Dirichlet systems with relativistic operator: differences vs differential

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Contents

A difference system

- $\mathbb{Z}[p,q] := \{p, p+1, \ldots, q\}$ for $p, q \in \mathbb{Z}$ $(p < q)$;
- $B_\sigma := \{x \in \mathbb{R}^N \,:\, |x| < \sigma\}$;
- $\phi : B_1 \to \mathbb{R}^N$ is the homeomorphism

$$\phi(y) = \frac{y}{\sqrt{1-|y|^2}} \quad (y \in B_1). \tag{1.1}$$

- We mainly discuss Dirichlet systems of difference equations of type:

$$\left\{ \begin{array}{ll} \Delta\left[\phi(\Delta u(n-1))\right] = f(n, u(n-1), u(n), u(n+1)) & (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) & \end{array} \right. \tag{1.2}$$

◇ $T \in \mathbb{Z}$ is positive and fixed;

◇ $\Delta u(n-1) := u(n) - u(n-1)$ is the usual forward difference operator;

◇ $f : \mathbb{Z}[1,T] \times (\mathbb{R}^N)^3 \to \mathbb{R}^N$ is continuous.

Notice, the unknown function u is $\mathbb{R}^N$-valued.

- Multiple solutions for systems having the form

$$\left\{ \begin{array}{ll} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda \nabla G(n, u(n)) & (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) & \end{array} \right. \tag{1.3}$$

◇ $\lambda > 0$ is a real parameter, $G(n,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is of class C^1 for all $n \in \mathbb{Z}[1,T]$.

$\diamond\diamond$ J. Mawhin, *Nonlinear Anal.* (2012), *J. Difference Equ. Appl.* (2016)
$\diamond\diamond$ X. Su and R. Ma, *Adv. Difference Equ.* (2020)

- The equation in (1.2) is a discrete variant of a system having the form

$$[\phi(u')]' = \widetilde{f}(t, u(t), u'(t)). \tag{1.4}$$

$$u \mapsto [\phi(u')]' \ - \ \textit{the relativistic acceleration operator}$$

The motion of a slowly accelerated charged particle under the influence of an electromagnetic field in a relativistic regime is described by the Lorentz force equation:

$$[\phi(u')]' = \overrightarrow{E}(t, u) + u' \times \overrightarrow{B}(t, u), \tag{1.5}$$

where $\overrightarrow{E}$, $\overrightarrow{B}$ denote the electric and magnetic field, respectively[1,2].

- The PDE variant of (1.4) is

$$\operatorname{div}[\phi(\nabla u)] = \widetilde{f}(x, u, \nabla u). \tag{1.6}$$

$$u \mapsto \operatorname{div}[\phi(\nabla u)] \ - \ \textit{the mean curvature operator in Minkowski space}$$

This also occurs in Born-Infeld nonlinear theory of electromagnetism[3] in the frame of Einstein's special relativity.

[1] H. Poincaré, *Rend. Circ. Mat. Palermo*, 1906
[2] M. Planck, *Verh. Deutsch. Phys. Ges.* 1906
[3] M. Born and L. Infeld, *Proc. R. Soc. Lond. Ser. A*, 1934

Solvability of system (1.2)

$$\begin{cases} \Delta\left[\phi(\Delta u(n-1))\right] = f(n, u(n-1), u(n), u(n+1)) & (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{1.2}$$

◇ $T \in \mathbb{Z}$ is positive and fixed;

◇ $\Delta u(n-1) := u(n) - u(n-1)$ is the usual forward difference operator;

◇ $f : \mathbb{Z}[1, T] \times (\mathbb{R}^N)^3 \to \mathbb{R}^N$ is continuous.

• $X_T := \{u : \mathbb{Z}[1, T] \to \mathbb{R}^N\}$ and $X^0_{T+2} := \{u : \mathbb{Z}[0, T+1] \to \mathbb{R}^N \mid u(0) = 0_{\mathbb{R}^N} = u(T+1)\}$ endowed with the inner product and the corresponding norm

$$(u|v)_T = \sum_{j=1}^{T} \langle u(j), v(j) \rangle, \quad \|u\|_T = \left(\sum_{j=1}^{T} |u(j)|^2 \right)^{1/2},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^N$.

• For $\ell \in X_T$, we define the operator $E_\ell : \mathbb{R}^N \to \mathbb{R}^N$ by

$$E_\ell(x) = \sum_{n=1}^{T+1} \phi^{-1} \left(\sum_{j=1}^{n-1} \ell(j) + x \right) \quad \left(x \in \mathbb{R}^N \right).$$

Notice, the inverse mapping $\phi^{-1} : \mathbb{R}^N \to B_1$ is given by

$$\phi^{-1}(y) = \frac{y}{\sqrt{1 + |y|^2}} \quad \left(y \in \mathbb{R}^N \right) \tag{2.1}$$

and it is strictly monotone.

Lemma 2.1. *For any $\ell \in X_T$, the equation $E_\ell(x) = 0_{\mathbb{R}^N}$ has an unique solution $Q(\ell) \in \mathbb{R}^N$. Moreover, the mapping $Q : X_T \to \mathbb{R}^N$ is continuous and satisfies*

$$|Q(\ell)| < (1+\sqrt{2}) \sum_{j=1}^{T} |\ell(j)| + 1, \tag{2.2}$$

for all $\ell \in X_T$.

Proof. Strictly monotonicity of E_ℓ $\Rightarrow$ uniqueness of the solution $Q(\ell)$.
Existence and (2.2): set

$$R_\ell := (1+\sqrt{2}) \sum_{j=1}^{T} |\ell(j)| + 1; \qquad c^n := \sum_{j=1}^{n-1} \ell(j) \quad (n \in \mathbb{Z}[1, T+1])$$

• $x \in \mathbb{R}^N$ with $|x| = R_\ell$:

$$\langle E_\ell(x), x \rangle = \sum_{n=1}^{T+1} \frac{\langle c^n + x, x \rangle}{\sqrt{1 + |c^n + x|^2}} \geq \cdots \geq \frac{\sqrt{2}}{2}(T+1). \tag{2.3}$$

Let the homotopy $\mathcal{H} : [0,1] \times \overline{B}_{R_\ell} \to \mathbb{R}^N$ be defined by

$$\mathcal{H}(t,x) = tx + (1-t)E_\ell(x), \quad \forall\ t \in [0,1],\ \forall\ x \in \overline{B}_{R_\ell}. \tag{2.4}$$

For any $t \in [0,1]$ and $x \in \partial B_{R_\ell}$, we have

$$\langle \mathcal{H}(t,x), x \rangle = t|x|^2 + (1-t)\langle E_\ell(x), x \rangle \geq tR_\ell^2 + (1-t)\frac{\sqrt{2}}{2}(T+1) > 0, \tag{2.5}$$

showing that $0_{\mathbb{R}^N} \notin \mathcal{H}([0,1] \times \partial B_{R_\ell})$.

