Optimal existence results for the Keller–Segel chemotaxis system and blowup for related parabolic toy models

Piotr Biler (Wrocław) and

Alexandre Boritchev, Lorenzo Brandolese (Lyon)

Łódź, October 5, 2022

Large global solutions of the parabolic-parabolic Keller-Segel system in $\mathbb{R}^d$ and blowup for related toy models

$$
\begin{aligned}
u_t - \Delta u + \nabla \cdot (u \nabla \varphi) &= 0, \\
\tau \varphi_t = \Delta \varphi + u&,
\end{aligned}
$$

$$u(x,0) = u_0(x) \geq 0,$$

$$(x,t) \in \mathbb{R}^d \times [0,T), \quad d \geq 2,$$

$u(x,t) \geq 0$ — the density of the population of microorganisms,
$\varphi(x,t)$ — the density of the chemical secreted by themselves that attracts them and makes them to aggregate.

Large global solutions of the parabolic-parabolic Keller-Segel system in $\mathbb{R}^d$ and blowup for related toy models

$$
\begin{aligned}
u_t - \Delta u + \nabla \cdot (u \nabla \varphi) &= 0, \\
\tau \varphi_t = \Delta \varphi + u&,
\end{aligned}
$$

$$u(x,0) = u_0(x) \geq 0,$$

$$(x,t) \in \mathbb{R}^d \times [0,T), \quad d \geq 2,$$

$u(x,t) \geq 0$ — the density of the population of microorganisms, $\varphi(x,t)$ — the density of the chemical secreted by themselves that attracts them and makes them to aggregate.

As $\tau \searrow 0$ solutions of the above (PP) system converge to those of the parabolic-elliptic KS system (PE) with $\Delta\varphi + u = 0$ (A. Raczyński 2009, PB + L. Brandolese 2009, P.-G. Lemarié-Rieusset 2013, ... , M. Kurokiba + T. Ogawa 2020, ...).

What about $\tau \gg 1$?

Motivations

• (an early result PB + L. Corrias + J. Dolbeault 2008)
The scaling transformation $u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$, $\quad \varphi_\lambda(t,x) = \varphi(\lambda^2 t, \lambda x)$ *for every* $\lambda > 0$, leaves the Keller–Segel system invariant. Each solution invariant under this scaling
$u(t,x) = u_\lambda(t,x), \quad \varphi(t,x) = \varphi_\lambda(t,x), \quad \lambda > 0$, is called a **self-similar solution** to system KS. It has the form

$$u(t,x) = \frac{1}{t} U\left(\frac{x}{\sqrt{t}}\right), \quad \varphi(t,x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

with $U(x) = u(1,x)$ and $\varphi(x) = \Phi(1,x)$.

Motivations

• (an early result PB + L. Corrias + J. Dolbeault 2008)
The scaling transformation $u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$, $\varphi_\lambda(t,x) = \varphi(\lambda^2 t, \lambda x)$ *for every* $\lambda > 0$, leaves the Keller–Segel system invariant. Each solution invariant under this scaling $u(t,x) = u_\lambda(t,x)$, $\varphi(t,x) = \varphi_\lambda(t,x)$, $\lambda > 0$, is called a **self-similar solution** to system KS. It has the form

$$u(t,x) = \frac{1}{t} U\left(\frac{x}{\sqrt{t}}\right), \quad \varphi(t,x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

with $U(x) = u(1,x)$ and $\varphi(x) = \Phi(1,x)$.

In $d = 2$ one has $u(0,x) = M\delta$ with $M \in [0, M(\tau))$;
$8\pi = M(\tau)$ for $\tau \in (0, \frac{1}{2})$, $M(\tau) \nearrow \infty$ as $\tau \to \infty$.

Nonuniqueness of self-similar solutions for large $\tau \gg 1$ and sufficiently large M.

For $d \geq 3$ if the limit $u_0(x) \equiv \lim_{t \to 0} \frac{1}{t} U\left(\frac{x}{\sqrt{t}}\right)$ exists (for example, in the sense of distributions), then the initial datum u_0 has to be homogeneous of degree -2.

