An improved range of exponents for absence of Lavrentiev phenomenon for double phase functionals

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Standard minimization problem

Let $1 < p < \infty$, consider

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p$$

as well as corresponding variational problem

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{F}(u).$$

Here, $u_0 \in C^\infty(\Omega)$ represents boundary data.

Absence of Lavrentiev phenomenon

We consider the problem: $\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \int_\Omega |\nabla u|^p$

1. Is there a minimizer? Yes, say u^*.
2. Is there a sequence of functions $\{u_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $u_0 + u_n \to u^*$ and $\mathcal{F}(u_0 + u_n) \to \mathcal{F}(u^*)$?
   Standard density argument.
3. What is regularity of the minimizer u^*?
   De Giorgi - Nash - Moser theory...

Question 2 implies

$$\inf_{u_0 + C_c^\infty(\Omega)} \int_\Omega |\nabla u|^p = \inf_{u_0 + W_0^{1,\infty}(\Omega)} \int_\Omega |\nabla u|^p = \inf_{u \in u_0 + W_0^{1,p}(\Omega)} \int_\Omega |\nabla u|^p.$$

Mania's example (1934)

Lavrentiev phenomenon is a situation when for some functional $\mathcal{F}$ and two spaces X, Y

$$\inf_{u \in X} \mathcal{F}(u) < \inf_{u \in Y} \mathcal{F}(u).$$

Consider

$$\mathcal{F}(u) = \int_0^1 (u(t)^3 - t)^2 \, (u'(t))^6 \, \mathrm{d}t$$

over functions with $u(0) = 0, u(1) = 1$.

The Lavrentiev phenomenon occurs between

$$X = W^{1,1}(0,1) \text{ and } Y = W^{1,\infty}(0,1).$$

B. Mania. Sopra un esempio di Lavrentieff. Boll. Un. Mat. Ital., 13:147–153, 1934.

It's not only an academic example.

F. Rindler. Calculus of variations. Springer, 2018.

Double phase functionals

Absence of LP is not trivial for functionals with x-dependent growth, the model example reads:

$$\mathcal{I}(u) = \int_\Omega |\nabla u|^p + a(x)\, |\nabla u|^q$$

with $1 < p < q < \infty$.

Here, $a : \Omega \to \mathbb{R}^+$ is a continuous function vanishing on some part of Ω.

One usually assumes Holder regularity: $a \in C^\alpha(\Omega)$.

We cannot obtain any valuable L^q estimates on ∇u on the set

$$\{x \in \Omega : a(x) \neq 0\}.$$

The functional $\mathcal{I}$ is strongly l.s.c. on both $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ and hence weakly l.s.c. by convexity.

There is a minimizer in $W^{1,p}(\Omega)$ (but not necessarily in $W^{1,q}(\Omega)$!).

It may happen that

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{I}(u) < \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{I}(u).$$

Positive and negative results

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{I}(u) = \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{I}(u).$$

- $q \leq p + \alpha \frac{p}{d}$ (simple mollification argument)

  M. Colombo and G. Mingione. Regularity for double phase variational problems. ARMA, 2015.

  A. Esposito, F. Leonetti, and P. Vincenzo Petricca. Absence of Lavrentiev gap for non-autonomous functionals with (p, q)-growth. Adv. Nonlinear Anal., 8(1):73–78, 2019.

- $q \leq p + \alpha$ (complicated proof based on regularity of minimizers and additional boundedness assumption)

  M. Colombo and G. Mingione. Bounded minimisers of double phase variational integrals. ARMA, 2015.

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{I}(u) < \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{I}(u).$$

- we discuss this at the end...

Main result

Our result concerns the positive part.

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{I}(u) = \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{I}(u).$$

- $q \leq p + \alpha \frac{p}{d}$ (simple mollification argument)
- $q \leq p + \alpha$ (complicated proof based on regularity of minimizers and additional boundedness assumption)

OUR RESULT: We show that a simple mollification argument works if

$$q \leq p + \alpha \max\left(\frac{p}{d}, 1\right).$$

Functional analytic setting

We would like to work in normed space of functions such that

$$\int_\Omega |\nabla u|^p + a(x)\,|\nabla u|^q < \infty$$

but it is hard to define the norm.

