Existence and asymptotic stability of solutions of differential equations with distributed delay: a topological approach

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Heat equation

$$u_t(t,x) = u_{xx}(t,x) + g\,(t,u(t,x))\ ,\ t \geq 0\,,\ x \in [0,1]$$

Flexible robotics equation

$$u_{tt}(x,t) = -a u_{xxxx}(x,t) + g(u(t,x))\,,\ x \in (0,1)\,,\ t \geq 0$$

Population dynamics equation

$$u_t(t,x) = -b(t,x)u(t,x) + g\,(t, K(u(t,x)))\ ,\ t \geq 0\,,\ x \in [0,1]$$

What's the connection?

Example of reformulation

$$u_t(t,x) \quad = \quad u_{xx}(t,x) \;+\; g\,(t,u(t,x))$$

↳

- $y : t \mapsto y(t) \in L^2([0,1])$

  $$y(t) : x \mapsto y(t)(x) := u(t,x)$$

- $A\,y(t)(x) := \dfrac{\partial^2}{\partial x^2} u(t,x)$
- $f\,(t, y(t))\,(x) := g\,(t, y(t)(x)) = g\,(t, u(t,x))$

$$y'(t) \quad = \quad A\,y(t) \;+\; f\,(t, y(t))$$

SEMILINEAR DIFFERENTIAL EQUATION

$y'(t) = A\,y(t) + f(t, y(t))\,,\ t \geq 0$ in a Banach space E

- studied combining the SEMIGROUP THEORY and FIXED POINT THEORY, even in the multivalued case: $y'(t) \in A\,y(t) + F(t, y(t))\,,\ t \geq 0$

Pazy, A. (1983)
Semigroups of Linear Operators and Applications to Partial Differential Equations

Kamenskii, M.; Obukhovskii, V.; Zecca, P. (2001)
Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces

Vrabie, I. (2003)
C_0-semigroups and applications

$$(P)\ \begin{cases} y'(t) = A\,y(t) + f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases}$$

Mild solution to (P): $y : [t_0, T] \to E$

$$y(t) = U(t - t_0)y_0 + \int_{t_0}^{t} U(t - s)f(s, y(s))ds,\ t \in [t_0, T]$$

Solution operator: $\Phi : C([t_0, T]; E) \to C([t_0, T]; E)$

$$\Phi(y)(t) = U(t - t_0)y_0 + \int_{t_0}^{t} U(t - s)f(s, y(s))ds,\ [t_0, T],\ y \in C([t_0, T]; E)$$

y mild solution to (P) $\iff$ y fixed point for Φ

The Sadovskii Fixed Point Theorem

Theorem (Sadovskii, 1967)

- *E Banach space*
- *M nonempty, convex, closed, bounded subset of E*
- *$f : M \to M$ α-condensing,*

  i.e. continuous & $\alpha(f(\Omega)) < \alpha(\Omega),\ \Omega \in \mathcal{P}_b(X)$ s. t. $\alpha(\Omega) > 0$

$\Rightarrow\ \exists \bar{x} \in M\ :\ f(\bar{x}) = \bar{x}$

α Kuratowski measure of noncompactness (1930)

$$\alpha(\Omega) := \inf\{\varepsilon > 0\ :\ \Omega \text{ can be covered by a finite number of sets with diameter less or equal to } \varepsilon\}$$

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Remark

- *f contraction of constant $k \in]0,1[\Rightarrow f$ α-condensing*

  $\Rightarrow$ *Sadovskii Th. contains. Banach-Caccioppoli Th. (metric theorem)*

- *f compact $\Rightarrow f$ α-condensing*

  $\Rightarrow$ *Sadovskii Th. contains Schauder Th. (topological theorem)*

Studies based on this approach

2005 - Cardinali T.; Rubbioni P.

- Semilinear evolution inclusions with nonautonomous linear term:
  $y'(t) \in A(t)y(t) + F(t, y(t))$ and $y(t_0) = y_0$
  $\{A(t)\}_{0 \le t \le T}$ generates an evolution system $\{T(t,s)\}_{0 \le s \le t \le T}$

2012 - Benchohra M.; Nieto J. J., Ouahab A.

- Impulsive semilinear differential inclusions:
  $y'(t) - A(t)y(t) \in F(t, y(t))$ and $y(t_0) = y_0$
  $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$ for $k \in \mathbb{N}$

2017 - Agarwal R. P.; Lupulescu, V., O'Regan D.

- Causal functional evolution equations:
  $D^\alpha y(t) = Ay(t) + (Qy)(t)$ and $y_{t_0} = \varphi : [-r, 0] \to E$
  Q causal operator: $y(t) = x(t)$ on $[t_0, \sigma] \Rightarrow Qy(t) = Qx(t)$ a.e. on $[t_0, \sigma], \sigma \in [t_0, T)$

Studies based on this approach

2017 - Cichoń M.; Satco B.

- Semilinear evolution inclusions involving measures:

  $dy \in -Aydt + F(t,y)dg$ and $y(t_0) = y_0$

  $g : [0,1] \to \mathbb{R}$ is a left-continuous non-decreasing function, $-A$ generates a C_0-semigroup of contractions

2017 - Bungardi S.; Cardinali T.; Rubbioni P.