- The invariance under a homotopy of the Brouwer degree $\Rightarrow$

$$\begin{aligned}\deg\left[E_\ell, B_{R_\ell}, 0_{\mathbb{R}^N}\right] &= \deg\left[\mathcal{H}(0,\cdot), B_{R_\ell}, 0_{\mathbb{R}^N}\right] = \deg\left[\mathcal{H}(1,\cdot), B_{R_\ell}, 0_{\mathbb{R}^N}\right]\\ &= \deg\left[I_{\mathbb{R}^N}, B_{R_\ell}, 0_{\mathbb{R}^N}\right] = 1\end{aligned}$$

$\Rightarrow$ the equation $E_\ell(x) = 0_{\mathbb{R}^N}$ has a (unique) solution $Q(\ell)$ and (2.2) holds true.

Continuity of the mapping $Q : X_T \to \mathbb{R}^N$: let $\{\ell^m\}$ be a sequence in X_T s.t. $\ell^m \xrightarrow{\|\cdot\|_T} \ell$ in X_T, as $m \to \infty$. We show that for any subsequence of $\{Q(\ell^m)\}$, still denoted by $\{Q(\ell^m)\}$, there exists a subsequence $\{Q(\ell^{m_k})\}$ of $\{Q(\ell^m)\}$ s.t. $Q(\ell^{m_k}) \to Q(\ell)$, as $k \to \infty$. This will conclude the proof. Since $\{\ell^m\}$ is bounded in $(X_T, \|\cdot\|_T)$, using (2.2) we get that $\{Q(\ell^m)\}$ is bounded in $\mathbb{R}^N$; hence, there is a subsequence $\{Q(\ell^{m_k})\}$ of $\{Q(\ell^m)\}$ and some $\alpha \in \mathbb{R}^N$ s.t. $Q(\ell^{m_k}) \to \alpha$, as $k \to \infty$. Now, for all $n \in \mathbb{Z}[1, T+1]$, one has

$$\sum_{j=1}^{n-1} \ell^{m_k}(j) + Q(\ell^{m_k}) \to \sum_{j=1}^{n-1} \ell(j) + \alpha, \tag{2.6}$$

which implies that

$$\sum_{n=1}^{T+1} \phi^{-1}\left(\sum_{j=1}^{n-1} \ell^{m_k}(j) + Q(\ell^{m_k})\right) \to \sum_{n=1}^{T+1} \phi^{-1}\left(\sum_{j=1}^{n-1} \ell(j) + \alpha\right), \tag{2.7}$$

i.e., $0_{\mathbb{R}^N} = E_{\ell^{m_k}}(Q(\ell^{m_k})) \to E_\ell(\alpha)$, as $k \to \infty$. Therefore, $E_\ell(\alpha) = 0_{\mathbb{R}^N}$ and by the uniqueness of solution $Q(\ell)$, we get $\alpha = Q(\ell)$. ∎

Theorem 2.1. *For any $\ell \in X_T$, problem*

$$\begin{cases} \Delta\left[\phi(\Delta u(n-1))\right] = \ell(n) & (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) \end{cases} \tag{2.8}$$

has an unique solution $u_\ell \in X^0_{T+2}$ and this is given by

$$u_\ell(n) = \sum_{i=0}^{n-1} \phi^{-1}\left(\sum_{j=1}^{i} \ell(j) + Q(\ell)\right) \quad (n \in \mathbb{Z}[0, T+1]). \tag{2.9}$$

Proof. Lemma 2.1 + direct computations. ∎

Theorem 2.2. *For any continuous function* $f : \mathbb{Z}[1, T] \times (\mathbb{R}^N)^3 \to \mathbb{R}^N$, *problem* (1.2) *has at least one solution.*

Proof. $N_f : X^0_{T+2} \to X_T$ (Nemytskii type operator),

$$N_f(u)(n) = f(n, u(n-1), u(n), u(n+1)) \quad (n \in \mathbb{Z}[1, T]). \tag{2.10}$$

- $\mathcal{D} : X^0_{T+2} \to X^0_{T+2}$ (fixed point operator),

$$\mathcal{D}(u)(n) := \sum_{i=0}^{n-1} \phi^{-1}\left(\sum_{j=1}^{i} N_f(u)(j) + Q(N_f(u))\right) \quad (n \in \mathbb{Z}[0, T+1]). \tag{2.11}$$

is well defined, continuous (as N_f and Q are continuous) and $\|\mathcal{D}(u)\|_T \leq T\sqrt{T}$ for all $u \in X^0_{T+2}$.

- Brouwer's fixed point theorem implies that $\mathcal{D}$ has a fixed point $u \in X^0_{T+2}$;

$$u(n) = \sum_{i=0}^{n-1} \phi^{-1}\left(\sum_{j=1}^{i} N_f(u)(j) + Q(N_f(u))\right) \tag{2.12}$$

and Theorem 2.1 with $\ell = N_f(u) \Rightarrow u$ satisfies

$$\Delta\left[\phi(\Delta u(n-1))\right] = N_f(u)(n) \quad (n \in \mathbb{Z}[1, T]). \tag{2.13}$$

∎

Variational solutions

- We introduce a variational formulation for a potential Dirichlet problem having the form

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \nabla F(n, u(n)), \qquad (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{3.1}$$

$\diamond$ $F(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is a mapping of class C^1 for all $n \in \mathbb{Z}[1, T]$.

- Denoting $|\Delta u|_\infty := \max_{n \in \mathbb{Z}[0,T]} |\Delta u(n)|$, we introduce

$$K := \{u \in X^0_{T+2} : \ |\Delta u|_\infty \leq 1\}$$

which is a convex and compact subset of X^0_{T+2}.

