The existence of symbolic dynamics for Kuramoto-Sivashinski PDE

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What is talk is about?

- We want to study a dynamics of dissipative PDEs. We want to apply finite dimensional tools from the dynamics of ODEs.
- We consider Galerkin projections and pass to the limit.
- We need explicit bounds of good quality that enable us to pass to the limit with the dimension of the Galerkin projection
- We want to do computer assisted proofs, if necessary.
- We need coordinates!!!
- we need theorems in finite dimensions, which behave well with respect to multidimensional perturbations (when adding 'contracting' directions)

Plan of the talk

1. Description of the main result: the existence of symbolic dynamics (chaos) for KS with $\nu = 0.1212$
2. What is the interval arithmetic, what is a computer assisted proof
3. Method of self-consistent bounds for the study of dynamics of dissipative PDEs, computer assisted proofs, examples of results
4. Some details ...

A Model Problem - Kuramoto-Sivashinsky PDE

Consider the Kuramoto-Sivashinsky (KS) ej.

$$u_t = -\nu u_{xxxx} - u_{xx} + (u^2)_x, \qquad \nu > 0$$

where $(t, x) \in [0, \infty) \times \mathbb{R}$ subject to periodic and odd boundary conditions

$$\begin{aligned} u(t,0) &= u(t,2\pi) \\ u(t,-x) &= -u(t,x) \end{aligned}$$

For various values of ν a variety of dynamics,
fixed points,
periodic orbits,
heteroclinic orbits,
chaotic dynamics,
have been observed numerically.
Goal: A rigorous means of proving these numerical results.

A Model Problem - Kuramoto-Sivashinsky PDE, Fourier expansion

Fourier expansion is: $u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}$
Substituting in KS and applying boundary conditions gives:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}$$

where $b_k = ia_k$ and $k = 1, 2, 3, \ldots$.
Linearization: $\dot{a}_k = k^2(1-\nu k^2)a_k$

- k-th mode is unstable for $k < \frac{1}{\sqrt{\nu}}$
- k-th mode is stable for $k > \frac{1}{\sqrt{\nu}}$
- the modes with $k >> \frac{1}{\sqrt{\nu}}$ should be irrelevant for the dynamics

A Model Problem - Kuramoto-Sivashinsky PDE, known results

Known analytical results:

- the existence of global attractor, the functions from attractor are analytic - Fourier series converge at geometric rate (Foias, Temam)
- the existence of finite dimensional inertial manifold (Foias, Nicolaenko, Sell, Temam, Rossa, Jolly) (not of much use in rigorous numerics)

No analytical results on dynamics more complicated than fixed points bifurcating from zero solution

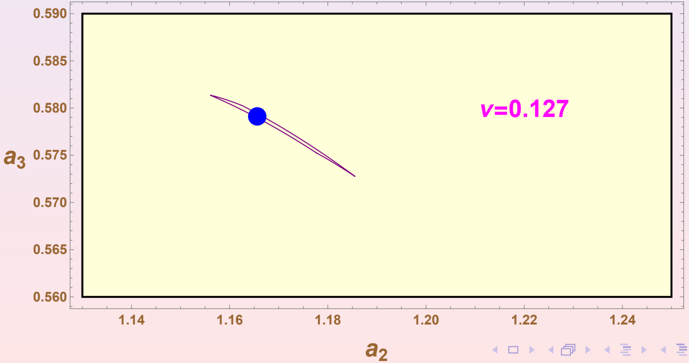
Our rigorous results for Kuramoto-Sivashinsky PDE

- the existence of multiple periodic orbits for various parameter values $\nu \approx 0.1215, 0.1212, 0.125, 0.032, 0.02991$, both stable and unstable orbits
- the existence of multiple fixed points for various values o f ν and their bifurcations
- the existence of attractive fixed points for various values of ν
- the existence of heteroclinic connection between zero and unimodal fixed point for $\nu = 0.75$
- the existence of symbolic dynamics for $\nu = 0.1212$ today

Rigorous results for periodic orbits Kuramoto-Sivashinsky PDE, CAP

- $\nu = 0.127 + [-1, 1] \cdot 10^{-7}$, $\nu = 0.125$, $\nu = 0.1215$, $\nu = 0.032$, (branches of) (symmetric) periodic orbits

Zgliczyński, FoCM'2004, TMNA'2010



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  Figueras, Gameiro, Lessard, de la Llave '2017
- $\nu = 0.1212$ - **chaos, countable infinity of periodic orbits**
  Zgliczyński, Wilczak, JDE'2021
- $\nu = 0.1212$ - **countable infinity of connecting orbits**
  Zgliczyński, Wilczak, 2024?

Kuramoto-Sivashinsky equations

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}$$

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2π-periodic, odd

$$u(t,x) = -2\sum_{k=1}^{\infty} a_k(t)\sin(kx)$$

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Infinite dimensional ODE

$$a_k' = k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{\infty} a_n a_{n+k}\right)$$

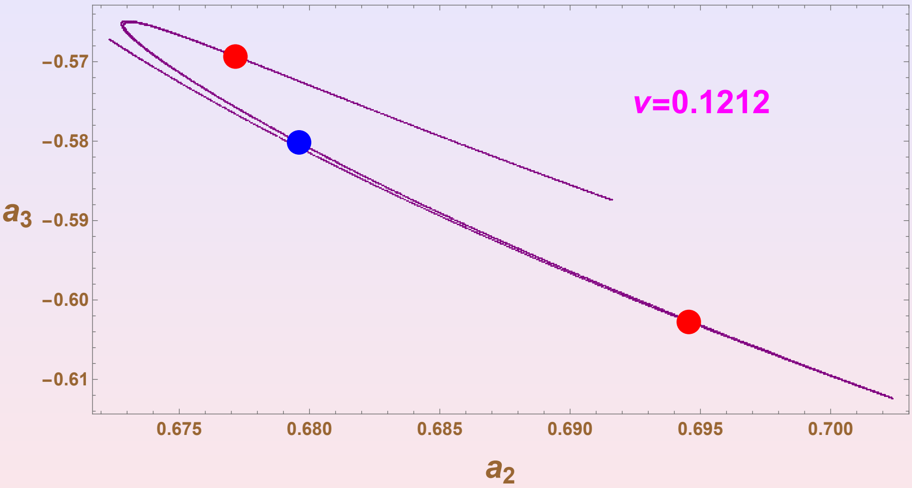
M-dimensional Galerkin projection

$$a_k' = k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{M-k} a_n a_{n+k}\right)$$