- Nonlocal semilinear integro-differential inclusions:

  $y'(t) \in A(t)y(t) + F(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds)$ and $y(t_0) = g(y)$

  k nonnegative continuous real function, g continuous E-valued function

2019 - Malaguti L.; Perrotta S.; Taddei V.

- Exact controllability in a Hilbert space:

  $y'(t) = Ay(t) + f(t, y(t)) + Bu(t)$

  $B : U \to E$ linear and bounded, E Banach

Studies based on this approach

Rubbioni, P.; *Asymptotic stability of solutions for some classes of impulsive differential equations with distributed delay*,
Nonlinear Anal. Real World Appl. (2021)

Rubbioni, P.; *Solvability for a Class of Integro-Differential Inclusions Subject to Impulses on the Half-Line*,
Mathematics (2022)

The population dynamics model

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x)\right)$$

- $u : [t_0, +\infty[\times [0,1] \to \mathbb{R}$
  - $u(t,x)$ population density at time t and place x
- $b : [t_0, +\infty[\times [0,1] \to \mathbb{R}^+$
  - $-b(t,x)$ removal coefficient = death and migration rate of the population
- $g : [t_0, +\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ population development law

The population dynamics model with distributed delay

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \delta_T(t,x,u)\right) \quad (T > 0)$$

1. $\delta_T(t,x,u) = \int_{-T}^{0} u(t+\theta, x)d\theta$

- at each time t the system has memory of the evolution of the state up to that moment, for a past of fixed width T

The distributed delay

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \delta_T(t,x,u)\right) \quad (T > 0)$$

2. $$\delta_T(t,x,u) = \int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds$$

- spanning effect provided by the memory kernel $k(t,s) = \dfrac{e^{-(t-s)/T}}{T}$
- exponential distribution of probability $\mathcal{k}(\tau) = me^{-m\tau}$,
  $m = \frac{1}{T}$ decay parameter, T historical average waiting time
- $\mathcal{k}$ decreasing $\Rightarrow$ k assigns a greater weight to the most recent events,
  increasingly fading the influence of those further back in time

The distributed delay

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \delta_T(t,x,u)\right) \quad (T > 0)$$

1. $\delta_T(t,x,u) = \int_{-T}^{0} u(t+\theta, x)\, d\theta$

2. $\delta_T(t,x,u) = \int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds$

Notice that

- δ_T induces a feedback control on the population's evolution
- $T > 0$ provides the width of the action of the delay:
  the larger T, the more the system's memory is extended to past events affecting its present state
- the process is set on the whole half-line $\Rightarrow$ no *a priori* bounds on T
  $\Rightarrow$ the relevance of the delay on the status of the solution trajectory can be chosen arbitrarily

Reformulation of the population dynamics equation in case 1.

$$\frac{\partial u}{\partial t}(t,x) = -b(t,x)u(t,x) + g\left(t, u(t,x), \int_{-T}^{0} u(t+s,x)ds\right)$$

- $y : t \mapsto y(t) \in L^2([0,1])$

  $$y(t) : x \mapsto y(t)(x) := u(t,x)$$
- $A(t)y(t)(x) := -b(t,x)u(t,x)$
- $f(t, y(t), y_t) := g\left(t, u(t,\cdot), \int_{-T}^{0} u(t+s,\cdot)ds\right)$
  - $y_t(s) := y(t+s),\ s \in [-T, 0]$

$$y'(t) = A(t)y(t) + f(t, y(t), y_t)$$

SEMILINEAR *FUNCTIONAL* DIFFERENTIAL EQUATION

The presence of impulses on the system

$$u\left(t_k^+, x\right) = u\left(t_k, x\right) + \mathcal{I}_k\left(\int_{-T}^{0} u(t_k+\theta, x)\, d\theta\right), \quad x \in [0,1], \; k \in \mathbb{N}$$

$$\downarrow$$

$$y(t_k^+) \quad = \quad y(t_k) \; + \; I_k(y_{t_k})$$

- the impulse functions depend on the whole dynamic of the problem up to the time when the impulse has to act
  $\Rightarrow$ the delay structure of the system is preserved also in this aspect of the problem
- they represent instantaneous external actions given at prescribed times $0 \leq t_0 < t_1 < \cdots \to +\infty$ and cause sudden behavioral changes on the system
- in Biology, the are called *regulation functions*

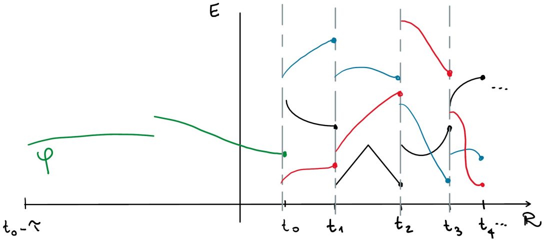
▶ Example: the administration of antibiotics on a bacterial population, or of pesticides on an insect infestation

The function spaces: the future

$$\mathcal{PC}([t_0,+\infty[,E) := \{\ y : [t_0,+\infty[\to E : y_{|\,]t_{k-1},t_k]} \text{ continuous},\ k \in \mathbb{N}^+\ ;\ \exists\, y(t_k^+) = \lim_{h\to 0^+} y(t_k+h) \in E,\ \ k \in \mathbb{N}\ \}$$