- Setting

$$\Phi(y) := 1 - \sqrt{1 - |y|^2}, \qquad (y \in \overline{B}_1),$$

we consider the functionals

$$\Psi : X^0_{T+2} \to (-\infty, +\infty], \qquad \Psi(u) = \begin{cases} \sum_{j=0}^{T} \Phi[\Delta u(j)], & \text{if } u \in K, \\ +\infty, & \text{if } u \in X^0_{T+2} \setminus K, \end{cases} \tag{3.2}$$

and

$$\mathbf{F} : X^0_{T+2} \to \mathbb{R}, \qquad \mathbf{F}(u) = -\sum_{j=1}^{T} F(j, u(j)).$$

- Ψ is proper, convex and lower semicontinuos, while $\mathbf{F}$ is of class C^1, its derivative being given by

$$\mathbf{F}'(u)(v) = -\sum_{j=1}^{T} \langle \nabla F(j,(u(j)), v(j) \rangle \qquad (u, v \in X^0_{T+2}). \tag{3.3}$$

- The action functional $I : X^0_{T+2} \to (-\infty, +\infty]$ associated to (3.1) will be

$$I = \Psi + \mathbf{F},$$

which fits the structure required by Szulkin's critical point theory[4].

Recall: An element $u \in K$ $(= D(\Psi))$ is said to be *critical point* of I if it satisfies the variational inequality

$$\langle \mathbf{F}'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0 \quad (v \in K).$$

A sequence $\{u_n\} \subset K$ is called a (PS)-*sequence* if $I(u_n) \to r \in \mathbb{R}$ and

$$\langle \mathbf{F}'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|_T \quad (v \in K),$$

where $\varepsilon_n \to 0+$. The functional I is said *to satisfy the* (PS) *condition* if every (PS)-sequence possesses a convergent subsequence in X^0_{T+2}.

[4] A. Szulkin, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1986

Lemma 3.1. *If $\ell \in X_T$, then u_ℓ from Theorem 2.1 is the unique solution in K of the variational inequality*

$$\sum_{j=0}^{T}\{\Phi[\Delta v(j)] - \Phi[\Delta u(j)]\} + \sum_{j=1}^{T}\langle \ell(j), v(j) - u(j)\rangle \geq 0 \quad (v \in K). \tag{3.4}$$

- The main result of this section is the following

Theorem 3.1. *Any critical point of I is a solution of problem* (3.1). *Moreover, I satisfies the (PS) condition, is bounded from below and attains its infimum at some $u_0 \in K$, which solves problem* (3.1).

Proof. Let $w \in K$ be a critical point of I. On account of (3.3), for any $v \in K$, one has

$$\sum_{j=0}^{T}\{\Phi[\Delta v(j)] - \Phi[\Delta w(j)]\} - \sum_{j=1}^{T}\langle \nabla F(j, w(j)), v(j) - w(j)\rangle \geq 0,$$

i.e., w is a solution of the variational inequality

$$\sum_{j=0}^{T}\{\Phi[\Delta v(j)] - \Phi[\Delta w(j)]\} + \sum_{j=1}^{T}\langle \ell_w(j), v(j) - w(j)\rangle \geq 0, \quad \forall\ v \in K, \tag{3.5}$$

with $\ell_w \in X_T$ given by $\ell_w(n) = -\nabla F(n, w(n))$ $(n \in \mathbb{Z}[1, T])$.

- Lemma 3.1 $\Rightarrow$ w solves problem (3.1).

- K compact $\Rightarrow$ I is bounded from below and satisfies the (PS) condition $\Rightarrow$ I attains its infimum at some $u_0 \in K$ $\Rightarrow$ u_0 is a critical point of I. ■

Parameterized potential systems of type (1.3)

- We are interested about multiple solutions for systems having the form:

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda \nabla G(n, u(n)) & (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{1.3}$$

$\diamond$ $\lambda > 0$ is a real parameter, $G(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is of class C^1 for all $n \in \mathbb{Z}[1, T]$.

- With Ψ defined in (3.2), the energy functional associated with (1.3) will be now $I_\lambda : X^0_{T+2} \to (-\infty, +\infty]$ given by

$$I_\lambda = \Psi + \mathbf{G}_\lambda,$$

with

$$\mathbf{G}_\lambda : X^0_{T+2} \to \mathbb{R}, \qquad \mathbf{G}_\lambda(u) = -\lambda \sum_{j=1}^{T} G(j, u(j)).$$

Notice that, by virtue of Theorem 3.1, system (1.3) is always solvable, I_λ satisfies the (PS) condition, is bounded from below and any critical point of I_λ is a solution of (1.3).

Theorem 4.1. *Assume that $G(\cdot, 0_{\mathbb{R}^N}) = 0$ and that there is some $\eta \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$ with $|\eta| \leq 1$ and $\sum_{j=1}^{T} G(j, \eta) > 0$. If*

$$\lim_{|x| \to 0} \frac{G(n, x)}{|x|^2} = 0 \qquad (n \in \mathbb{Z}[1, T]), \tag{4.1}$$

then there exists $\Lambda > 0$ s.t. (1.3) has at least two nontrivial solutions for all $\lambda > \Lambda$. If, in addition, $G(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is even for all $n \in \mathbb{Z}[1, T]$, then (1.3) has at least two distinct pairs of nontrivial solutions, for all $\lambda > \Lambda$.

Proof. (*Ideea*) *Step 1.* We prove the existence of a constant $\Lambda > 0$ s.t., for all $\lambda > \Lambda$, one has

$$\inf_{X^0_{T+2}} I_\lambda < 0. \tag{4.2}$$

Then, as $I_\lambda(0_{X^0_{T+2}}) = 0$, Theorem 3.1 and (4.2) will ensure that (1.3) has a solution which is a nontrivial minimizer of I_λ.