For L^p spaces there is an equivalent norm

$$\|f\|_p = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^p \leq 1\right\}$$

We define the norm with:

$$\inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^p + a(x)\,\left|\frac{f(x)}{\lambda}\right|^q \leq 1\right\}$$

Functional analytic setting

We let

$$\varphi(x,\xi) = |\xi|^p + a(x)\,|\xi|^q.$$

We define the norm with

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(x, \left|\frac{f(x)}{\lambda}\right|\right) \leq 1\right\}$$

and the related Banach space is

$$W_0^{1,\varphi}(\Omega) = \{u \in W_0^{1,1}(\Omega) : \|\nabla u\|_\varphi < \infty\},$$
$$W^{1,\varphi}(\Omega) = \{u \in W^{1,1}(\Omega) : \|\nabla u\|_\varphi < \infty\}.$$

and the norm

$$\|u\| := \|u\|_1 + \|\nabla u\|_\varphi.$$

What is $W^{1,\varphi}(\Omega)$?

A space of integrable functions such that

$$\int_\Omega |\nabla u|^p + a(x)\,|\nabla u|^q < \infty.$$

A space between $W^{1,p}$ and $W^{1,q}$:

$$W^{1,p}(\Omega) \subset W^{1,\varphi}(\Omega) \subset W^{1,q}(\Omega).$$

For convergence $u_n \to u$ in $W^{1,\varphi}(\Omega)$ we need

- $u_n \to u$ in L^1,
- $\nabla u_n \to \nabla u$ wrt $\|\cdot\|_\varphi$

I. Chlebicka, P. Gwiazda, A. Wróblewska-Kamińska, and A. Świerczewska-Gwiazda. Partial Differential Equations in anisotropic Musielak-Orlicz spaces. Springer Monographs in Mathematics, 2021.

We want to rewrite problem of Lavrentiev phenomenon in the setting of Musielak-Orlicz-Sobolev spaces.

Theorem. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ and f be a measurable function. Then, $f_n \to f$ in $L^1(\Omega)$ if and only if $f_n \to f$ in measure and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable, i.e.

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall A \subset \Omega \qquad \lambda(A) < \delta \implies \sup_{n\in\mathbb{N}} \int_A |f_n| \, \mathrm{d}\lambda < \varepsilon.$$

Think about $f_n = n\mathbb{1}_{[0,1/n]}$.

One consequence of VCT

Suppose that

- $f_n \to f$ in measure,
- there are g_n such that $|f_n| \leq g_n$ and $g_n \to g$ in L^1.

Then, $f_n \to f$ in L^1.

Compare with dominated covergence...

Convergence in $\|\cdot\|_\varphi$

We have

$$\|f - f_n\|_\varphi \to 0 \iff \int_\Omega \varphi(x, |f - f_n|) \to 0$$

i.e.

$$\int_\Omega |f - f_n|^p + a(x)\,|f - f_n|^q \to 0.$$

More on convergence in $\|\cdot\|_\varphi$

$\|f - f_n\|_\varphi \to 0$ with $\|f\|_\varphi < \infty$ if and only if

1. $f_n \to f$ in measure,
2. $\{\varphi(x, f_n)\}_{n\in\mathbb{N}}$ is uniformy integrable.

$$\varphi(x, f_n - f) = \varphi\left(x, 2\,\frac{f_n - f}{2}\right) \le 2^q\,\varphi\left(x, \frac{f_n - f}{2}\right).$$

Then, we use convexity

$$\varphi\left(x, \frac{f_n - f}{2}\right) \; = \; \varphi\left(x, \frac{f_n}{2} - \frac{f}{2}\right) \; \le \; \frac{1}{2}\varphi\left(x, f_n\right) + \frac{1}{2}\varphi\left(x, f\right)$$

The opposite direction in a similar manner.