In particular, we will construct radial, nonnegative self-similar solutions to system KS corresponding to initial data of the form $u_0(x) = \frac{M}{|x|^2}$ for some constant $M > 0$.
When $\tau = 0$ they exist for $M \in [0, 2(d-2))$, while $u_C(x) = 2(d-2)|x|^{-2}$ is a **discontinuous** singular stationary solution (C=Chandrasekhar).

For $d \geq 3$ if the limit $u_0(x) \equiv \lim_{t\to 0} \frac{1}{t} U\left(\frac{x}{\sqrt{t}}\right)$ exists (for example, in the sense of distributions), then the initial datum u_0 has to be homogeneous of degree -2.

In particular, we will construct radial, nonnegative self-similar solutions to system KS corresponding to initial data of the form $u_0(x) = \frac{M}{|x|^2}$ for some constant $M > 0$.
When $\tau = 0$ they exist for $M \in [0, 2(d-2))$, while $u_C(x) = 2(d-2)|x|^{-2}$ is a **discontinuous** singular stationary solution (C=Chandrasekhar).

Note that for sufficiently small, not necessarily radial but homogeneous initial datum such a construction can be made using a fixed point argument, in the framework of mild solutions. Even for sign changing initial data.

• another motivation

When $\tau = 0$ blowup of “big” nonnegative solutions occur.

• For $d = 2$ “big” means $\int u_0(x)\,\mathrm{d}x > 8\pi$ [many results, various approaches].

• another motivation

When $\tau = 0$ blowup of "big" nonnegative solutions occur.

• For $d = 2$ "big" means $\int u_0(x)\,\mathrm{d}x > 8\pi$ [many results, various approaches].

There is a (perfect) dichotomy:

If $\int u_0(x)\,\mathrm{d}x \le 8\pi$, then the solution is global-in-time and uniformly bounded in any L^p, $1 \le p \le \infty$.

• another motivation

When $\tau = 0$ blowup of "big" nonnegative solutions occur.

• For $d = 2$ "big" means $\int u_0(x)\,\mathrm{d}x > 8\pi$ [many results, various approaches].

There is a (perfect) dichotomy:
If $\int u_0(x)\,\mathrm{d}x \le 8\pi$, then the solution is global-in-time and uniformly bounded in any L^p, $1 \le p \le \infty$.

• For $d \ge 3$ "big" means (when $u_0 \ge 0$) (PB + J. Zienkiewicz 2019)

$$\sup_{R>0} R^{2-d} \int_{\{|x|<R\}} u_0(x)\,\mathrm{d}x \gg 1.$$

This is the **homogeneous Morrey** space $M^s(\mathbb{R}^d)$ norm with $s = \frac{d}{2}$: $\quad \|u\|_{M^s} \equiv \sup_{R>0,\, x\in\mathbb{R}^d} R^{d(1/s-1)} \int_{\{|y-x|<R\}} |u(y)|\,\mathrm{d}y.$

• another motivation

When $\tau = 0$ blowup of "big" nonnegative solutions occur.

• For $d = 2$ "big" means $\int u_0(x)\,\mathrm{d}x > 8\pi$ [many results, various approaches].

There is a (perfect) dichotomy:

If $\int u_0(x)\,\mathrm{d}x \le 8\pi$, then the solution is global-in-time and uniformly bounded in any L^p, $1 \le p \le \infty$.

• For $d \ge 3$ "big" means (when $u_0 \ge 0$) (PB + J. Zienkiewicz 2019)

$$\sup_{R>0} R^{2-d} \int_{\{|x|<R\}} u_0(x)\,\mathrm{d}x \gg 1.$$

This is the **homogeneous Morrey** space $M^s(\mathbb{R}^d)$ norm with $s = \frac{d}{2}$: $\quad \|u\|_{M^s} \equiv \sup_{R>0,\, x\in\mathbb{R}^d} R^{d(1/s-1)} \int_{\{|y-x|<R\}} |u(y)|\,\mathrm{d}y.$

Notice that the local existence of solutions to the Cauchy problem requires some regularity of u_0 ($u_0 \in M^{\frac{d}{2}}(\mathbb{R}^d)$) and a size condition on local singularities, i.e.

$$\limsup_{R\to 0} R^{2-d} \int_{\{|x|<R\}} u_0(x)\,\mathrm{d}x < (\text{some constant})(d),$$

• For $\tau > 0$ blowup results are rather scarce, and they are obtained under specific integrability conditions imposed on u_0 (M. Winkler 2013, 2020) such as $\int u_0 \log u_0 < \infty$.