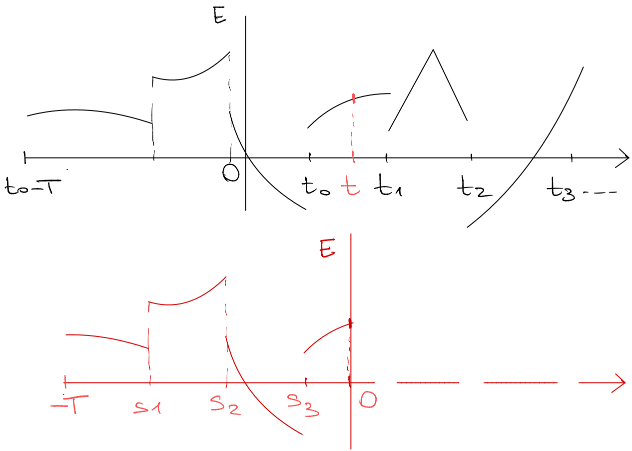
- $0 \le t_0 < t_1 < \cdots < t_k < \ldots$ and $t_k \to +\infty$
- $\mathcal{PC}([t_0,+\infty[,E)$ seminormed, endowed with the family of seminorms

$$\|y\|_{\infty,k} = \sup_{t\in[t_0,t_k]} \|y(t)\|,\ k \in \mathbb{N}^+$$



What about the past?

- $y_t(s) = y(t+s)$, $s \in [-T, 0]$ $\rightsquigarrow$ $y_t \in \mathcal{PC}_T$



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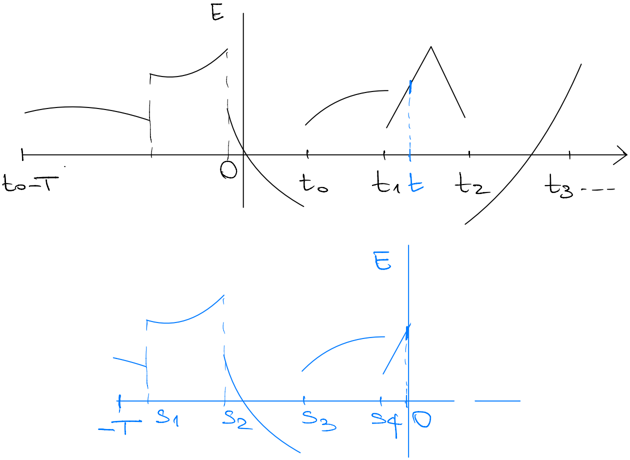

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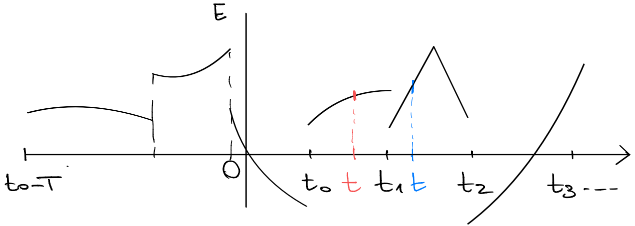

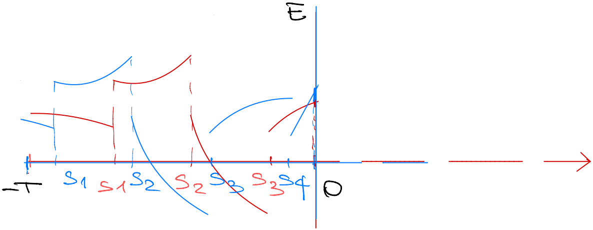

The function spaces: the past

$$\mathcal{PC}_T := \{\ y\ :\ [-T,0] \to E\ :\ y \text{ has at most a finite number of jump discontinuities } \}$$

- the jump points of its elements are function-depending
- it is a NORMED space when endowed with

$$\|y\|_{\mathcal{PC}_T} := \frac{1}{T}\int_{-T}^{0} \|y(s)\| ds$$

- $\|\cdot\|_{\mathcal{PC}_T}$ is weaker than the Chebyshev sup-norm, $\|y\|_{\infty\_0} = \sup_{-T\le s\le 0} \|y(s)\|$
- the mapping $t \mapsto y_t$ is continuous if in the space $\mathcal{PC}_T$ with $\|\cdot\|_{\mathcal{PC}_T}$ (but not with $\|\cdot\|_{\infty\_0}$)

The set the solutions belong to

$$\mathcal{S}([t_0 - T, +\infty[, E) := \{y : [t_0 - T, +\infty[\to E : y_{t_0} \in \mathcal{PC}_T \text{ and } y_{|[t_0,+\infty[} \in \mathcal{PC}([t_0, +\infty[, E)\}$$

The problem

$$(FDP)_\varphi \begin{cases} y'(t) = A(t)y(t) + f(t, y(t), y_t) \ , \ t \geq t_0 \, , \ t \neq t_k \, , \ k \in \mathbb{N} \\ y_{t_0} = \varphi \in \mathcal{PC}_T \\ y(t_0^+) = \varphi(0) + I_0(\varphi) \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}) \, , \ k \in \mathbb{N}^+ \end{cases}$$

- E Banach space
- $I_k : \mathcal{PC}_T \to E$, $k \in \mathbb{N}$, impulse functions, $\{t_k\}_k \nearrow +\infty$
- $f : [t_0, +\infty[\times E \times \mathcal{PC}_T \to E$