Density of $C_c^\infty(\Omega)$ in $W_0^{1,\varphi} \implies$ absence of LP.

Let $u_0 \in W^{1,q}(\Omega)$. We want to prove

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{I}(u) = \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{I}(u).$$

Inequality $\leq$ is trivial. To see $\geq$, we take $u^* \in W_0^{1,p}(\Omega)$ to be a minimizer of $\mathcal{I}$.

1. $u^* \in W^{1,\varphi}$ because $\mathcal{I}(u) < \infty$,
2. $u_0 \in W^{1,\varphi}$ because $u_0 \in W^{1,q}$
3. $u^* - u_0 \in W_0^{1,\varphi}$ so there is a sequence of $\{u_n\}_{n \in \mathbb{N}}$ such that $u_n + u_0 \to u^*$ in $W^{1,\varphi}$ i.e. $\mathcal{I}(u_n + u_0) \to \mathcal{I}(u^*)$
4. Then,

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{I}(u) = \mathcal{I}(u^*) = \lim_{n \to \infty} \mathcal{I}(u_n + u_0) \geq \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{I}(u).$$

Density of $W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$ in $W_0^{1,\varphi}(\Omega)$.

Let $u \in W^{1,\varphi}(\Omega)$. Consider truncation operator

$$T_k(u) := \begin{cases} u & \text{if } |u| \leq k, \\ \frac{u}{|u|} k & \text{if } |u| > k. \end{cases}$$

CLAIM: Then, $T_k(u) \to u$ in $W^{1,\varphi}(\Omega)$.

Indeed, $\nabla T_k(u) = \mathbb{1}_{|u| \leq k} \nabla u \to \nabla u$ a.e. As

$$\varphi(x, |\nabla T_k(u)|) \leq \varphi(x, |\nabla u|),$$

the sequence $\{\varphi(x, \nabla T_k(u))\}_{k \in \mathbb{N}}$ is uniformly integrable.

This is actually the main novelty of our work.

Proof of density $C_c^\infty(\Omega)$ in $W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$

Let $u \in W_0^{1,\varphi}(\Omega)$ and u^ε be its usual mollification.

We will prove that the sequence $\{\varphi(x, \nabla u^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable.

As $\nabla u^\varepsilon \to \nabla u$ a.e., we obtain

$$\|\nabla u^\varepsilon - \nabla u\|_\varphi \to 0.$$

Does it prove $C_c^\infty(\Omega)$ is dense in $W_0^{1,\varphi}(\Omega)$?

$\{\varphi(x, \nabla u^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable.

$$0 \leq \varphi(x, \nabla u^\varepsilon) = \varphi\left(x, \int_{B_\varepsilon} \nabla u(x-y)\,\eta_\varepsilon(y)\mathrm{d}y\right)$$
$$\leq \int_{B_\varepsilon} \varphi\left(x, \nabla u(x-y)\right)\eta_\varepsilon(y)\mathrm{d}y$$

If we could replace $\varphi\left(x, \nabla u(x-y)\right)$ with $\varphi\left(x-y, \nabla u(x-y)\right)$, the (RHS) is

$$\int_{B_\varepsilon} \varphi\left(x-y, \nabla u(x-y)\right)\eta_\varepsilon(y)\mathrm{d}y = \varphi(x, \nabla u(x)) * \eta_\varepsilon.$$

As $\varphi(x, \nabla u(x)) \in L^1(\Omega)$, its mollification converges in $L^1(\Omega)$ and $\{\varphi(x, \nabla u^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable.

Autonomous approximation

Fix $x \in \Omega$ and consider

$$\widetilde{\varphi}(\xi) = \inf_{y \in B_\varepsilon(x) \cap \Omega} \varphi(x, \xi).$$

By continuity, there is $x^* \in B_\varepsilon(x) \cap \Omega$ such that

$$\widetilde{\varphi}(\xi) = \varphi(x^*, \xi).$$

It follows that $\xi \mapsto \widetilde{\varphi}(\xi)$ is convex.