Our goals:

• Show existence of global-in-time solutions with arbitrarily large data (not very regular) and suitably big τ (extending the result in two dimensions by PB + I. Guerra + G. Karch 2015).

• Prove finite time blowup of solutions with even larger data.

• For $\tau > 0$ blowup results are rather scarce, and they are obtained under specific integrability conditions imposed on u_0 (M. Winkler 2013, 2020) such as $\int u_0 \log u_0 < \infty$.

Our goals:

• Show existence of global-in-time solutions with arbitrarily large data (not very regular) and suitably big τ (extending the result in two dimensions by PB + I. Guerra + G. Karch 2015).

• Prove finite time blowup of solutions with even larger data.

The first goal is achieved.

• For $\tau > 0$ blowup results are rather scarce, and they are obtained under specific integrability conditions imposed on u_0 (M. Winkler 2013, 2020) such as $\int u_0 \log u_0 < \infty$.

Our goals:

• Show existence of global-in-time solutions with arbitrarily large data (not very regular) and suitably big τ (extending the result in two dimensions by PB + I. Guerra + G. Karch 2015).

• Prove finite time blowup of solutions with even larger data.

The first goal is achieved.
The second goal is not.

• For $\tau > 0$ blowup results are rather scarce, and they are obtained under specific integrability conditions imposed on u_0 (M. Winkler 2013, 2020) such as $\int u_0 \log u_0 < \infty$.

Our goals:

• Show existence of global-in-time solutions with arbitrarily large data (not very regular) and suitably big τ (extending the result in two dimensions by PB + I. Guerra + G. Karch 2015).

• Prove finite time blowup of solutions with even larger data.

The first goal is achieved.
The second goal is not.

However, we proposed two toy models, both consisting of two parabolic equations, for which the above scenario is confirmed.

They are related to the well known semilinear heat equation (NLH) $u_t = \Delta u + u^2$ in a way similar to that as (PP) is related to (PE).

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d, \ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM}$$

They are related to the well known semilinear heat equation (NLH) $u_t = \Delta u + u^2$ in a way similar to that as (PP) is related to (PE).

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM}$$

$$\begin{cases} u_t = \Delta u + (\Delta\varphi)^2, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM'}$$

They are related to the well known semilinear heat equation (NLH) $u_t = \Delta u + u^2$ in a way similar to that as (PP) is related to (PE).

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM}$$

$$\begin{cases} u_t = \Delta u + (\Delta\varphi)^2, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM'}$$

$$\begin{cases} u_t = \Delta u + u^2, \\ u(0) = u_0, \end{cases} \qquad x \in \mathbb{R}^d,\ t > 0. \tag{NLH}$$

They are related to the well known semilinear heat equation (NLH) $u_t = \Delta u + u^2$ in a way similar to that as (PP) is related to (PE).

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM}$$

$$\begin{cases} u_t = \Delta u + (\Delta\varphi)^2, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{TM'}$$

$$\begin{cases} u_t = \Delta u + u^2, \\ u(0) = u_0, \end{cases} \qquad x \in \mathbb{R}^d,\ t > 0. \tag{NLH}$$

(TM) is obtained by omitting the term $-\nabla u \cdot \nabla\varphi$ in (PP)/(PE).

They are related to the well known semilinear heat equation (NLH) $u_t = \Delta u + u^2$ in a way similar to that as (PP) is related to (PE).

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, & \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, & \end{cases} \tag{TM}$$

$$\begin{cases} u_t = \Delta u + (\Delta\varphi)^2, & \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, & \end{cases} \tag{TM'}$$

$$\begin{cases} u_t = \Delta u + u^2, & \\ u(0) = u_0, & \end{cases} \qquad x \in \mathbb{R}^d,\ t > 0. \tag{NLH}$$

(TM) is obtained by omitting the term $-\nabla u \cdot \nabla\varphi$ in (PP)/(PE). And for $\tau = 0$ the term $-u\Delta\varphi$ is just u^2, as is for (TM').