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- E Banach space
- $I_k : \mathcal{PC}_T \to E$, $k \in \mathbb{N}$, impulse functions, $\{t_k\}_k \nearrow +\infty$
- $f : [t_0, +\infty[\times E \times \mathcal{PC}_T \to E$
- $\{A(t)\}_{t\geq 0}$ family of densely defined linear operators on the Banach space E generating a strongly continuous evolution system on the half-line $\{T(t,s)\}_{t\geq s\geq 0}$
  $\downarrow$
  i.e. $\frac{\partial T}{\partial t}(t,s) = A(t)T(t,s)$ and $\frac{\partial T}{\partial s}(t,s) = -T(t,s)A(s)$

The existence theorem on $[t_0, +\infty[$

Theorem (Rubbioni P. - NARWA, 2021)

- $\{A(t)\}_{t\geq 0}$ *generating* $\{T(t,s)\}_{t\geq s\geq 0}$
- $I_k : \mathcal{PC}_T \to E,\ k \in \mathbb{N}$

(hf) $f : [t_0, +\infty] \times E \times \mathcal{PC}_T \to E$

(h1) $f(t,\cdot,\cdot)$ *continuous* $\forall t \geq t_0$ *and* $f\left(\cdot, y(\cdot), y_{(\cdot)}\right)$ *measurable* $\forall y \in \mathcal{S}([t_0 - T, +\infty[, E)$

(h2) $\exists \alpha \in L^{1,+}_{loc}([t_0, +\infty[)$ *such that for a.e.* $t \geq t_0$ *and all* $y \in E, \mathcal{y} \in \mathcal{PC}_T$

$$\|f(t,y,\mathcal{y})\| \leq \alpha(t)(1 + \|y\| + \|\mathcal{y}\|_{\mathcal{PC}_T})$$

(h3) $\exists h \in L^{1,+}_{loc}([t_0, +\infty[)$ *such that for a.e.* $t \geq t_0$ *and every* $\Omega_1 \subset E$, $\Omega_2 \subset \mathcal{PC}_T$ *bounded sets*

$$\chi(f(t,\Omega_1,\Omega_2)) \leq h(t)\left[\chi(\Omega_1) + \sup_{-T\leq\theta\leq 0} \chi(\Omega_2(\theta))\right]$$

$\Longrightarrow \exists\ y$ *delayed impulsive mild solution of* $(FDP)_\varphi$

i.e. $y(t) = T(t,t_0)[\varphi(0) + I_0(\varphi)] + \sum_{t_0<t_i<t} T(t,t_i)I_i(y_{t_i}) + \int_{t_0}^{t} T(t,s)f(s,y(s),y_s)\,ds,\ t \geq t_0$

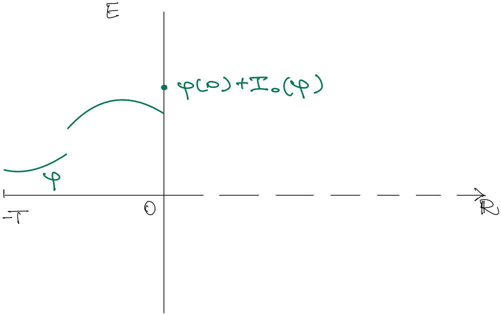
Idea of the proof

→ To implement an “extension-with-memory” process and then to generate an impulsive mild solution starting from the mild solutions of an ordered iterative sequence of **non-impulsive** Cauchy problems

Sketch of the proof

(1) In $[t_0, t_1]$, consider

$$(P1) \begin{cases} y'(t) = A(t)y(t) + f(t, y(t), y_t)\ , \quad t \in [t_0, t_1], \\ y_{t_0} = \varphi =: \varphi^0, \\ y(t_0^+) = \varphi^0(0) + I_0(\varphi^0) \end{cases}$$



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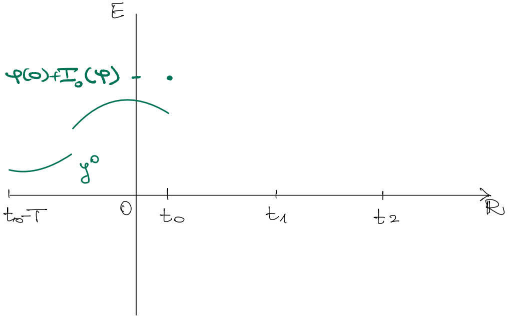

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- $\Gamma_1 : C([t_0, t_1], E) \to C([t_0, t_1], E)$

$$\Gamma_1(y)(t) = T(t, t_0)\left[\varphi^0(0) + I_0(\varphi^0)\right] + \int_{t_0}^{t} T(t,s) f(s, y(s), y[y^0]_s)\, ds$$

$$\left.\begin{array}{l} y \in C([t_0, t_1], E) \\ y^0(t) = \varphi^0(t - t_0) \end{array}\right] \quad \rightsquigarrow \quad y[y^0](t) := \begin{cases} y(t), & t \in]t_0, t_1] \\ y^0(t), & t \in [t_0 - T, t_0] \end{cases}$$

- $\Gamma_1(y)(t) = T(t,t_0)\left[\varphi^0(0) + I_0(\varphi^0)\right] + \int_{t_0}^{t} T(t,s)f(s,y(s),y[y^0]_s)\,ds$
- Sadovski type Fixed Point Theorem

$$\downarrow$$

$\exists y^1$ on $[t_0,t_1]$ such that $y^1 = \Gamma_1\left(y^1\right)$, i.e.

