} > \sqrt{\frac{h}{g}}.$$

The algorithm continues until the first step k at which, for $\lambda = \lambda\left(\overline{\sigma}_k\right)$, one has

$$\frac{z\,(T)}{x\,(T)} \leq \sqrt{\frac{h}{g}} + \delta$$

for an acceptable margin $0 < \delta < \overline{z}_0\,(T)\,/\overline{x}_0\,(T) - \sqrt{h/g}$. Then the vector $\lambda\left(\overline{\sigma}_k\right) = \left(1 - \overline{\sigma}_k\right)\underline{\lambda}_0 + \overline{\sigma}_k\overline{\lambda}_0$ can be a good approximation for the control λ.

Referring to the general framework, we have:

$$W = C\left([0,T];(0,+\infty)^3\right), \; w = (x,y,z),$$
$$\Lambda = \{\lambda(\sigma): \; \sigma \in [0,1]\}, \; \text{where} \; \lambda(\sigma) = (1-\sigma)\underline{\lambda}_0 + \sigma\overline{\lambda}_0,$$

$$D = \left\{(w,\lambda): \; \frac{z(T)}{x(T)} = \sqrt{\frac{h}{g}}\right\}$$

$$\underline{D} = \left\{(w,\lambda): \; \frac{z(T)}{x(T)} \leq \sqrt{\frac{h}{g}}\right\}$$

$$\overline{D} = \left\{(w,\lambda): \; \frac{z(T)}{x(T)} \geq \sqrt{\frac{h}{g}}\right\}$$

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Thank you for your attention!