There are many results on blowup for (NLH) obtained with the use of various methods (moment $\equiv$ eigenfunction method, convexity $\equiv$ energy method, monotonicity) based on specific properties of the single parabolic equation (NLH) (such as variational structure).

There are many results on blowup for (NLH) obtained with the use of various methods (moment $\equiv$ eigenfunction method, convexity $\equiv$ energy method, monotonicity) based on specific properties of the single parabolic equation (NLH) (such as variational structure).

Surprisingly, for the systems of two parabolic equations those approaches (almost) fail.

There are many results on blowup for (NLH) obtained with the use of various methods (moment $\equiv$ eigenfunction method, convexity $\equiv$ energy method, monotonicity) based on specific properties of the single parabolic equation (NLH) (such as variational structure).

Surprisingly, for the systems of two parabolic equations those approaches (almost) fail.
Instead, we applied a Fourier transform argument showing that a singularity may appear for $\widehat{u}(.,t)$.

Cross-diffusion structures of (PP), (TM) make their analyses delicate.

The main results

Mild formulation of the evolution problem

$u = U_0 + B(u,u)$, $U_0(t) = e^{t\Delta}u_0$, $Lz(t) = \frac{1}{\tau}\int_0^t \nabla \mathrm{e}^{\frac{1}{\tau}(t-s)\Delta} z(s)\,\mathrm{d}s$,

$$B(u,z)(t) = -\int_0^t \nabla \mathrm{e}^{(t-s)\Delta} \cdot (u(s)Lz(s))\,\mathrm{d}s,$$

Pseudomeasures

$\mathcal{PM}^a(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{PM}^a} = \sup_{\xi\in\mathbb{R}^d} |\xi|^a|\widehat{f}(\xi)| < \infty\}$,

$\widehat{f}$ denotes the Fourier transform of the tempered distribution f

The main results

Mild formulation of the evolution problem

$u = U_0 + B(u,u)$, $U_0(t) = e^{t\Delta}u_0$, $Lz(t) = \frac{1}{\tau}\int_0^t \nabla \mathrm{e}^{\frac{1}{\tau}(t-s)\Delta} z(s)\,\mathrm{d}s$,

$$B(u,z)(t) = -\int_0^t \nabla \mathrm{e}^{(t-s)\Delta}\cdot(u(s)Lz(s))\,\mathrm{d}s,$$

Pseudomeasures

$\mathcal{PM}^a(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{PM}^a} = \sup_{\xi\in\mathbb{R}^d} |\xi|^a|\widehat{f}(\xi)| < \infty\}$,

$\widehat{f}$ denotes the Fourier transform of the tempered distribution f

$$\mathcal{Y}_a = \{u \in L^\infty_{\mathrm{loc}}(0,\infty;\mathcal{S}'(\mathbb{R}^d)) : \sup_{t>0,\,\xi\in\mathbb{R}^d} t^{1+(a-d)/2}|\xi|^a|\widehat{u}(\xi,t)| < \infty\},$$

$\mathcal{Y}_{d-2} \equiv \mathcal{X} = L^\infty(0,\infty;\mathcal{PM}^{d-2})$. Clearly, $\frac{1}{|x|^2} \in \mathcal{PM}^{d-2}$.

The main results

Mild formulation of the evolution problem

$u = U_0 + B(u,u),\ U_0(t) = e^{t\Delta}u_0,\ Lz(t) = \frac{1}{\tau}\int_0^t \nabla \mathrm{e}^{\frac{1}{\tau}(t-s)\Delta} z(s)\,\mathrm{d}s,$

$$B(u,z)(t) = -\int_0^t \nabla \mathrm{e}^{(t-s)\Delta}\cdot(u(s)Lz(s))\,\mathrm{d}s,$$

Pseudomeasures

$\mathcal{PM}^a(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{PM}^a} = \sup_{\xi\in\mathbb{R}^d} |\xi|^a|\widehat{f}(\xi)| < \infty\},$

$\widehat{f}$ denotes the Fourier transform of the tempered distribution f

$$\mathcal{Y}_a = \{u \in L^\infty_{\mathrm{loc}}(0,\infty;\mathcal{S}'(\mathbb{R}^d)) : \sup_{t>0,\,\xi\in\mathbb{R}^d} t^{1+(a-d)/2}|\xi|^a|\widehat{u}(\xi,t)| < \infty\},$$

$\mathcal{Y}_{d-2} \equiv \mathcal{X} = L^\infty(0,\infty;\mathcal{PM}^{d-2})$. Clearly, $\frac{1}{|x|^2} \in \mathcal{PM}^{d-2}$.