Step 2. Let $\lambda > \Lambda$ be fixed and $u_{\lambda,1} \in K$ be the corresponding (nontrivial) minimizer of I_λ with

$$I_\lambda(u_{\lambda,1}) < 0. \tag{4.3}$$

We show that there exist $\alpha > 0$ and $\rho \in (0, \|u_{\lambda,1}\|_T)$, with

$$I_\lambda(u) \geq \alpha, \quad \text{for all } u \in K, \text{ with } \|u\|_T = \rho.$$

- Mountain Pass Theorem $\Rightarrow \exists$ a (nontrivial) critical point $u_{\lambda,2}$ of I_λ, with

$$I_\lambda(u_{\lambda,2}) > 0. \tag{4.4}$$

Obviously, $u_{\lambda,1} \neq u_{\lambda,2}$ (see (4.3) and (4.4)). ∎

Example 4.1. If $w : \mathbb{Z}[1, T] \to \mathbb{R}$ is s.t. $0 \not\equiv w \geq 0$ and $a \in \mathbb{R}$ is a constant, then there exists $\Lambda > 0$ so that the system

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda \begin{pmatrix} w(n)\left(|u(n)|u(n)_1 + a\,u(n)_1^2\right) \\ \vdots \\ w(n)\left(|u(n)|u(n)_N + a\,u(n)_N^2\right) \end{pmatrix} & (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) \end{cases}$$

has at least two nontrivial solutions for all $\lambda > \Lambda$. This fact holds because Theorem 4.1 applies with

$$G(n, x) = \frac{w(n)}{3}\left(|x|^3 + a\sum_{i=1}^{N} x_i^3\right) \quad (x = (x_1, \cdots, x_N) \in \mathbb{R}^N,\ n \in \mathbb{Z}[1, T]).$$

If $a = 0$ then at least two distinct pairs of nontrivial solutions exist, for all $\lambda > \Lambda$.

- To produce more solutions, we make use of the eigenvalues of the operator $-\Delta^2$ on

$$\mathcal{X}^0_{T+2} := \{u : \mathbb{Z}[0, T+1] \to \mathbb{R} \ : \ u(0) = 0 = u(T+1)\}.$$

It is known[5] that these are

$$\lambda_m = 4\sin^2\left(\frac{m\pi}{2(T+1)}\right) \qquad (m \in \mathbb{Z}[1, T]). \tag{4.5}$$

Notice that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_T < 4$.

[5] J.F. Elliott, *Master Thesis*, Univ. Tennessee, 1953

Theorem 4.2. *Assume that the C^1 mapping $G(n,\cdot):\mathbb{R}^N\to\mathbb{R}$ is even, $G(n,0_{\mathbb{R}^N})=0$ and that*

$$\liminf_{|x|\to 0}\frac{2G(n,x)}{|x|^2}\geq 1 \qquad (n\in\mathbb{Z}[1,T]). \tag{4.6}$$

Then, the following hold true:

(i) if

$$\lambda > 8\sin^2\left(\frac{m\pi}{2(T+1)}\right)(=2\lambda_m) \quad (m\in\mathbb{Z}[1,T]), \tag{4.7}$$

then (1.3) *has at least mN distinct pairs of nontrivial solutions;*

(ii) if

$$\lambda > 4\sin^2\left(\frac{\pi}{2(T+1)}\right)(=\lambda_1), \tag{4.8}$$

then (1.3) *has at least one pair of nontrivial solutions.*

Proof. (i) According to an extension of a result of D.G. Clark[6] to Szulkin type functionals[7], it suffices to show that there exists a compact, symmetric set A_m, with $0_{X^0_{T+2}}\notin A_m$ and having the Krasnoselskii genus $\gamma(A_m)$ at least mN, s.t.

$$\sup_{v\in A_m} I_\lambda(v)<0. \tag{4.9}$$

[6] D.C. Clark, *Indiana Univ. Math. J.*, 1972

[7] see Theorem 4.3 in A. Szulkin, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1986

Since $\lambda > 2\lambda_m$, we can choose $\varepsilon \in (0,1)$ so that $\lambda > 2\lambda_m/(1-\varepsilon)$ and by (4.6), there exists $\delta \in (0, 1/2]$ s.t.

$$2G(j,x) \geq (1-\varepsilon)|x|^2, \quad \text{for all } j \in \mathbb{Z}[1,T] \text{ and } x \in \mathbb{R}^N, \text{ with } |x| \leq \delta. \tag{4.10}$$

• We consider an orthonormal basis $e^1, \dots, e^T$ in $\mathcal{X}^0_{T+2}$, s.t. e^k is an eigenfunction corresponding to λ_k ($k \in \mathbb{Z}[1,T]$) and let the subspace of X^0_{T+2},

$$E_m := \left\{ \alpha^1 e^1 + \ldots + \alpha^m e^m \;:\; \alpha^1, \ldots, \alpha^m \in \mathbb{R}^N \right\},$$

equipped with the norm

$$\left\| \alpha^1 e^1 + \ldots + \alpha^m e^m \right\|_{E_m} = \left(\sum_{k=1}^m |\alpha^k|^2 \right)^{\frac{1}{2}}.$$

Notice that $E_m = e^1\mathbb{R}^N \oplus e^2\mathbb{R}^N \oplus \cdots \oplus e^m\mathbb{R}^N$ and hence, $\dim E_m = mN$. We set

$$A_m(\rho) := \{ v \in E_m : \; \|v\|_{E_m} = \rho \},$$

with $\rho \in (0, \delta]$.
Then, it is easily seen that the odd mapping $H : A_m(\rho) \to S^{mN-1}$ given by

$$H\left(\sum_{k=1}^m \alpha^k e^k \right) = \left(\frac{\alpha^1_1}{\rho}, \ldots, \frac{\alpha^1_N}{\rho}, \cdots, \frac{\alpha^m_1}{\rho}, \cdots, \frac{\alpha^m_N}{\rho} \right)$$

is a homeomorphism between $A_m(\rho)$ and S^{mN-1} (= the $mN-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^{mN}$). Hence, we have $\gamma(A_m(\rho)) = mN$.