Comparing $\widetilde{\varphi}(\xi)$ with $\varphi(x,\xi)$

Let $|x-z| \le \varepsilon$ and $\xi = \nabla u^{\varepsilon}(x)$. Then,

$$\frac{\varphi(x,\xi)}{\varphi(z,\xi)} = \frac{|\xi|^p + a(x)\,|\xi|^q}{|\xi|^p + a(z)\,|\xi|^q} = \frac{1 + a(x)\,|\xi|^{q-p}}{1 + a(z)\,|\xi|^{q-p}} =$$
$$= \frac{1 + a(z)\,|\xi|^{q-p} - a(z)\,|\xi|^{q-p} + a(x)\,|\xi|^{q-p}}{1 + a(z)\,|\xi|^{q-p}}$$
$$\le 1 + |a(x) - a(z)|\,|\xi|^{q-p} \le 1 + |a|_{\alpha}\,|x-z|^{\alpha}\,|\xi|^{q-p}$$

As $|x-z| \le \varepsilon$, we need $\|\nabla u^{\varepsilon}\|_{\infty}^{q-p} \le C\,\varepsilon^{-\alpha}$.

Estimate on convolution

We need $\|\nabla u^\varepsilon\|_\infty^{q-p} \le C\,\varepsilon^{-\alpha}$.

Young's convolutional inequality:

$$\|f * g\|_\infty \le \|f\|_p\,\|g\|_q, \qquad 1 = \frac{1}{p} + \frac{1}{q}.$$

Application of $u \in L^\infty$:

$$\|u * \nabla\eta_\varepsilon\|_\infty \le \|u\|_\infty\,\|\nabla\eta_\varepsilon\|_1 \le \|u\|_\infty\,\frac{C}{\varepsilon} \le C\varepsilon^{-\alpha/(q-p)}$$

We need $q - p \le \alpha$

Application of $\nabla u \in L^p$:

$$\|\nabla u * \eta_\varepsilon\|_\infty \le \|\nabla u\|_p\,\|\eta_\varepsilon\|_{p'} \le \varepsilon^{-d/p}\|\nabla u\|_p \le C\varepsilon^{-\alpha/(q-p)}$$

We need $q - p \le \alpha\frac{p}{d}$.

The final proof

Under this exponent regime we have

$$\varphi(x,\xi) \le C\,\widetilde{\varphi}(\xi).$$

Then, we estimate

$$\begin{aligned} 0 \le \varphi(x,\nabla u^{\varepsilon}) &\le \widetilde{\varphi}(\nabla u^{\varepsilon}) \\ &= \widetilde{\varphi}\left(\int_{B_\varepsilon} \nabla u(x-y)\,\eta_\varepsilon(y)\mathrm{d}y\right) \\ &\le \int_{B_\varepsilon} \widetilde{\varphi}\,(\nabla u(x-y)\,)\,\eta_\varepsilon(y)\mathrm{d}y \\ &\le \int_{B_\varepsilon} \varphi\,(x-y,\nabla u(x-y)\,)\,\eta_\varepsilon(y)\mathrm{d}y \end{aligned}$$

as $x-y \in B_\varepsilon(x)$.

Conclusion

We don't observe Lavrentiev phenomenon if

$$q - p \leq \alpha \max\left(1, \frac{p}{d}\right).$$

Counterexamples exists when

$$q - p > \alpha \max\left(1, \frac{p-1}{d-1}\right).$$

A. K. Balci, L. Diening, and M. Surnachev. New examples on Lavrentiev gap using fractals. Calc. Var. PDE 2020.

Our work is optimal when $p \leq d$

For $p > d$ there is still some room for improvement.

Final comments

1. The presented proof is in fact inaccurate. Some care is needed to make functions compactly supported.
2. After some modifications, the reasoning can be extended to variable exponent functionals

$$\mathcal{H}(u) = \int_\Omega |\xi|^{p(x)} + a(x)\,|\xi|^{q(x)}.$$

3. As $\varphi(x, |\nabla u|)$ depends only on the length of the gradient ∇u, the reasoning can be extended to vectorial problems

THANK YOU!