$$y^1(t) = T(t,t_0)\left[\varphi^0(0) + I_0(\varphi^0)\right] + \int_{t_0}^{t} T(t,s)f(s,y^1(s),y^1[y^0]_s)\,ds,\ \ t \in [t_0,t_1]$$

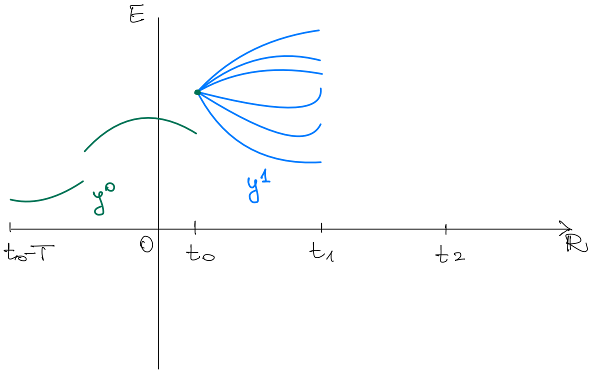
$\Rightarrow$ the function $y^1[y^0] : [t_0 - T, t_1] \to E$

$$y^1[y^0](t) = \begin{cases} y^1(t), & t \in]t_0,t_1] \\ y^0(t), & t \in [t_0 - T, t_0] \end{cases}$$

is a mild solution to $(P1)$

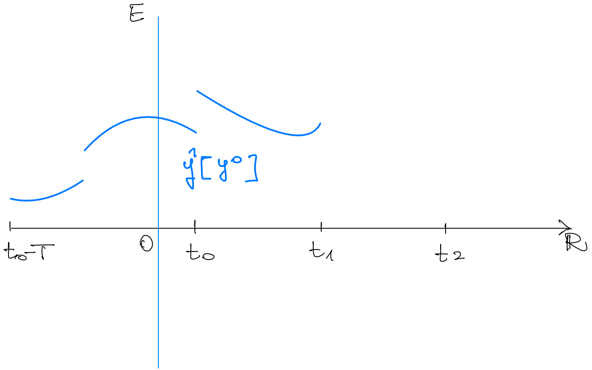
- mild solution to $(P1)$:

$$y^1[y^0](t) := \begin{cases} y^1(t) = T(t,t_0)\left[\varphi^0(0) + I_0(\varphi^0)\right] + \int_{t_0}^{t} T(t,s)f(s,y^1(s),y^1[y^0]_s)\,ds, & t \in]t_0,t_1] \\ y^0(t) = \varphi^0(t-t_0), & t \in [t_0-T,t_0] \end{cases}$$



(2) In $[t_1, t_2]$, consider

$$(P2)\begin{cases} y'(t) = A(t)y(t) + f(t, y(t), y_t), & t \in [t_1, t_2] \\ y_{t_1} = \varphi^1 := y^1[y^0]_{t_1} \\ y(t_1^+) = \varphi^1(0) + I_1(\varphi^1) \end{cases}$$

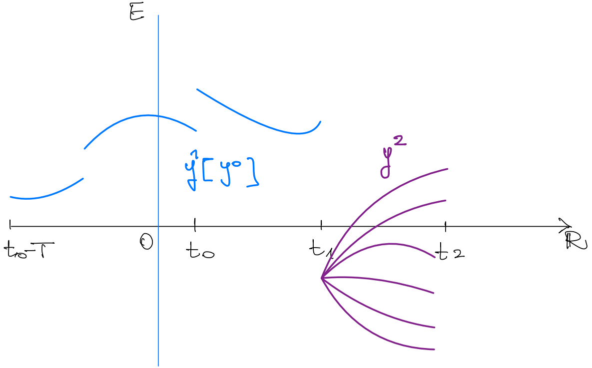


- Sadovski type Fixed Point Theorem $\rightsquigarrow \exists y^2$ on $[t_1,t_2]$ such that

$$y^2(t) = T(t,t_1)[\varphi^1(0) + I_1(\varphi^1)] + \int_{t_1}^{t} T(t,s)f(s,y^2(s),y^2[y^1[y^0]]_s)\,ds,\ t \in [t_1,t_2]$$

$$\Rightarrow\ y^2[y^1[y^0]](t) := \begin{cases} y^2(t), & t \in]t_1,t_2] \\ y^1[y^0](t), & t \in [t_0 - T, t_1] \end{cases}$$

is a mild solution to $(P2)$

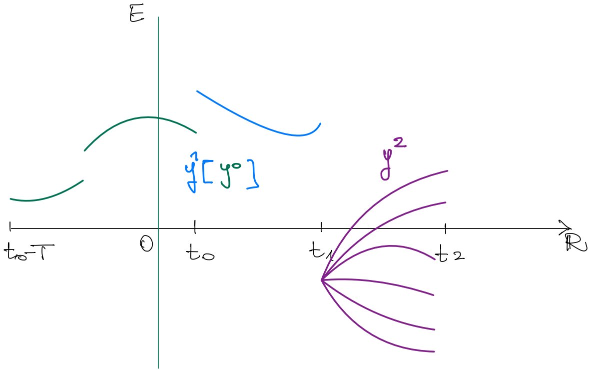


- Sadovski type Fixed Point Theorem $\rightsquigarrow \exists y^2$ on $[t_1, t_2]$ such that