Let $v = \sum_{k=1}^{m} \alpha^k e^k \in A_m(\rho)$. Using that

$$\sum_{j=0}^{T} \Delta e^i(j)\Delta e^k(j) = \lambda_k \delta_{ik} \qquad (i, k \in \mathbb{Z}[1, T]), \tag{4.11}$$

one has

$$\begin{aligned}
\sum_{j=0}^{T} |\Delta v(j)|^2 &= \sum_{j=0}^{T} \left| \Delta \left(\sum_{k=1}^{m} \alpha^k e^k(j) \right) \right|^2 = \sum_{j=0}^{T} \left[\sum_{i=1}^{N} \left(\sum_{k=1}^{m} \alpha_i^k \Delta e^k(j) \right)^2 \right] \\
&= \sum_{j=0}^{T} \left[\sum_{i=1}^{N} \left(\sum_{k=1}^{m} (\alpha_i^k)^2 (\Delta e^k(j))^2 + \sum_{\substack{l,k=1 \\ l \neq k}}^{m} \alpha_i^k \alpha_i^l \Delta e^k(j) \Delta e^l(j) \right) \right] \\
&= \sum_{k=1}^{m} \left[\sum_{i=1}^{N} (\alpha_i^k)^2 \sum_{j=0}^{T} (\Delta e^k(j))^2 \right] \\
&\quad + \sum_{\substack{l,k=1 \\ l \neq k}}^{m} \left[\sum_{i=1}^{N} \alpha_i^k \alpha_i^l \sum_{j=0}^{T} \Delta e^k(j) \Delta e^l(j) \right] \\
&= \sum_{k=1}^{m} \lambda_k |\alpha^k|^2 \leq \lambda_m \|v\|_{E_m}^2 = \lambda_m \rho^2.
\end{aligned} \tag{4.12}$$

This gives $|\Delta v|_\infty \leq \sqrt{\lambda_m}\rho \leq 2\delta \leq 1$, meaning that $A_m(\rho) \subset K$.

Also, it is clear that

$$\begin{aligned}\sum_{j=1}^{T}|v(j)|^2 &= \sum_{j=1}^{T}\left|\sum_{k=1}^{m}\alpha^k e^k(j)\right|^2 = \sum_{j=1}^{T}\left[\sum_{i=1}^{N}\left(\sum_{k=1}^{m}\alpha_i^k e^k(j)\right)^2\right]\\ &= \sum_{j=1}^{T}\left[\sum_{i=1}^{N}\left(\sum_{k=1}^{m}(\alpha_i^k)^2(e^k(j))^2 + \sum_{\substack{l,k=1\\ l\neq k}}^{m}\alpha_i^k\alpha_i^l e^k(j)e^l(j)\right)\right]\\ &= \sum_{k=1}^{m}\sum_{i=1}^{N}(\alpha_i^k)^2 = \sum_{k=1}^{m}|\alpha^k|^2 = \rho^2. \qquad (4.13)\end{aligned}$$

Then, from $|v(j)| \leq \rho \leq \delta$ ($j \in \mathbb{Z}[1, T]$), together with (4.10), (4.12) and (4.13), we estimate I_λ on $A_m(\rho)$ as follows

$$\begin{aligned}I_\lambda(v) &= \Psi(v) + \mathbf{G}_\lambda(v) \leq \sum_{j=0}^{T}|\Delta v(j)|^2 - \frac{\lambda}{2}(1-\varepsilon)\sum_{j=1}^{T}|v(j)|^2\\ &\leq \rho^2\lambda_m - \frac{\lambda}{2}(1-\varepsilon)\rho^2 = \rho^2\frac{2\lambda_m - \lambda(1-\varepsilon)}{2} < 0.\end{aligned}$$

This yields

$$\sup_{v\in A_m(\rho)} I_\lambda(v) < 0,$$

showing that (4.9) holds true with $A_m = A_m(\rho)$ and the proof of (i) is accomplished.

(ii) Let $d \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$ be s.t. $\varphi_1 := de^1$ satisfies $|\Delta\varphi_1|_\infty \leq 1$. Then $\varphi_1 \in K \setminus \{0_{\mathbb{R}^N}\}$ and

$$\sum_{j=0}^{T} |\Delta\varphi_1(j)|^2 = \lambda_1 \sum_{j=1}^{T} |\varphi_1(j)|^2.$$

This implies

$$\lim_{s\to 0^+} \frac{\sum_{j=0}^{T}\left[1-\sqrt{1-|s\Delta\varphi_1(j)|^2}\right]}{\frac{1}{2}\sum_{j=1}^{T}|s\varphi_1(j)|^2} = \lim_{s\to 0^+} \frac{\sum_{j=0}^{T}\frac{s|\Delta\varphi_1(j)|^2}{\sqrt{1-|s\Delta\varphi_1(j)|^2}}}{s\sum_{j=1}^{T}|\varphi_1(j)|^2} = \lambda_1. \tag{4.14}$$

Now, let $\lambda > \lambda_1$ and let us fix some $\varepsilon > 0$ with $\lambda_1 < \lambda - \varepsilon$. From (4.14) there exists $s_1 = s_1(\lambda, \varepsilon) \in (0, 1)$ s.t.

$$\sum_{j=0}^{T}\left[1-\sqrt{1-|s\Delta\varphi_1(j)|^2}\right] < \frac{\lambda-\varepsilon}{2}\sum_{j=1}^{T}|s\varphi_1(j)|^2 \qquad (s \in (0, s_1)). \tag{4.15}$$

On the other hand, from (4.6), there is some $\delta = \delta(\lambda, \varepsilon) > 0$ so that

$$2G(j,x) \geq \left(1 - \frac{\varepsilon}{\lambda}\right) |x|^2, \text{ for all } j \in \mathbb{Z}[1,T] \text{ and } x \in \mathbb{R}^N, \text{ with } |x| \leq \delta. \tag{4.16}$$

Then, choosing $s_2 = s_2(\lambda, \varepsilon) \in (0, s_1)$ with

$$\max_{n \in \mathbb{Z}[1,T]} |s_2 \varphi_1(n)| \leq \delta,$$

from (4.15) and (4.16), we infer

$$I_\lambda(s_2\varphi_1) \leq \sum_{j=0}^{T} \left[1 - \sqrt{1 - |s_2 \Delta \varphi_1(j)|^2}\right] - \frac{\lambda - \varepsilon}{2} \sum_{j=1}^{T} |s_2 \varphi_1(j)|^2 < 0 = I_\lambda(0_{\mathbb{R}^N})$$

and hence, if $\lambda > \lambda_1$, the even functional I_λ attains its infimum at some $u_\lambda \in K \setminus \{0_{\mathbb{R}^N}\}$. Therefore, by Theorem 3.1 system (1.3) has a pair of nontrivial solutions $(u_\lambda, -u_\lambda)$ and the proof is now complete. ■