$\Pi_M = \{a_1 = 0 \wedge a_1' < 0\}$ - Poincaré section

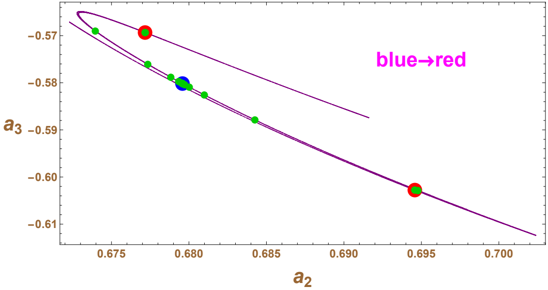
$P_M : \Pi_M \to \Pi_M$ - Poincaré map

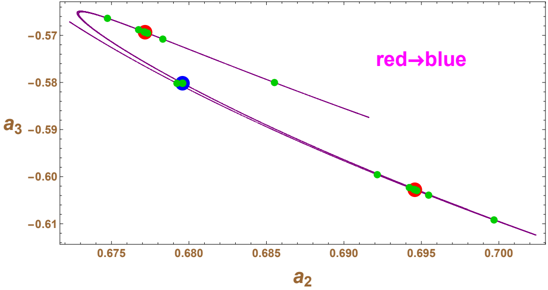
Observed chaotic attractor for P_M



Click here to run animation

Approximate heteroclinic orbits





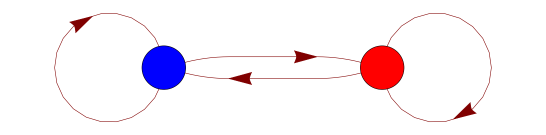
Main result

Consider full infinite dimensional system:

$\Pi = \{a_1 = 0 \wedge a_1' < 0\} \qquad P : \Pi \to \Pi$

Theorem (PZ,DW, JDE'2021)

- *There is an invariant set $\mathcal{H} \subset \Pi$ on which P is semiconjugated to a subshift of finite with positive topological entropy*
- *$\mathcal{H}$ contains countable infinity of periodic orbits*



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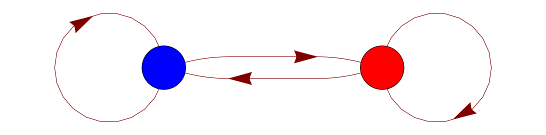

Wall time: 40 minutes on 64CPUs

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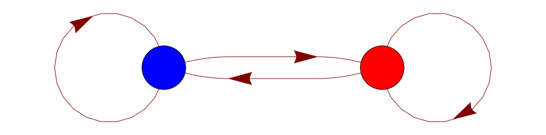


Wall time: 40 minutes on 64CPUs

Corollary: the same result for all Galerkin projections $M > 23$.

C^1 - result

Theorem (PZ, DW, '2024)

Periodic orbits **B** *and* **R** *are hyperbolic and*

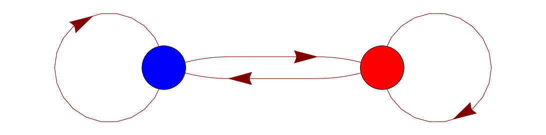
every biinfinte homoclinic/heteroclinic paths on the graph

$$\downarrow$$

homoclinic/heteroclinic solution **u** *of the KS equation*

$$\alpha(\mathbf{u}), \omega(\mathbf{u}) \in \{\mathbf{B}, \mathbf{R}\}$$

- countable infinity of periodic orbits with unbounded periods
- countable infinity of homo/hetero orbits between **B** and **R**



Why $\nu = 0.1212$? Why odd boundary conditions?

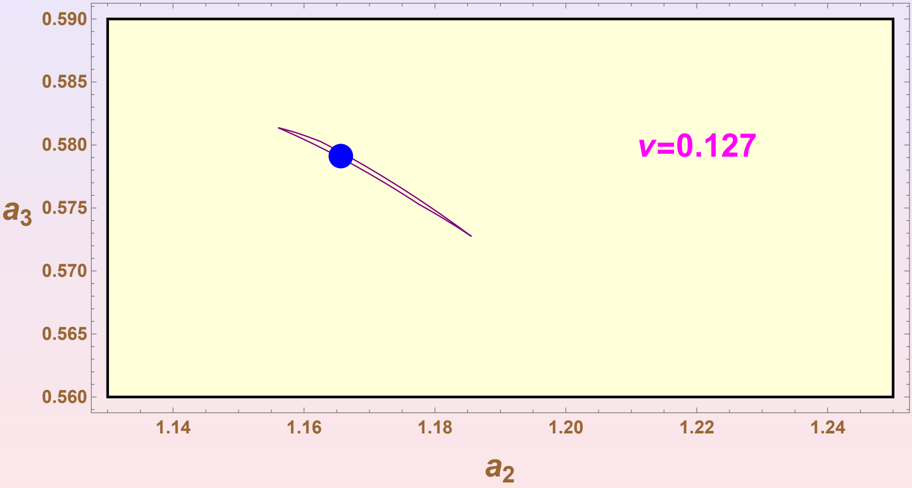
- the Feigenbaum route to chaos through successive period doubling bifurcations apparently happens for ν decreases toward $\nu = 0.1212$.
- KS with periodic boundary conditions has the translational symmetry, which implies that for fixed value of ν periodic orbits are members of one-parameter families of periodic orbits. This is bad for geometric methods
  The restriction to the subspace of odd functions breaks this symmetry and gives a hope that the dynamically interesting objects are topologically isolated.