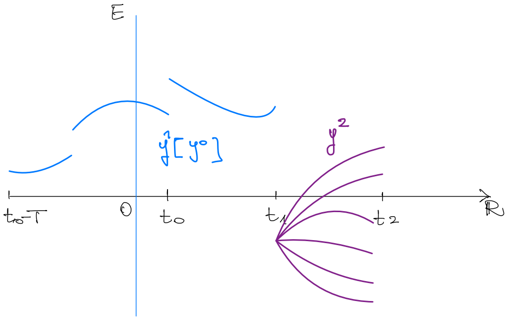
$$y^2(t) = T(t,t_1)[\varphi^1(0) + I_1(\varphi^1)] + \int_{t_1}^{t} T(t,s)f(s, y^2(s), y^2[y^1[y^0]]_s)\,ds,\ t \in [t_1, t_2]$$

$$\Rightarrow\ y^2[y^1[y^0]](t) := \begin{cases} y^2(t), & t \in]t_1, t_2] \\ y^1[y^0](t), & t \in [t_0 - T, t_1] \end{cases} = \begin{cases} y^2(t), & t \in]t_1, t_2] \\ y^1(t), & t \in]t_0, t_1] \\ y^0(t), & t \in [t_0 - T, t_0] \end{cases}$$

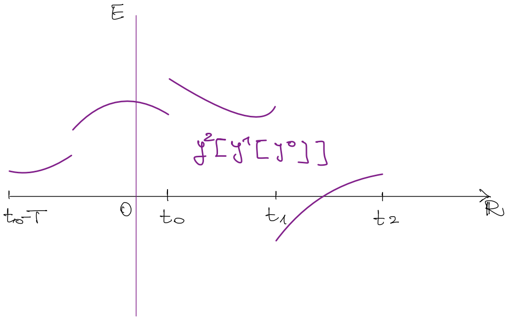
is a mild solution to $(P2)$



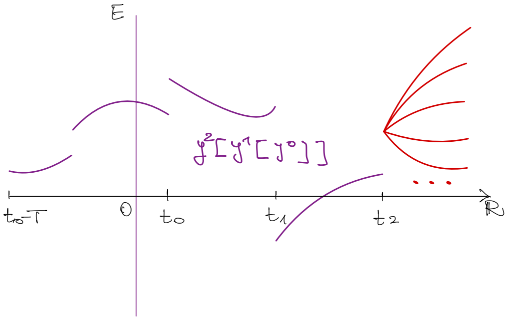
(3) Iterate the process:



(3) Iterate the process:



(3) Iterate the process:



(4) ▶ define $y^\infty : [t_0 - T, +\infty[\to E$

$$y^\infty(t) = \begin{cases} y^k(t), & t \in]t_{k-1}, t_k], \; k \in \mathbb{N}^+ \\ y^0(t), & t \in [t_0 - T, t_0]. \end{cases}$$

▶ check that it is a mild solution of $(FDP)_\varphi$ □

Non impulsive case

$$(FDCP) \begin{cases} y'(t) = A(t)y(t) + f(t, y(t), y_t) \ , \ t \geq t_0 \\ y_{t_0} = \varphi \in \mathcal{PC}_T \end{cases}$$

Corollary

$\{A(t)\}_{t\geq 0}$ *generating* $\{T(t,s)\}_{t\geq s\geq 0}$ + (hf)

$\Longrightarrow \exists\, y$ *mild solution of* $(FDCP)$: $y(t) = T(t,t_0)\varphi(0) + \int_{t_0}^{t} T(t,s)f(s,y(s),y_s)\,ds, \ t \geq t_0$

Remark

In this case, the solution is a continuous function

Asymptotic stability

Conrad F., Morgül Ö.;
On the stabilization of a flexible beam with a tip mass, SIAM J. Control Optim. (1998)

Olszowy L., Wędrychowicz S.,
On the existence and asymptotic behaviour of solutions of an evolution equation and an application to the Feynman-Kac theorem, Nonlinear Anal. (2011)

Banas J., Rzepka B.,
An application of a measure of noncompactness in the study of asymptotic stability, Appl. Math. Lett. 16 (2003)

Luo Z.-H., Guo B.-Z., Morgül Ö.,
Stability and Stabilization of Infinite Dimensional Systems with Applications, Springer-Verlag, London, 1999

Banas J., Jleli M., Mursaleen M., Samet B., Vetro C. -Editors,
Advances in Nonlinear Analysis via the Concept of Measures of Noncompactness, Springer, Singapore, 2017

Uniform asymptotic stability

Definition

Fixed Ω a nonempty bounded subset of $\mathcal{PC}_T$

the mild solutions of $y'(t) = A(t)y(t) + f(t, y(t), y_t)$ are said to be *uniformly asymptotically stable on Ω* if

$$\forall \varepsilon > 0 \; \exists t(\varepsilon) > 0 \text{ such that } \forall t \geq t(\varepsilon) \Rightarrow \|z(t) - y(t)\| \leq \varepsilon$$

$$\forall z \text{ solution of } (P)_z, \; y \text{ solution of } (P)_y, \; \forall z, y \in \Omega$$