An application: existence/non-existence and multiplicity of nontrivial solutions for the Dirichlet system involving Fisher-Kolmogorov type nonlinearities

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \lambda(1-w(n)|u(n)|^q)u(n) \qquad (n \in \mathbb{Z}[1,T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{4.17}$$

where $q > 0$ is fixed and and $w : \mathbb{Z}[1,T] \to \mathbb{R}$. Note, in this case one has

$$G(n,x) = \frac{|x|^2}{2} - w(n)\frac{|x|^{q+2}}{q+2} \qquad (x \in \mathbb{R}^N,\ n \in \mathbb{Z}[1,T]). \tag{4.18}$$

Corollary 4.1. *(i) If*

$$\lambda > 8\sin^2\left(\frac{m\pi}{2(T+1)}\right) (= 2\lambda_m) \quad (m \in \mathbb{Z}[1,T])\,, \tag{4.19}$$

then (4.17) *has at least mN distinct pairs of nontrivial solutions.*

(ii) If

$$\lambda > 4\sin^2\left(\frac{\pi}{2(T+1)}\right) (= \lambda_1)\,, \tag{4.20}$$

then (4.17) *has at least one pair of nontrivial solutions.*

(iii) If

$$0 < \lambda \leq 4\sin^2\left(\frac{\pi}{2(T+1)}\right) \tag{4.21}$$

and $w \geq 0$, then the only solution of (4.17) *is the trivial one.*

Proof. • Theorem 4.2 and (4.18) $\Rightarrow$ (i) and (ii).

• (iii) by contradiction: assume that, for some $\lambda \in (0, \lambda_1]$, a function $u \in X^0_{T+2}$ is a nontrivial solution of (4.17). Summation by parts formula + Poincaré inequality $\Rightarrow$

$$\begin{aligned}\lambda \sum_{j=1}^{T} |u(j)|^2 (1 - w(j)|u(j)|^q) &= -\sum_{j=1}^{T} \langle \Delta\left[\phi(\Delta u(j-1))\right], u(j)\rangle \\ &= \sum_{j=0}^{T} \langle \phi(\Delta u(j)), \Delta u(j)\rangle = \sum_{j=0}^{T} \frac{|\Delta u(j)|^2}{\sqrt{1-|\Delta u(j)|^2}} \\ &\geq \sum_{j=0}^{T} |\Delta u(j)|^2 \geq \lambda_1 \sum_{j=1}^{T} |u(j)|^2. \qquad (4.22)\end{aligned}$$

This implies

$$0 \geq -\lambda \sum_{j=1}^{T} w(j)|u(j)|^{q+2} \geq (\lambda_1 - \lambda) \sum_{j=1}^{T} |u(j)|^2$$

and, in the case $0 < \lambda < \lambda_1$, we obtain the contradiction

$$0 \geq (\lambda_1 - \lambda) \sum_{j=1}^{T} |u(j)|^2 > 0.$$

If $\lambda = \lambda_1$, from (4.22) it follows

$$\lambda_1 \sum_{j=1}^{T} |u(j)|^2 \geq \lambda_1 \sum_{j=1}^{T} |u(j)|^2 (1 - w(j)|u(j)|^q)$$

$$= \sum_{j=0}^{T} \frac{|\Delta u(j)|^2}{\sqrt{1 - |\Delta u(j)|^2}} \geq \sum_{j=0}^{T} |\Delta u(j)|^2 \geq \lambda_1 \sum_{j=1}^{T} |u(j)|^2$$

$$\Longrightarrow \qquad \sum_{j=0}^{T} |\Delta u(j)|^2 \left(\frac{1}{\sqrt{1 - |\Delta u(j)|^2}} - 1 \right) = 0$$

$\Rightarrow \Delta u(0) = \Delta u(1) = \ldots = \Delta u(T) = 0_{\mathbb{R}^N} \Rightarrow u$ is a constant vector and then, as $u \in K$, we infer that $u \equiv 0_{\mathbb{R}^N}$ – a contradiction, again. Thus, (4.17) has only the trivial solution. ■

Multiple solutions for the non-parametric system (3.1)

$$\begin{cases} -\Delta \left[\phi(\Delta u(n-1)) \right] = \nabla F(n, u(n)), & (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1), \end{cases} \tag{3.1}$$

Corollary 4.2. *Assume that, for all $n \in \mathbb{Z}[1, T]$, the C^1-mapping $F(n, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is even and $F(n, 0_{\mathbb{R}^N}) = 0$.*

(i) If, for some $m \in \mathbb{Z}[1, T]$, F satisfies

$$\liminf_{|x| \to 0} \frac{F(n, x)}{|x|^2} > 4 \sin^2 \left(\frac{m\pi}{2(T+1)} \right) (= \lambda_m) \qquad (n \in \mathbb{Z}[1, T]), \tag{4.23}$$

then (3.1) has at least $m\,N$ distinct pairs of nontrivial solutions.
(ii) If

$$\liminf_{|x| \to 0} \frac{F(n, x)}{|x|^2} > 2 \sin^2 \left(\frac{\pi}{2(T+1)} \right) \left(= \frac{\lambda_1}{2} \right) \qquad (n \in \mathbb{Z}[1, T]), \tag{4.24}$$

holds true, then problem (3.1) has at least one pair of nontrivial solutions.