About the approach

- a mixture of rigorous numerics and geometric methods in dynamics
- the topological part : exploits an apparent existence of transversal heteroclinic connections of two periodic orbits in both directions. The two approximate heteroclinic orbits connecting the periodic orbits are then used to obtain the topological horseshoe for some higher iterate of the Poincaré map. Cone conditions near selected periodic orbits give us homo- and heteroclinic connections
- rigorous numerics: uses topology to obtain a priori bounds for short time steps

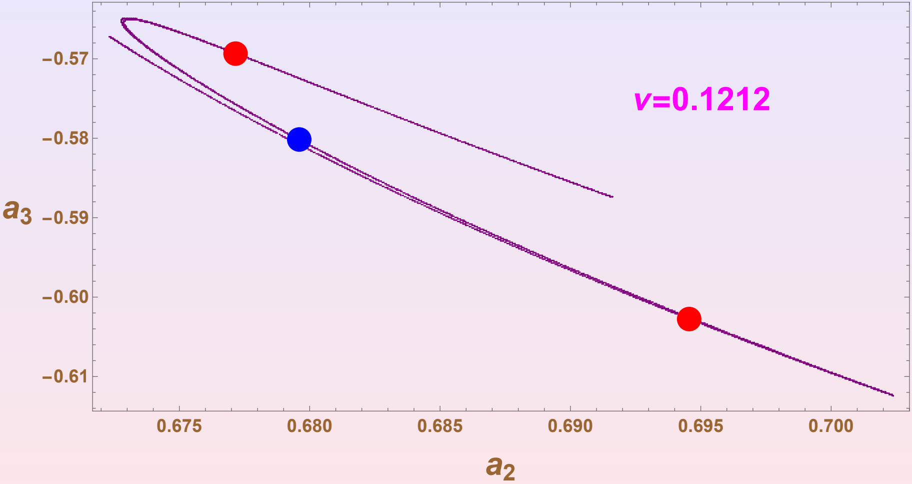
Proof of stable periodic orbit

Result reproduced from Zgliczyński FoCM'2004



u_i	$P_i(u)$	λ_i
$[-1,1]\cdot 10^{-5}$	$[-5.45, 5.45]\cdot 10^{-6}$	0.5258
$[-1,1]\cdot 10^{-5}$	$[-9.85, 9.81]\cdot 10^{-7}$	0.0903
$[-1,1]\cdot 10^{-5}$	$[-5.86, 4.67]\cdot 10^{-9}$	$3.5\cdot 10^{-8}$
$[-1,1]\cdot 10^{-5}$	$[-6.61, 4.32]\cdot 10^{-9}$	$1.65\cdot 10^{-8}$
$[-1,1]\cdot 10^{-5}$	$[-8.02, 5.65]\cdot 10^{-9}$	$-3.77\cdot 10^{-9}$
$[-1,1]\cdot 10^{-5}$	$[-6.62, 8.19]\cdot 10^{-9}$	$-4.01\cdot 10^{-11}$
$[-1,1]\cdot 10^{-5}$	$[-7.30, 9.62]\cdot 10^{-9}$	$-8.94\cdot 10^{-10}$
$[-1,1]\cdot 10^{-5}$	$[-2.15, 1.53]\cdot 10^{-9}$	$-6.69\cdot 10^{-11}$
$\dots$	$\dots$	
$k > 23$	$k > 23$	
$10^{-\mathbf{5}}(1.5)^{-k}$	$5.01\cdot 10^{-\mathbf{8}}(1.5)^{-k}$	

Unstable periodic orbit for $\nu = 0.1212$



Data from the proof of blue fixed point

u_i	$P_i(u)$	λ_i
$3.8[-1,1]\cdot 10^{-6}$	$[-8.09, 8.09]\cdot 10^{-6}$	-1.7704
$1.9[-1,1]\cdot 10^{-7}$	$[-4.33, 4.59]\cdot 10^{-8}$	-0.06511
$1.9[-1,1]\cdot 10^{-7}$	$[-2.35, 1.68]\cdot 10^{-8}$	$-2.92\cdot 10^{-16}$
$1.9[-1,1]\cdot 10^{-7}$	$[-0.718, 1.13]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-0.982, 1.40]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-1.33, 2.04]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-2.86, 3.64]\cdot 10^{-9}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-2.75, 1.67]\cdot 10^{-9}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-3.64, 4.31]\cdot 10^{-10}$	≈ 0
$\dots$	$\dots$	
$k > 23$	$k > 23$	
$1.9\cdot 10^{-7}(1.5)^{-k}$	$2.64\cdot 10^{-9}(1.5)^{-k}$	

Some computer assisted proofs in dynamics

- Langford - 1982, the proof of Feigenbaum universality conjectures
- Eckmann, Koch, Wittwer - 1984, universality for area-preserving maps
- Grebogi, Hammel, Yorke 1987 - rigorous numerical shadowing of trajectories
- Neumaier, Rage, Schlier - 1994, - chaos in the molecular Thiele-Wilson model
- Mischaikow and Mrozek - chaos in Lorenz equations, 1995
- Palmer, Coomes, Kocak, Stoffer, Kichgraber - 1996-2003 - chaos via shadowing for Henon map, PCR3BP
- W. Tucker - 2001 - geometric model for Lorenz attractor

CAPD - Krakow/Rutgers group

• Mischaikow (Rutgers), Mrozek, Zgliczynski, Wilczak, Galias, Kapela, Pilarczyk
• proofs of chaos (semiconjugacy with Bernoulli shift) for Lorenz equations, Rössler equations, Hénon map, Chua circuit, PCR3BP
• homo- and heteroclinic orbits for PCR3BP, Hénon map, Michelson system
• Kuramoto-Sivashinsky PDE: existence of multiple steady states and its bifurcations, periodic orbits, heteroclinic connections between fixed points
• N-body problem: the existence of choreographies
• period doubling bifurcations for Rössler system

General scheme of CAP in dynamics

• a problem - $\mathcal{P}$, for example the question of existence of the horseshoe for Poincaré map for ODE
• abstract theorem, $\mathcal{T}$, implying a solution of problem $\mathcal{P}$, provided we can verify $\mathcal{Z}$ - the assumptions in $\mathcal{T}$
• the reduction $\mathcal{Z}$ to finite computations, $\mathcal{O}$
• finite rigorous computation of $\mathcal{O}$ checking $\mathcal{Z}$
• If $\mathcal{Z}$ is true, then theorem $\mathcal{T}$ gives positive answer to our problem $\mathcal{P}$