Hypothesis on $\{A(t)\}_{t\geq 0}$

(A) $\{A(t)\}_{t\geq 0}$ family of linear operators, $A(t) : D(A) \subset E \to E$,
$D(A)$ a dense subset of the Banach space E not depending on t,
generating an *exponentially stable* evolution system on the half-line $\{T(t,s)\}_{t\geq s\geq 0}$

- $\omega_* < 0$
- $\omega_* := \inf\left\{\omega \in \mathbb{R} : \exists D \geq 0 \text{ such that } \|T(t,s)\|_{\mathcal{L}(E)} \leq De^{\omega(t-s)},\ t \geq s \geq 0\right\}$
  the growth bound of the family $\{T(t,s)\}_{t\geq s\geq 0}$

Van Minh N., Räbiger F., Schnaubelt R., Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line. Integral Equations Operator Theory (1998)

$\Rightarrow$ exist $\bar{\omega} < 0$ and $\bar{D} > 1$ s. t. $\|T(t,s)\|_{\mathcal{L}(E)} \leq \bar{D}e^{\bar{\omega}(t-s)}$, $t \geq s \geq 0$

Hypotheses on $f : [t_0, +\infty[\times E \times \mathcal{PC}_T \to E$

(h4) $f(\cdot, y, \mathcal{y})$ is measurable, for every $y \in E$ and $\mathcal{y} \in \mathcal{PC}_T$

(h5) there exist $c_1, c_2 > 0$ such that

$$\|f(t,z,\mathcal{z}) - f(t,y,\mathcal{y})\| \leq c_1\|z-y\| + c_2\|\mathcal{z}-\mathcal{y}\|_{\mathcal{PC}_T}, \text{ for all } t \geq t_0,\ z,y \in E,\ \mathcal{z},\mathcal{y} \in \mathcal{PC}_T$$

Hypotheses on $f : [t_0, +\infty[\times E \times \mathcal{PC}_T \to E$

(h4) $f(\cdot, y, \mathcal{y})$ is measurable, for every $y \in E$ and $\mathcal{y} \in \mathcal{PC}_T$

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Remark

- If E separable, then

(h4) + (h5) $\Rightarrow$

> (h1) $f(t,\cdot,\cdot)$ is continuous $\forall t \ge t_0$ and $f(\cdot, y(\cdot), y_{(\cdot)})$ is measurable $\forall y \in \mathcal{S}([t_0 - T, +\infty[, E)$

- (h5) $\Rightarrow$

> (h2) $\exists \alpha : [t_0, +\infty[\to \mathbb{R}^+$ locally integrable s.t. for a.e. $t \ge t_0, \forall y \in E, \mathcal{y} \in \mathcal{PC}_T$
>
> $$\|f(t,y,\mathcal{y})\| \le \alpha(t)(1 + \|y\| + \|\mathcal{y}\|_{\mathcal{PC}_T})$$
>
> $+$
>
> (h3) $\exists h : [t_0, +\infty[\to \mathbb{R}^+$ locally integrable s.t.
>
> $$\chi(f(t,\Omega_1,\Omega_2)) \le h(t)\left[\chi(\Omega_1) + \sup_{-T\le\theta\le 0}\chi(\Omega_2(\theta))\right]$$
>
> for a.e. $t \ge t_0$, all bounded $\Omega_1 \subset E$, $\Omega_2 \subset \mathcal{PC}_T$

Hypotheses on $f : [t_0, +\infty[\times E \times \mathcal{PC}_T \to E$

(h4) $f(\cdot, y, \mathcal{y})$ is measurable, for every $y \in E$ and $\mathcal{y} \in \mathcal{PC}_T$

(h5) there exist $c_1, c_2 > 0$ such that

$$\|f(t,z,\mathcal{z})-f(t,y,\mathcal{y})\| \leq c_1\|z-y\|+c_2\|\mathcal{z}-\mathcal{y}\|_{\mathcal{PC}_T}, \text{ for all } t \geq t_0,\ z,y \in E,\ \mathcal{z},\mathcal{y} \in \mathcal{PC}_T$$

Remark

- If E separable, then (h4) + (h5) $\Rightarrow$ (h1)
- (h5) $\Rightarrow$ (h2) + (h3)

$\Rightarrow (P)_{\mathcal{y}}$ has a (unique) mild solution, $\forall \mathcal{y} \in \mathcal{PC}_T$

Theorem (Rubbioni P. - NARWA, 2021)

E separable real Banach space; $\Omega \subset \mathcal{PC}_T$ nonempty and bounded

(A) $\{A(t)\}_{t\geq 0}$ *generates exponentially stable evolsys* $\{T(t,s)\}_{t\geq s\geq 0}$

(h4) $f\left(\cdot, y, \mathcal{y}\right)$ *is measurable,* $\forall y \in E,\ \mathcal{y} \in \mathcal{PC}_T$

(h5) $\exists c_1, c_2 > 0$ *such that* $\forall t \geq t_0,\ z, y \in E,\ \mathcal{z}, \mathcal{y} \in \mathcal{PC}_T$

$$\|f(t,z,\mathcal{z}) - f(t,y,\mathcal{y})\| \leq c_1\|z-y\| + c_2\|\mathcal{z}-\mathcal{y}\|_{\mathcal{PC}_T}$$

(hi) $\exists\ \{a_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+$ *such that*