Proof. (4.23) $\Rightarrow \exists\ \overline{\lambda} > 0$ s.t.

$$\liminf_{|x| \to 0} \frac{2F(n, x)}{|x|^2} \geq \overline{\lambda} > 2\lambda_m \qquad (n \in \mathbb{Z}[1, T]).$$

Theorem 4.2 (i) $\Rightarrow$ problem

$$\begin{cases} -\Delta\left[\phi(\Delta u(n-1))\right] = \overline{\lambda}\ \nabla \left(\dfrac{F(n, u(n))}{\overline{\lambda}} \right) & (n \in \mathbb{Z}[1, T]), \\ u(0) = 0_{\mathbb{R}^N} = u(T+1) & \end{cases}$$

has at least mN distinct pairs of nontrivial solutions. A similar argument works when (4.24) is fulfilled. ∎

Example 4.2. Assume that the C^1-mapping $F(n,\cdot):\mathbb{R}^N \to \mathbb{R}$ is even and $F(n,0_{\mathbb{R}^N})=0$, for all $n\in\mathbb{Z}[1,T]$. Let $R>0$, $a:=\min_{i\in\{1,\dots,N\}}\{a_i\}>0$ and $\alpha\in(0,1]$ be s.t.

$$F(n,x)\geq (a_1x_1^2+a_2x_2^2+\cdots+a_Nx_N^2)^\alpha \quad \text{as } |x|\leq R \qquad (n\in\mathbb{Z}[1,T]).$$

- If, either $\alpha=1$ and $a>4\sin^2\left(\frac{T\pi}{2(T+1)}\right)$ or $\alpha\in(0,1)$, then (3.1) has at least $T\,N$ distinct pairs of nontrivial solutions.
- If $\alpha=1$ and there is some $m_0\in\{1,\dots,T-1\}$ s.t.

$$4\sin^2\left(\frac{m_0\pi}{2(T+1)}\right)<a\leq 4\sin^2\left(\frac{(m_0+1)\pi}{2(T+1)}\right),$$

then (3.1) has at least $m_0\,N$ distinct pairs of nontrivial solutions.

Multiplicity for differential systems

$$\begin{cases} -\left[\phi(u')\right]' = \lambda \nabla G(u) \text{ in } [0, T] \\ u(0) = 0_{\mathbb{R}^N} = u(T), \end{cases} \tag{5.1}$$

◇ $G : \mathbb{R}^N \to \mathbb{R}$ of class C^1, even, $G(0_{\mathbb{R}^N}) = 0$ and satisfies

(H) $\liminf_{|x| \to 0} \dfrac{2G(x)}{|x|^2} \geq 1.$

$$K := \{v \in W^{1,\infty}([0, T]; \mathbb{R}^N) \ : \ \|v'\|_\infty \leq 1, \ v(0) = 0 = v(T)\},$$

- $\Psi : C([0, T]; \mathbb{R}^N) \to (-\infty, +\infty]$,

$$\Psi(v) = \begin{cases} \displaystyle\int_0^T \left[1 - \sqrt{1 - |v'|^2}\right] dt, & \text{if } v \in K \\ +\infty, & \text{otherwise} \end{cases} \tag{5.2}$$

- The energy functional $I_\lambda : C([0, T]; \mathbb{R}^N) \to (-\infty, +\infty]$ is now given by

$$I_\lambda(v) = \Psi(v) - \lambda \int_0^T G(v(t))dt \tag{5.3}$$

The eigenvalues and corresponding eigenfunctions of the classical scalar Dirichlet problem $(-u'' = \lambda u$, $u(0) = 0 = u(T))$:

$$\lambda_m^c = \left(\frac{m\pi}{T}\right)^2, \qquad e_m^c(t) = \sqrt{\frac{2}{T}} \sin \frac{m\pi t}{T}, \quad m = 1, 2, \ldots$$

Theorem 5.1. *Under the above hypotheses on G, the following hold true:*
(i) if

$$\lambda > 2\left(\frac{m\pi}{T}\right)^2 (= 2\lambda_m^c) \quad (m \geq 1) \tag{5.4}$$

then problem (5.1) *has at least mN distinct pairs of nontrivial solutions;*
(ii) if

$$\lambda > \left(\frac{\pi}{T}\right)^2 (= \lambda_1^c) \tag{5.5}$$

then problem (5.1) *has at least one pair of nontrivial solutions.*

Corollary 5.1. *Let $a \geq 0 < q$.*
(i) If (5.4) *is fulfilled, then the problem*

$$\begin{cases} -\left[\phi(u')\right]' = \lambda u(1 - a|u|^q) \text{ in } [0, T] \\ u(0) = 0_{\mathbb{R}^N} = u(T), \end{cases} \tag{5.6}$$

has at least mN distinct pairs of nontrivial solutions.
(ii) If (5.5) *is satisfied, then* (5.6) *has at least one pair of nontrivial solutions.*
(iii) If $\lambda \in (0, \pi^2/T^2]$ $(= (0, \lambda_1^c])$, the only solution of (5.6) *is the trivial one.*

Corollary 5.2. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be even, of class C^1 and $F(0_{\mathbb{R}^N}) = 0$.*
(i) If

$$\liminf_{|x|\to 0} \frac{F(x)}{|x|^2} > \frac{m^2\pi^2}{T^2} \; (= \lambda_m^c) \quad (m \geq 1),$$

then the problem

$$\begin{cases} -\left[\phi(u')\right]' = \nabla F(u) \text{ in } [0, T] \\ u(0) = 0_{\mathbb{R}^N} = u(T), \end{cases} \tag{5.7}$$

has at least mN distinct pairs of nontrivial solutions.
(ii) If

$$\liminf_{|x|\to 0} \frac{F(x)}{|x|^2} > \frac{\pi^2}{2T^2} \; (= \lambda_1^c/2) \,,$$

then (5.7) has at least one pair of nontrivial solutions.
(iii) If

$$\lim_{|x|\to 0} \frac{F(x)}{|x|^2} = +\infty,$$

then (5.7) has infinitely many distinct pairs of nontrivial solutions.

Example 5.1. For any $\alpha \in (-1, 0)$, the system

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1-|u'|^2}}\right)' = u|u|^\alpha \text{ in } [0, T] \\ u(0) = 0_{\mathbb{R}^N} = u(T), \end{cases} \tag{5.8}$$

has infinitely many distinct pairs of nontrivial solutions. To see this, Corollary 5.2 *(iii)* applies with

$$F(x) = \frac{1}{\alpha+2}|x|^{\alpha+2} \quad (x \in \mathbb{R}^N).$$

The PDE case

$$\begin{cases} -\mathrm{div}\,[\phi(\nabla u)] = \lambda g(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{6.1}$$

◇ Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$ of class C^2;

◇ $\lambda > 0$ is a real parameter;

◇ $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function.

By a *solution* of (6.1) we mean a function $u \in W^{2,p}(\Omega)$ for some $p > N$, with $\|\nabla u\|_\infty < 1$, which satisfies the equation a.e. in Ω and vanishes on $\partial\Omega$.