Some difficulties:

• computer is finite, the continuum can not be in rigorous way represented in computer (round-off errors)
• not every theorem can be verified in finite computations
• computer can be used to verification of theorems, whose assumptions can be reduced to a finite number of (strong) inequalities, which can be verified in finite computation

Interval arithmetics

Arithmetics on closed intervals. For example:

- $[1, 3] \langle + \rangle [3, 17] = [4, 20]$
- $[-1, 1] \langle \cdot \rangle [3, 4] = [-4, 4]$
- $1 \langle / \rangle 3 = [0.33333, 0.33334]$

Rigorous interval arithmetics can be realized on the computer i.e. for each arithmetic operator $\diamondsuit \in \{+, -, \cdot, /\}$ the following is true

$$[a_-, a^+] \diamondsuit [b_-, b_+] \subset [a_-, a^+] \langle \diamondsuit \rangle [b_-, b_+]$$

Interval arithmetics

Interval arithmetics does for us two things:

- takes care of round-off errors
- enables us to evaluate functions (maps) on sets (not just single points !!!)

Example: finding (proving) zero of an analytic function

$f(x) = \sum_{n=0}^{\infty} a_n x^n$

Problem: Prove that f has a zero in interval $(1, 2)$

Numerical simulation: Apparently $f(x)$ is increasing on $[1, 2]$ and $f(1) < 0$ i $f(2) > 0$. From the intermediate value thm. it follows that f has a zero in $(1, 2)$

Reduction to finite computation:

• $f_M(x) = \sum_{n=0}^{M} a_n x^n$ - a function computable in finite number of steps

• analytical estimate: $|f_M(x) - f(x)| < \epsilon$ dla $x \in [1, 2]$, this is done by a mathematician

• rigorous check on the computer that
$f_M(1) < -\epsilon$ i $f_M(2) > \epsilon$

Example: the existence of an attracting periodic orbit

$x' = f(x)$, $x \in \mathbb{R}^3$
Two-dimensional Poincaré map, P, on section Θ.
Numerical fact: Apparently, all orbits starting in some open set U converge to periodic orbit γ.

Brouwer Theorem: If D is homeomorphic with the closed ball, $D \subset \Theta$ and $P(D) \subset \text{int}D$ (interior of D), then there exists $x \in D$ such that $P(x) = x$.
In particular, the trajectory of x is periodic.

Reduction to finite computations:

Condition: $P(D) \subset \text{int}D$ - represents a finite number of inequalities, if D - a parallepiped or ball

Phase space discretization: $D \subset \sum_{i=1}^{M} D_i$,

D_i small enough, to compute $P(D_i)$ with a reasonable overestimation

$M \approx \frac{L^2 \cdot \text{Area}(D)}{4\epsilon^2}$, where

ϵ - an error margin

L - a Lipschitz constant (rigorous) for P

$|P(x) - P(y)| < L|x - y|$

L obtained in interval computations is usually much larger than L seen in nonrigorous simulations (the wrapping effect)

Total computation time: $= M\cdot$ computation time of $P(D_i)$

The sources of errors (overestimations) in rigorous computations of ODEs:

• round-off errors - *interval arithmetics*

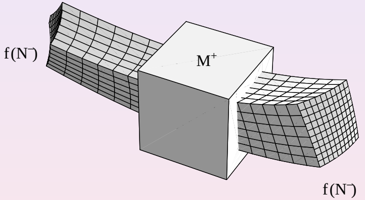
• the numerical method error (the time discretization error) - *explicit formulas for error terms*

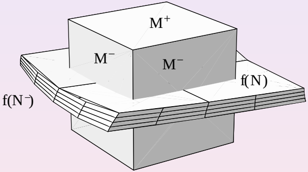
• the space discretizaton error and the propagation error (- *SERIOUS PROBLEM*)

• the errors connected to the intersection with the section in the computation of Poincaré map

Covering relations

P. Zgliczyński, M. Gidea, JDE'2004





One unstable direction **Two unstable directions**

h-sets on the plane - definition

h-set N on the plane:

- $c, u, s \in \mathbb{R}^2$, u, s - linearly independent
- $|N| = c + [-1,1]u + [-1,1]s$ - the support of N
- $N^+ = c + [-1,1]u + \{-1,1\}s$ - horizontal edges N
- $N^{le} = c - u + [-1,1]s$, $N^{re} c + u + [-1,1]s$ - 'left' and 'right' edfe of N
- $S(N)_l = c + (-\infty, 1)u + (-\infty, \infty)s$,
  $S(N)_r = c + (1, \infty)u + (-\infty, \infty)s$ - 'left' and 'right' side of N

Covering relation - Definition with one unstable direction

N, M - h-sets, $f : |N| \to \mathbb{R}^2$ - continuous

We say, that $N \overset{f}{\Longrightarrow} M$ (N f-covers M) if

- $f(|N|) \subset \text{interior}(S(M)_l \cup |M| \cup S(M)_r)$
- one of the conditions (O) or (R) is satisfied
  (O) $f(N^{le}) \subset S(M)_l$ i $f(N^{re}) \subset S(M)_r$
  (R) $f(N^{le}) \subset S(M)_r$ i $f(N^{re}) \subset S(M)_l$

Main theorem on covering relations

Theorem.(P.Z. and Gidea)
$N_0, N_1, \ldots, N_k$ - h-sets. $f_i : |N_i| \to \mathbb{R}^2$ -continuous for $i = 0, \ldots, k-1$.
Assume, that

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \ldots \overset{f_{k-1}}{\Longrightarrow} N_k.$$

Then there exists $x \in \text{int}|N_0|$ such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_0(x) \in \text{int}|N_{i+1}|, \quad i = 0, \ldots, k-1.$$