Theorem Let $d \geq 2$, $u_0 \in \mathcal{PM}^{d-2}$, $\varphi_0 = 0$, $\tau \geq \mathrm{e}^3$.
If $\|u_0\|_{\mathcal{PM}^{d-2}} < 3^3\kappa_d\,\tau/(\mathrm{e}\ln\tau)^3$, then (PP) has a global-in-time solution. This solution belongs to $\mathcal{X}\cap\mathcal{Y}_{d-4/\ln\tau}$, and is unique in the ball of $\mathcal{Y}_{d-4/\ln\tau}$ centered at the origin, with radius $0 < r \lesssim \tau/(\ln\tau)^3$.

Besov spaces

Almost optimal spaces, optimal initial data: $\mathcal{E}_p :=$
$\Big\{ u \in L^\infty(0,\infty; L^p(\mathbb{R}^d)),\ \lvert\lvert\lvert u \rvert\rvert\rvert_p := \sup_{t>0} t^{1-d/(2p)} \|u\|_p < \infty \Big\}$

Besov spaces

Almost optimal spaces, optimal initial data: $\mathcal{E}_p :=$
$\left\{ u \in L^\infty(0,\infty; L^p(\mathbb{R}^d)),\ |||u|||_p := \sup_{t>0} t^{1-d/(2p)}\|u\|_p < \infty \right\}$

Theorem Let $d \geq 2$, $\max(d/2,\ 2d/(d+1)) < p < 2d$, $u_0 \in \dot{B}^{-(2-d/p)}_{p,\infty}$ and $\varphi_0 = 0$. Let q such that

$$|1/p - 1/d| < 1/q \leq \min(1/p, 1-1/p), \qquad 1/q < 1/d.$$

Then there exist constants $C_{p,q}, \kappa_{p,q} > 0$, independent of τ and u_0, such that if

$$\|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}} < C_{p,q}\tau^{1/2-d/2(1/p-1/q)},$$

then (PP) has a mild solution $u \in \mathcal{E}_p$, such that $|||u|||_p \leq \kappa_{p,q}\|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}}$. Moreover, mild solutions of (PP) satisfying

$$|||u|||_p \leq R, \quad R < \kappa_{p,q}C_{p,q}\,\tau^{1/2-d/2(1/p-1/q)},$$

are unique.

Existence results for toy models (TM) and (TM')

Almost the same as for (PP), with similar proofs.

Blowup of large solutions for both toy models

We are interested in nonnegative solutions of big size, say $\approx \tau$.

An idea of S. Montgomery-Smith for "cheap" Navier-Stokes equations: blowup of the Fourier transform of solutions of (TM).

Blowup of large solutions for both toy models

We are interested in nonnegative solutions of big size, say $\approx \tau$.

An idea of S. Montgomery-Smith for "cheap" Navier-Stokes equations: blowup of the Fourier transform of solutions of (TM).

Consider $w_0 \in L^2(\mathbb{R}^d)$ defined by $\widehat{w}_0(\xi) = \mathbb{I}_{B_0}(\xi)$, where $\mathbb{I}_E$ denotes the indicator function of a measurable set E, and B_0 is the ball with center $\frac{3}{4}(1, 0 \ldots, 0)$ and radius $\frac{1}{4}$. Thus, the support of $\widehat{w}_0$ is contained in the annulus $E_0 = \{\frac{1}{2} \leq |\cdot| \leq 1\}$.

Blowup of large solutions for both toy models

We are interested in nonnegative solutions of big size, say $\approx \tau$.

An idea of S. Montgomery-Smith for "cheap" Navier-Stokes equations: blowup of the Fourier transform of solutions of (TM).