We denote

$$G(t) := \int_0^t g(\tau)d\tau \quad (t \in \mathbb{R})$$

and assume that

$(\widetilde{H})$ $\liminf_{t\to 0} \dfrac{2G(t)}{t^2} \geq 1.$

$$K := \{v \in W^{1,\infty}(\Omega) \ : \ \|\nabla v\|_\infty \leq 1, \ v|_{\partial\Omega} = 0\},$$

- $\Psi : C(\overline{\Omega}) \to (-\infty, +\infty]$,

$$\Psi(v) = \begin{cases} \int_\Omega \left[1 - \sqrt{1 - |\nabla v|^2}\right] dx, & \text{if } v \in K \\ +\infty, & \text{otherwise} \end{cases} \tag{6.2}$$

- The energy functional $I_\lambda : C(\overline{\Omega}) \to (-\infty, +\infty]$ is defined by

$$I_\lambda(v) = \Psi(v) - \lambda \int_\Omega G(v(x))dx \tag{6.3}$$

The eigenvalues of $-\Delta$ on $H_0^1(\Omega)$: $0 < \lambda_1^\Delta < \lambda_2^\Delta \leq \ldots \leq \lambda_m^\Delta \leq \ldots$

Theorem 6.1. *Under the above hypotheses on g, the following hold true:*
(i) if

$$\lambda > 2\lambda_m^\Delta \quad (m \geq 1) \tag{6.4}$$

then problem (6.1) has at least m distinct pairs of nontrivial solutions;
(ii) if

$$\lambda > \lambda_1^\Delta \tag{6.5}$$

then problem (6.1) has at least one pair of nontrivial solutions.

Corollary 6.1. *Let $a \geq 0 < q$.*

(i) If (6.4) is fulfilled, then the problem

$$\begin{cases} -div[\phi(\nabla u)] = \lambda u(1 - a|u|^q) & in\ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{6.6}$$

has at least m distinct pairs of nontrivial solutions.

(ii) If (6.5) is satisfied, then (6.6) has at least one pair of nontrivial solutions.

(iii) If $\lambda \in (0, \lambda_1^\Delta]$, the only solution of (6.6) is the trivial one.

Remark 6.1. Corollary 6.1 with $a = 0 \Rightarrow$ the spectrum of the operator

$$u \mapsto -\text{div}\left[\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right]$$

is the interval $(\lambda_1^\Delta, +\infty)$.

Remark 6.2.

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2}x^2 + \cdots + \frac{(2k-3)!!}{(k-1)!2^{k-1}}x^{k-1} + \ldots \quad \text{for } |x| < 1$$

Take $x := |\nabla u|^2 \Rightarrow$

$$-\text{div}\left[\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right] = -\Delta u - \frac{1}{2}\underbrace{\text{div}(|\nabla u|^2 \nabla u)}_{\Delta_4} - \ldots$$

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Thank you for your attention !

Appendix

- $(Y, \|\cdot\|)$ be a real Banach space and $\mathcal{J} : Y \to (-\infty, +\infty]$ be a functional having the structure

$$\mathcal{J} = \mathcal{F} + \psi, \tag{7.1}$$

with $\mathcal{F} \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ proper, convex and lower semicontinuous. An element $u \in D(\psi)$ is said to be *a critical point* of $\mathcal{J}$ if it satisfies the variational inequality

$$\langle \mathcal{F}'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad (v \in D(\psi)).$$

A number $c \in \mathbb{R}$ such that $\mathcal{J}^{-1}(c)$ contains a critical point is called *a critical value* of the functional $\mathcal{J}$. A sequence $\{u_n\} \subset D(\psi)$ is called a (PS)-*sequence* if $\mathcal{J}(u_n) \to r \in \mathbb{R}$ and

$$\langle \mathcal{F}'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \quad (v \in D(\psi)),$$

where $\varepsilon_n \to 0$. The functional $\mathcal{J}$ is said *to satisfy the* (PS) *condition* if every (PS)-sequence possesses a convergent subsequence in Y.

Theorem A.1. *If $\mathcal{J}$ is of type* (7.1)*, satisfies the (PS) condition and is bounded from below, then*

$$c := \inf_Y \mathcal{J}$$

is a critical value of $\mathcal{J}$.

Theorem A.2. *Assume that $\mathcal{J}$ has the structure* (7.1), *satisfies (PS) condition and that*

(i) $\mathcal{J}(0) = 0$ *and there exist* $\alpha, \rho > 0$ *such that* $\mathcal{J}(u) \geq \alpha$ *if* $\|u\| = \rho$;

(ii) $\mathcal{J}(e) \leq 0$ *for some* $e \in Y$, *with* $\|e\| > \rho$.

Then, $\mathcal{J}$ *has a critical value* $c \geq \alpha$ *which can be characterized by*

$$c = \inf_{\theta \in \Theta} \sup_{t \in [0,1]} \mathcal{J}(\theta(t)),$$

where $\Theta = \{\theta \in C([0,1]; Y) \ : \ \theta(0) = 0, \ \theta(1) = e\}$.

Let Σ be the collection of all symmetric and closed subsets of $Y \setminus \{0\}$. The Krasnoselskii *genus* of a nonempty set $A \in \Sigma$, denoted $\gamma(A)$, is defined as being the smallest integer k with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k \setminus \{0\}$. If such an integer does not exist, then $\gamma(A) := +\infty$. It is known that if $A \in \Sigma$ is homeomorphic to S^{k-1} (= the $k-1$ dimension unit sphere in the Euclidean space $\mathbb{R}^k$) by an odd homeomorphism, then $\gamma(A) = k$.

Denote by Γ the collection of all nonempty compact, symmetric subsets of Y, endowed with the Hausdorff-Pompeiu distance and set

$$\Gamma_k := cl\{A \in \Gamma : \ 0 \notin A, \ \gamma(A) \geq k\},$$

where cl is the closure in Γ.

Theorem A.3. *Let $\mathcal{I}$ be of type (7.1) with $\mathcal{F}$ and ψ even. Also, suppose that $\mathcal{I}$ is bounded from below, satisfies the (PS) condition and $\mathcal{I}(0) = 0$. If*

$$\inf_{A \in \Gamma_k} \sup_{v \in A} \mathcal{I}(v) < 0,$$

then the functional $\mathcal{I}$ has at least k distinct pairs of nontrivial critical points.