If moreover $N_k = N_0$, then x can be chosen so that

$$f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0(x) = x.$$

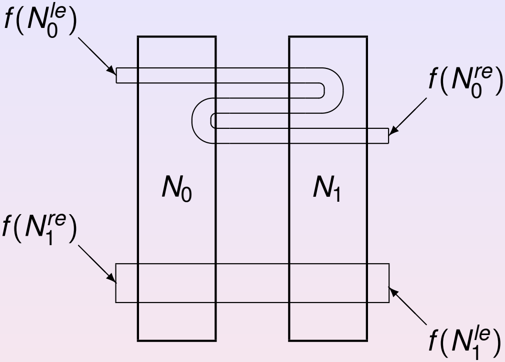


Figure: Topological Smale's horseshoe. $N_i \overset{f}{\Longrightarrow} N_j$, $i, j = 0, 1$

Symbolic dynamics exists, semiconjugacy with Bernouli shift

Cone conditions

Lemma

N– h-set, $u := u(N)$, $N \overset{f}{\Longrightarrow} N$.
$Q : \mathbb{R}^{u+s} \to \mathbb{R}$ – continuous.
*If (**cone condition**)*

$$Q(f(u_1) - f(u_2)) > Q(u_1 - u_2), \quad u_1 \neq u_2 \in N, f(u_1), f(u_2) \in N$$

then

- *f has unique fixed point $u_* \in N$*
- *every full forward trajectory of f in N must converge to u_**
- *every full backward trajectory of f in N must converge to u_**

Cone conditions

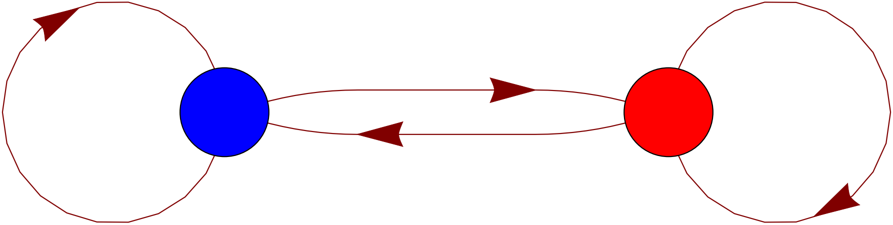
Proof:

- existence of $u_* = f(u_*)$ from $N \overset{f}{\Longrightarrow} N$
- uniqueness: if $u = f(u)$, $v = f(v)$ and $u \neq v$ then

$$Q(u - v) = Q(f(u) - f(v)) > Q(u - v)$$

- convergence: $L(u) = Q(u - u_*)$ is Lyapunov function (increases along orbits in N), hence $f^n(u) \to u_*$ for full forward orbits in N

Homoclinic orbits



$\mathbf{N_0} \Longrightarrow \mathbf{N_0}$ + **cone conditions** on $\mathbf{N_0}$ + non-constant sequence

$$\mathbf{N_0} \Longrightarrow N_{i_1} \Longrightarrow \cdots \Longrightarrow N_{i_k} = \mathbf{N_0}$$

lead to a homoclinic orbit.

How to check cone conditions?

$$Q(x,y) = \sum_{1\leq k\leq m} q_k x_k^2 - \|y\|_\infty^2$$

Compute **a**, **c**, **d**:

$$\begin{aligned}
dx^T \left((D_x f_x)^T Q_{M,1} D_x f_x \right) dx - \|D_x f_y\|^2 \|dx\|^2 - Q_{N,1}(dx) &\geq \mathbf{a}\|dx\|^2, \\
\left(\left(\sum_{k\leq m} |q_k| \cdot \|D_x f_k\| \cdot \|D_y f_k\| \right) + \|D_x f_y\| \cdot \|D_y f_y\| \right) &\leq \mathbf{c}, \\
1 - \left(\left(\sum_{k\leq m} |q_k| \cdot \|D_y f_k\|^2 \right) + \|D_y f_y\|^2 \right) &> \mathbf{d},
\end{aligned}$$

Lemma

If $a > 0, d > 0$, $c > 0$ and $ad > c^2$ then for $u_1, u_2 \in N$, $u_1 \neq u_2$

$$Q_M(f(u_1) - f(u_2)) > Q_N(u_1 - u_2)$$

Dissipative PDEs

'Definition' of Dissipative PDEs: - apparently all interesting (asymptotic) dynamics is 'finite dimensional'.

apparently= In numerical simulations increasing the dimension of Galerkin projection does not change the dynamics.

Examples:

$$u_t(t,x) = Lu + N(u, Du, \ldots, D^r u) + f(t,x),$$

L - Laplacian or its power with correct sign, $r < p$, where p the order of L, N-polynomial (or analytic), $f \in C^\infty$ or analytic.

We consider always *periodic boundary conditions*, we use the Fourier coefficients as coordinates.

Challenge of infinite dimension for computer assisted proofs

Our PDE $\dot{x} = f(x)$, x - a sequence of Fourier coefficients

Problem:

- how to represent in finite form elements of our phasespace? What is our phasespace?

Our approach: We restrict our attention to the sets of the form

$$W \oplus \Pi_{|k|>m}^{\infty}[a_k^-, a_k^+], \quad a_k^{\pm} = \pm C/|k|^s$$

$W \subset \mathbb{R}^{m_1}$ - compact set, s - large enough.

If $u(t,x) \in \mathbb{R}^{d_1}$, then

$$W \oplus \Pi_{|k|>m}^{\infty}\overline{B}_{d_1}\left(0, C/|k|^s\right),$$

Why $W \oplus \Pi_{|k|>m}^{\infty}[-C/|k|^s, C/|k|^s]$?

Let $T = \Pi_{k>m}^{\infty}[a_k^-, a_k^+]$, where $a_k^{\pm} = \pm C/|k|^s$

$$W \oplus T = \{(a_k)_{k \in \mathbf{N}} \mid (a_1, \ldots, a_m) \in W, \; a_k \in [a_k^-, a_k^+], \text{for } k > m\}$$

- any continuous periodic function of class C^s is contained in some $W \oplus T$
- $W \oplus T$ is compact in topology of component-wise convergence, l_2 etc
- on $W \oplus T$ our vector field becomes very smooth. Everything what is needed converges if s is big enough
- on $W \oplus T$ our PDE defines local semiflow, the local flows for Galerkin projections on $W \oplus T$ have uniformly bounded Lipschitz constants on a compact time intervals and converge uniformly to the semiflow for full system (logarithmic norms, one-sided Lipschitz condition)

Why $W \oplus \Pi_{k>m}^{\infty}[-C/|k|^s, C/|k|^s]$? - continued

Most important: While $W \oplus T$ is not invariant under the flow of our PDE, it may have an important dynamical property - an isolation for the tail.