Consider $w_0 \in L^2(\mathbb{R}^d)$ defined by $\widehat{w}_0(\xi) = \mathbb{I}_{B_0}(\xi)$, where $\mathbb{I}_E$ denotes the indicator function of a measurable set E, and B_0 is the ball with center $\frac{3}{4}(1, 0 \dots, 0)$ and radius $\frac{1}{4}$. Thus, the support of $\widehat{w}_0$ is contained in the annulus $E_0 = \{\frac{1}{2} \leq |\cdot| \leq 1\}$.

Theorem Let $\tau > 0$, $A > 0$, and $u_0 \in \mathcal{S}(\mathbb{R}^d)$, such that

$$\widehat{u}_0(\xi) \geq A\widehat{w}_0(\xi).$$

Let t^* be the maximal lifetime of the (unique) classical solution to (TM). There exists a constant $\kappa_d > 0$ (only dependent on d) such that if

$$A > \kappa_d \, \mathrm{e}^{1/\tau} \, \tau,$$

then $t^* < 1$.

Notice that the right-hand side above behaves like τ as $\tau \gg 1$. Thus, the best possible size condition to be put on the initial data, in order to obtain the global existence for (TM), would be of the form $\|u_0\| \lesssim \tau$, no matter the choice of the norm, and irrespectively of the functional setting where one constructs the solution.

Notice that the right-hand side above behaves like τ as $\tau \gg 1$. Thus, the best possible size condition to be put on the initial data, in order to obtain the global existence for (TM), would be of the form $\|u_0\| \lesssim \tau$, no matter the choice of the norm, and irrespectively of the functional setting where one constructs the solution.

Finally, consider (TM') in a smooth bounded domain $\Omega \subset \mathbb{R}^d$, supplemented with

$$u(x,t) = \varphi(x,t) = 0 \quad \text{for each} \quad x \in \partial\Omega, \quad t \geq 0.$$

There exist positive solutions of system (TM') for $\tau \geq 2$ with $u_0 \geq 0$, $\varphi_0 \geq 0$ of order τ^2, τ, resp., satisfying moreover

$$\int_\Omega \psi(x) u_0(x)\,\mathrm{d}x \geq \frac{3}{2}\lambda\tau^2,$$

$$\int_\Omega \psi(x) \varphi_0(x)\,\mathrm{d}x \geq \frac{3}{2}\tau,$$

where $\psi \geq 0$ is the normalized eigenfunction of Δ with the first eigenvalue λ, which cannot be continued past a moment $T > 0$.

The proof of that result involves the equation for φ

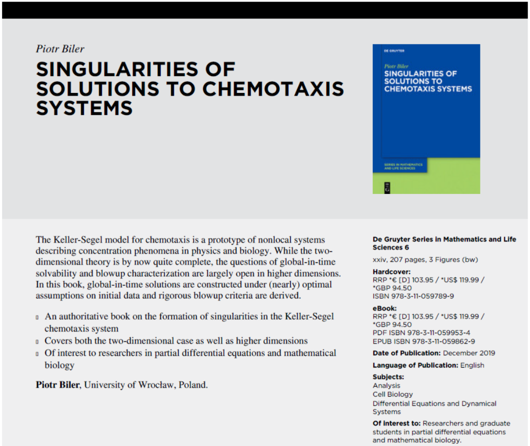
$$\tau\varphi_{tt} = (\tau+1)\Delta\varphi_t - \Delta^2\varphi + (\Delta\varphi)^2$$

and the evolution of moment $J(t) = \int_\Omega \psi(x)\varphi(x,t)\,\mathrm{d}x$ is studied

$$\tau\ddot{J}(t) = -\lambda(\tau+1)\dot{J}(t) - \lambda^2 J(t) + \int_\Omega \psi(\Delta\varphi)^2$$

$$\geq -\lambda(\tau+1)\dot{J}(t) - \lambda^2 J(t) + \lambda^2 J(t)^2.$$

References



Piotr Biler

SINGULARITIES OF SOLUTIONS TO CHEMOTAXIS SYSTEMS

The Keller-Segel model for chemotaxis is a prototype of nonlocal systems describing concentration phenomena in physics and biology. While the two-dimensional theory is by now quite complete, the questions of global-in-time solvability and blowup characterization are largely open in higher dimensions. In this book, global-in-time solutions are constructed under (nearly) optimal assumptions on initial data and rigorous blowup criteria are derived.