In other words: the set $W \oplus T$ has to be conditionally invariant for all Galerkin projections:
Let $u(t)$ be a solution for any Galerkin projection of our problem, then:

If $u(t_0) \in W \oplus T$, and $P_M u(t_0 + t) \in W$ for $t \in [0, h]$, then $u(t_0 + [0, h]) \in W \oplus T$.
Draw picture and explain

Why it is a easy to find a good tail = self-consistent bounds

$$u_t = Lu + N(u, Du, \ldots, D^r u)$$

$x \in \mathbf{T}^n$ (periodic boundary conditions),
L - linear, diagonal, N - polynomial
Fourier expansion $u(t) = \sum_{k \in \mathbf{Z}^n} u_k(t) e^{ik \cdot x}$

Lemma. Let $s > s_0$. If $|u_k| \leq C/|k|^s$, $|u_0| \leq C$, then there exists $D = D(C, s)$

$$|N_k| \leq \frac{D}{|k|^{s-r}}, \qquad |N_0| \leq D$$

This is in fact a statement about regularity. u is of "class C^s" (plus bounds) then $N(u)$ is of "class C^{s-r}" (plus bounds)

Isolation property

Sketch of the proof of the isolation property:

Assume $L(u)_k = -|k|^p u_k$, $p > r$.

Assume $|u_k| \leq \frac{C}{|k|^s}$, $|u_{k_0}| = \frac{C}{|k_0|^s}$, then

$$\frac{d|u_{k_0}|}{dt} \leq -|k_0|^p |u_{k_0}| + |N_{k_0}(u)| \leq$$
$$-C|k_0|^{p-s} + D|k_0|^{r-s}$$
$$\frac{d|u_{k_0}|}{dt} < 0, \qquad |k_0| > M$$

Some related work

- "Classical" PDEs setting, use of Schauder or Banach fixed points in suitable function spaces : Nakao, Yamamoto, Plum, McKenna, Watanabe, Lessard and others. restricted to static problems, no dynamics
- functional analytic approach: Arioli and Koch - results on fixed points and bifurcations for Kuramoto-Sivashinski PDE , periodic orbit
- self-consistent bounds - good for dynamics of dissipative PDEs (and static problems too).
  - fixed points and heteroclinic connections for Cahn-Hillard (gradient system): Maier-Papper, Mischaikow, Wanner
  - periodic orbits for Kuramoto-Sivashinski PDE in 1D - P. Z.
  - others: Swift-Hohenberg eq. - steady states : Hiraoka, Ogawa, Mischaikow, Day
  - bifurcations of steady states for KS eq. - P.Z.
  - global attractor consisting of single orbit for viscous Burgers equation in 1D with periodic and non-periodic forcing - J. Cyranka and P.Z.

Rigorous integration of dissipative PDEs - the general idea

$$u_t = Lu + N(u, Du, \ldots, D^r u) + f(x), \tag{1}$$

$u \in \mathbb{R}^n$, $x \in \mathbf{T}^d$, L is a linear, N - a polynomial (or analytic), f smooth enough.
L is diagonal in the Fourier basis $\{e^{kx}\}_{k \in \mathbf{Z}^d}$

$$Le^{ikx} = \lambda_k e^{ikx}, \tag{2}$$

$$\begin{aligned}
\lambda_k &= -\nu(|k|)|k|^p && (3)\\
0 &< \nu_0 \leq \nu(|k|) \leq \nu_1, \qquad \text{for } |k| > K_- && (4)\\
p &> r. && (5)
\end{aligned}$$

1 We replace PDE by an infinite ladder of ODEs for Fourier coefficients of $u(t, x)$.

$$\frac{du_k}{dt} = \lambda_k u_k + N_k(u), \qquad \text{for all } k \in \mathbf{Z}^d. \tag{6}$$

2 we split 'the phase space' for (6) into two parts: the finite dimensional part, X, containing the Fourier modes most relevant for the dynamics of (1) and the tail in $X^\perp$. Now problem (6) is replaced by two problems (7) and (8).

3 The first part consist of a finite dimensional differential inclusion for $p \in X$, given by

$$\frac{dp}{dt} \in P(Lp + N(p + T)), \qquad p \in X \tag{7}$$

P is a projection onto X. The second part is concerned with the evolution of T

$$\lambda_k u_{k,j} + N^-_{k,j} < \frac{du_{k,j}}{dt} < \lambda_k u_{k,j} + N^+_{k,j}, \quad \text{"}k \text{ not in } X\text{"} \tag{8}$$

where $N^\pm_{k,j}$ are suitably chosen constants.

Obviously, to infer from (7) and (8) any information on the behavior of solutions of the full system (6) one needs some **consistency conditions** and **fast decay of Fourier coefficients.**

Tails $T = \Pi_{k>m}[-\frac{C}{k^s}, \frac{C}{k^s}]$ do the job through **the isolation property.**

Our algorithm gives uniform and compact bounds for all Galerkin projections of PDE. The solution of PDE is obtained through passing to the limit with the dimension of Galerkin projection.