- An authoritative book on the formation of singularities in the Keller-Segel chemotaxis system
- Covers both the two-dimensional case as well as higher dimensions
- Of interest to researchers in partial differential equations and mathematical biology

Piotr Biler, University of Wrocław, Poland.

De Gruyter Series in Mathematics and Life Sciences 6

xxiv, 207 pages, 3 Figures (bw)

Hardcover:
RRP *€ [D] 103.95 / *US$ 119.99 / *GBP 94.50
ISBN 978-3-11-059789-9

eBook:
RRP *€ [D] 103.95 / *US$ 119.99 / *GBP 94.50
PDF ISBN 978-3-11-059953-4
EPUB ISBN 978-3-11-059862-9

Date of Publication: December 2019

Language of Publication: English

Subjects:
Analysis
Cell Biology
Differential Equations and Dynamical Systems

Of interest to: Researchers and graduate students in partial differential equations and mathematical biology.

P. Biler, *Singularities of solutions in chemotaxis systems*,
Series in Mathematics and Life Sciences, De Gruyter, Berlin, 2020.
ISBN 978-3-11-059789-9.

P. Biler, A.Boritchev, L. Brandolese, *Large global solutions of the parabolic-parabolic Keller–Segel system in higher dimensions*, 1–23. arXiv: 2203.09130. hal.archives-ouvertes.fr/hal-03609520.

P. Biler, A. Boritchev, L. Brandolese, *Sharp well-posedness and blowup results for parabolic systems of the Keller–Segel type*, 1–24. arXiv:220610399. hal.archives-ouvertes.fr/hal-03699868.

P. Biler, L. Corrias, J. Dolbeault, *Large mass self-similar solutions of the parabolic-parabolic Keller–Segel model*, J. Math. Biology **63** (2011), 1–32.

P. Biler, I. Guerra, G. Karch, *Large global-in-time solutions of the parabolic-parabolic Keller–Segel system on the plane*, Commun. Pure Appl. Analysis **14** (2015), 2117–2126.

P. Biler, G. Karch, J. Zienkiewicz, *Large global-in-time solutions to a nonlocal model of chemotaxis*, Adv. Math. **330** (2018), 834–875.

P. Biler, J. Zienkiewicz, *Blowing up radial solutions in the minimal Keller–Segel model of chemotaxis*, J. Evol. Equ. **19** (2019), 71–90. arXiv:1807.02633.

V. Calvez, L. Corrias, M. A. Ebde, *Blow-up, concentration phenomenon and global existence for the Keller–Segel model in high dimension*, Commun. Partial Differential Equations, **37** (2012), 561–584.

P.-G. Lemarié-Rieusset, *Small data in an optimal Banach space for the parabolic-parabolic and parabolic-elliptic Keller–Segel equations in the whole space*, Adv. Differ. Eq. **18** (2013), 1189–1208.

S. Montgomery-Smith, *Finite time blow up for a Navier–Stokes like equations*, Proc. Amer. Math. Soc. **129** (2001), 3025–3029.

Y. Naito, *Blow-up criteria for the classical Keller–Segel model of chemotaxis in higher dimensions*, J. Differential Equations **297** (2021), 144–174.

P. Quittner, Ph. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, 2nd ed., 2019, Birkhäuser/Springer, Cham, xvi+725; ISBN: 978-3-030-18220-5.

Ph. Souplet, M. Winkler, *Blow-up profiles for the parabolic-elliptic Keller–Segel system in dimensions $n \geq 3$*, Comm. Math. Phys. **367** (2019), 665–681.

T. Takeuchi, *The Keller–Segel system of parabolic-parabolic type in homogeneous Besov spaces framework*, J. Differential Equations **298** (2021), 609–640.

M. Winkler, *Single-point blow-up in the Cauchy problem for the higher-dimensional Keller–Segel system*, Nonlinearity, **33**, 2020, 5007–5048.