The method of self-consistent bounds

H - Hilbert space,
$e_1, e_2, \ldots$ - an orthogonal basis in H
The corresponding projections are

$$p_m = P_m a := (a_1, a_2, \ldots, a_m)$$
$$q_m = Q_m a := (a_{m+1}, a_{m+2}, \ldots)$$

The problem:

$$\dot{a} = F(a) \tag{9}$$

F is not continuous, with dense domain in H.
$F_k \circ P_n$ is a C^1-function for $n, k \in \mathbf{N}$
Later $F(a) = L(a) + N(a)$, L - linear, N- nonlinear
$e_1, e_2, \ldots$ - eigenvectors of L - very helpful

The method of self-consistent bounds

Def. Fix m, M ($m \leq M$). *A compact set $W \subset P_m(H)$ and a sequence of pairs $\{a_k^{\pm} \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbf{Z}^+\}$ are self-consistent a-priori bounds for F if:*

- *C1* *For $k > M$, $a_k^- < 0 < a_k^+$.*
- *C2* *Let $\hat{a}_k := \max |a_k^{\pm}|$ and set $\hat{u} = \sum_{k=0}^{\infty} \hat{a}_k e_k$. Then, $\hat{u} \in H$, ($\{\hat{a}_k\} \in l_2$)*
- *C3* *The function $u \mapsto F(u)$ is continuous on*

$$W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset H.$$

Moreover, if we define
$\hat{f}_k = \max_{u \in W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]} |F_k(u)|$ and set $\hat{f} = \sum \hat{f}_k e_k$, then $\hat{f} \in H$. ($\{\hat{f}_k\} \in l_2$)

Notation: $T = \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]$ - Tail

I. ISOLATION for $n > m$

For $a \in W \oplus T$ and $k > m$ holds

$$\begin{aligned} a_k = a_k^+ \quad &\Rightarrow \quad \dot{a}_k < 0 \\ a_k = a_k^- \quad &\Rightarrow \quad \dot{a}_k > 0 \end{aligned}$$

C1,C2,C3 - give the convergence of Galerkin projections

I - gives a priori bounds

C1,C2,C3,I - easy to satisfy. It is enough to take $|a_k| \leq \frac{C}{|k|^s}$ for s large enough (different choices are also possible)

Finite dimensional part

Basic Differential Inclusion:

$$\dot{p} \quad \in \quad P_m F(p) + \Gamma_m, \quad p \in \mathbb{R}^m, \tag{10}$$

where $\Gamma_m = \{P_m F(p+q) - P_m F(p) \mid p \in W, q \in T\}$
We say a multivalued map $p_I : [0,h] \to H$ is *upper attainable set (uas) map* for (10) if the following is true

- any C^1 function satisfying (10) and defined on the maximum interval of existence is defined on $[0,h]$
- if a C^1-function $p : [0,h] \to X_m$ satisfies (10), then $p(t) \in p_I(t)$ for $t \in [0,h]$

Theorem: Assume $W \oplus T$ are self-consistent bounds for F. If $p_I : [0, t_1] \to X_m = P_m(H)$ is uas map for (10), such that $p_I([0, t_1]) \subset W$. Then for any $q_0 \in T$, the problem $u' = F(u)$ (and all its Galerkin projections $u' = P_n F(u)$, $n > M$) has a solution $u(t) = (p(t), q(t))$ for $t \in [0, t_1]$, such that

$$p(t) \in p_I(t), \qquad q(t) \in T, \qquad \text{for } t \in [0, t_1]$$

Logarithmic norms

Logarithmic norm: $Q \in R^{n\times n}$

$$\mu(Q) = \lim_{h>0, h\to 0} \frac{\|I + hQ\| - 1}{h}$$

can be negative !!!

- for Euclidean norm

$$\mu(Q) = \text{the largest eigenvalue of} \quad 1/2(Q + Q^T).$$

- for max norm $\|x\| = \max_k |x_k|$

$$\mu(Q) = \max_k (q_{kk} + \sum_{i, i\neq k} |q_{ki}|)$$

- for norm $\|x\| = \sum_k |x_k|$

$$\mu(Q) = \max_i (q_{ii} + \sum_{k, k\neq i} |q_{ki}|)$$

Convergence of Galerkin projections.

Logarithmic norms - Fundamental lemma

Lemma: Let $\phi(t,x)$ be a flow induced by

$$x' = f(x).$$

Assume that Z is a convex set,

$$y([0,T]), \varphi([0,T],x_0) \in Z$$

$$\mu\left(\frac{\partial f}{\partial x}(\eta)\right) \leq l, \quad \text{for } \eta \in Z$$

$$\left\| \frac{dy}{dt}(t) - f(y(t)) \right\| \leq \delta.$$

Then for $0 \leq t \leq T$ we have

$$\|\varphi(t,x_0) - y(t)\| \leq e^{lt}\|y(0) - x_0\| + \delta \frac{e^{lt} - 1}{l}, \quad \text{if } l \neq 0.$$

For $l = 0$ we have

$$\|\varphi(t,x_0) - y(t)\| \leq \rho + \delta t.$$

Convergence of Galerkin projections.

$$x' = F(x) = Lx + N(x) \tag{11}$$

$e_1, e_2, \ldots$ - eigenvectors for L, $Le_k = \lambda_k e_k$, $\lambda_k \to -\infty$
W,$\{a_k^{\pm}\}$ - self-consistent bounds,
$T = \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$
W - convex
$P_n(W \oplus T)$ is a trapping region (an isolating block with $W^- = \emptyset$) for n-dim Galerkin projections of (11), $n > m$
Condition D: there exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \ldots$, $a \in W \oplus T$

$$\sum_{i=1}^{\infty} \left| \frac{\partial N_k}{\partial x_i} \right| (a) + \lambda_k \leq l$$

Idea: the logarithmic norms for all Galerkin projections are uniformly bounded, because the diagonal part dominates Df.

Convergence of Galerkin projections:

Theorem:

1. **Uniform convergence and existence** For $x_0 \in W \oplus T$, let $x_n : [0, \infty] \to P_n(W \oplus T)$ be a solution of $x' = P_n(F(x)), x(0) = P_n x_0$.
   Then x_n converges uniformly on compact intervals to a function $x^* : [0, \infty] \to W \oplus T$, which is a solution of (11) and $x^*(0) = x_0$. The convergence of x_n on compact time intervals is uniform with respect to $x_0 \in W \oplus T$.
2. **Lipschitz constant.** Let $x : [0, \infty] \to W \oplus T$ and $y : [0, \infty] \to W \oplus T$ be solutions of (11), then

$$|y(t) - x(t)| \leq e^{lt}|x(0) - y(0)|$$

Convergence of Galerkin projections - comments

- We got a semiflow on $W \oplus T$
- A computable expression for a Lipschitz constant of the induced semiflow
  Application: If $W \oplus T$ - a trapping region isolating a fixed point and $l < 0$, then we have an attracting fixed point - gives the verified basin of attraction
- we have a formula for the error of Galerkin projection

Interval arithmetics - problems

- dependency:

for $x = [-1, 1]$ holds

$$x \langle - \rangle x = [-2, 2]$$

Another example:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$[1 - \sinh(t), \cosh(t)] \subset \langle e^{-[0,t]} \rangle$$

$$\mathrm{diam}\,(\langle e^{-[0,t]} \rangle) \geq e^t - 1, \quad \mathrm{diam}\,(e^{-[0,t]}) = 1 - e^{-t}$$

- wrapping

the result of evaluation of multidimensional map is product of intervals, disastrous results when considering f^n for n-large, ODEs

Wrapping effect

Harmonic oscillator

$$x' = -y, \qquad y' = x$$

Time shift by h, φ_h , rotation by h - ISOMETRY.
In the ideal (round-off error free) interval arithmetics we obtain

$$\lim_{n->\infty} \langle \varphi_{\frac{2\pi}{n}} \rangle ([-\delta, \delta]^2) = e^{2\pi}[-\delta, \delta]^2$$

($e^{2\pi} \approx 536$) while we would expect that

$$\lim_{n->\infty} \langle \varphi_{\frac{2\pi}{n}} \rangle = \varphi_{2\pi} = \text{identyczność}$$

DISASTER - SERIOUS OVERFLOW SOON

REASON:

- after each step the result is the following form $I_1 \times I_2$, where I_1, I_2 are intervals

These are not the reasons

- round-off errors
- the numerical method error

Hence increasing of the precision of the computations and improvement of numerical method via taking higher order and/or smaller time step does not guarantee any improvement.
Conclusion: Naive application of interval arithmetics to the integration of ODEs if very ineffective.

$$x' = f(x) \tag{12}$$

$|f(x) - f(y)| \leq L|x - y|$.
Let $\varphi(t, x_0)$ be a solution of (12) with an initial condition $x(0) = x_0$.
Then

$$|\varphi(t, x) - \varphi(t, y)| \leq e^{Lt}|x - y|, \qquad t \geq 0$$

This is very bad estimate
Could be considerably improved (but still not enough) by using logaritmic norms.

Examples:

- $x' = -10x$, predicts error-growth: e^{10t}
- for the Lorenz attractor (from the proof by Galias and P. Z.), gives an estimate for Lipschitz constant for the Poincare map $L > 10^9$, while from simulations it is clear that $L \approx 5 - 6$
- in the proof for Rössler system (P.Z.), gives an estimate for the Lipschitz constant of Poincare map $L > 5 \cdot 10^{41}$, while from simulations $L \approx 2 - 3$ **cosmic computation time**

• *set division*: Let $S_t = \varphi(t, S)$. When S_t becomes too large, one should divide it into smaller pieces and compute further the evolution of each piece separately

• *Lohner algorithm:* in order to avoid wrapping effect one should choose good coordinate frame in each step. *This is what we are using most of the time. Package CAPD*

• *Taylor models:* COSY - Berz, Makino. Propagate Taylor series of high order by ODE. Slow, but quite robust.

Rigorous integration for ODEs - One step of the Lohner algorithm

$x' = f(x)$ induces $\varphi(t, x_0)$ - t-time, x_0 - initial condition,
$\Phi(h, x)$ - numerical method, Taylor method of order p

Input:

- t_k - time,h_k - time step
- $[x_k] \subset \mathbb{R}^n$, such that $\varphi(t_k, [x_0]) \subset [x_k]$

Output:

- $t_{k+1} = t_k + h_k$
- $[x_{k+1}] \subset \mathbb{R}^n$, such that $\varphi(t_{k+1}, [x_0]) \subset [x_{k+1}]$

1. Rough estimate of $\varphi([0, h_k], [x_k])$
   $[W_1] \subset \mathbb{R}^n$ compact and convex

$$\varphi([0, h_k], [x_k]) \subset [W_1]$$

2. $[A_k] = \frac{\partial \Phi}{\partial x}([x_k])$
3. $[x_{k+1}]$ ($m([x_k])$ - mid-point of $[x_k]$)

$$\begin{aligned} [x_{k+1}] &= \Phi(h_{k+1}, m([x_k])) + \\ &\quad [A_k]([x_k] - m([x_k])) + \mathrm{Rem}([W_1]) \end{aligned}$$

Reduction of the wrapping effect

$$[x_k] = x_k + [r_k], \qquad x_k = m([x_k]), [r_k] = [x_k] - x_k$$

The equation to evaluate:

$$[r_{k+1}] = [A_k][r_k] + [z_{k+1}]$$

We choose different coordinate frame: $[r_k] = B_k[\hat{r}_k]$,

$$[r_{k+1}] = [A_k][r_k] + [z_{k+1}] =$$
$$B_{k+1}\left(B_{k+1}^{-1}[A_k]B_k[\hat{r}_k] + B_{k+1}^{-1}[z_{k+1}]\right)$$

$$[r_0] = [B_0][\hat{r}_0], \quad [B_0] = \{Id\}$$
$$[\hat{r}_{k+1}] = \left([B_{k+1}^{-1}][A_k][B_k]\right)[\hat{r}_k] + [B_{k+1}^{-1}][z_{k+1}]$$
$$[r_{k+1}] = [B_{k+1}][\hat{r}_{k+1}]$$

Usually B_{k+1} is a Q-factor from QR decomposition of $U \in [A_k][B_k]$,
Even better:

$$[r_{k+1}] = C_{k+1}[r_0] + [\tilde{r}_{k+1}]$$
$$[\tilde{r}_{k+1}] = [A_k][\tilde{r}_k] + [z_{k+1}] + ([A_k]C_k - C_{k+1})[r_0],$$
$$[\tilde{r}_0] = 0$$
$$\text{and } C_0 = Id, \quad C_{k+1} \in [A_k]C_k$$

$[\tilde{r}_k]$ is evaluated using previous method