- $\|I_k(\mathcal{z}) - I_k(\mathcal{y})\| \leq a_k\|\mathcal{z}-\mathcal{y}\|_{\mathcal{PC}_T},\ \forall \mathcal{z}, \mathcal{y} \in \mathcal{PC}_T$
- $\sum_{k=0}^{\infty} a_k$ *converges*

(hc) $c_1 + c_2 e^{-\bar{\omega}T} < |\bar{\omega}|/\bar{D}$

$\Rightarrow$ *the mild solutions of* $y'(t) = A(t)y(t) + f\left(t, y(t), y_t\right)$ *are uniformly asymptotically stable on* Ω

Sketch of the proof

- consider $z, y \in \Omega \rightsquigarrow$ mild solution z to $(P)_z$ and y to $(P)_y$

$$\Downarrow$$

$$\begin{aligned} e^{-\bar{\omega}t}\|z(t)-y(t)\| \leq\ & \bar{D}e^{-\bar{\omega}t_0}\left\|z(0)-y(0)+I_0(z)-I_0(y)\right\| \\ & + \sum_{t_0<t_i<t} a_i\bar{D}e^{-\bar{\omega}T} \sup_{-T\leq s\leq 0} e^{-\bar{\omega}(t_i+s)}\|z(t_i+s)-y(t_i+s)\| \\ & + \int_{t_0}^{t} c_1\bar{D}e^{-\bar{\omega}r}\|z(r)-y(r)\|dr \\ & + \int_{t_0}^{t} c_2\bar{D}e^{-\bar{\omega}T} \sup_{-T\leq s\leq 0} e^{-\bar{\omega}(r+s)}\|z(r+s)-y(r+s)\|dr \end{aligned}$$

- apply the next Gronwall-Bellman type Lemma to

$$m(t) := e^{-\bar{\omega}t}\|z(t)-y(t)\|$$

Lemma on a Gronwall-Bellman-type impulsive inequality with delay

Lemma (Rubbioni P. - NARWA, 2021)

- $0 \le t_0 < t_1 < \cdots < t_k < \ldots \to +\infty$
- $m : [t_0 - T, +\infty[\to \mathbb{R}$ be a nonnegative piecewise continuous function with left-continuous jump-discontinuities at each t_k, and at a finite number of points $s_i \in [t_0 - T, t_0]$
- exist $c \ge 0$, $\beta_k \ge 0$, $k \in \mathbb{N}^+$, and $p, q : [t_0, +\infty[\to \mathbb{R}^+$ continuous, such that

$$m(t) \le c\, m(t_0) + \sum_{t_0<t_i<t} \beta_i \sup_{-T\le s\le 0} m(t_i+s) + \int_{t_0}^{t} p(r)m(r)dr + \int_{t_0}^{t} q(r) \sup_{-T\le s\le 0} m(r+s)dr, \ t \ge t_0$$

Then,

$$m(t) \le \tilde{c}\, \|m\|_{\infty\_t_0} \prod_{t_0<t_i<t} (1+\beta_i)\, e^{\int_{t_0}^{t}[p(r)+q(r)]dr}, \ t \ge t_0$$

where $\tilde{c} = \max\{1, c\}$

Idea of the proof of G.-B. Lemma

- consider the auxiliary function $M : [t_0 - T, +\infty[\to E$ as

$$M(t) = \sup_{r \in [t_0 - T, t]} m(r)$$

- apply to M the Gronwall-Bellman impulsive inequality (without delay) by

Baĭnov D., Covachev V., Stamova I., Estimates of the solutions of impulsive quasilinear functional-differential equations. Ann. Fac. Sci. Toulouse Math. (1991)

Back to the proof of uniform asymptotic stability

- consider $z, y \in \Omega \rightsquigarrow$ mild solution z to $(P)_z$ and y to $(P)_y$

$$\Downarrow$$

$$\begin{aligned} e^{-\bar{\omega}t}\|z(t)-y(t)\| \quad \leq \quad & \bar{D}e^{-\bar{\omega}t_0}\left\|z(0)-y(0)+I_0(z)-I_0(y)\right\| \\ & + \sum_{t_0<t_i<t} a_i\bar{D}e^{-\bar{\omega}T} \sup_{-T\leq s\leq 0} e^{-\bar{\omega}(t_i+s)}\|z(t_i+s)-y(t_i+s)\| \\ & + \int_{t_0}^{t} c_1\bar{D}e^{-\bar{\omega}r}\|z(r)-y(r)\|dr \\ & + \int_{t_0}^{t} c_2\bar{D}e^{-\bar{\omega}T} \sup_{-T\leq s\leq 0} e^{-\bar{\omega}(r+s)}\|z(r+s)-y(r+s)\|dr \end{aligned}$$

$$\Rightarrow \; e^{-\bar{\omega}t}\|z(t)-y(t)\| \leq C(z,y) \prod_{t_0<t_i<t}\left(1+a_i\bar{D}e^{-\bar{\omega}T}\right) e^{\bar{D}\left(c_1+c_2e^{-\bar{\omega}T}\right)(t-t_0)}$$

- $C(z,y) := \widetilde{D}e^{-\bar{\omega}t_0}\sup_{-T\leq\theta\leq 0}\|z(\theta)-y(\theta)\|$
- $\exists R>0 : C(z,y) \leq R \quad (\Leftarrow \Omega \subset \mathcal{PC}_T\, bounded\ )$

$$\Rightarrow \; \|z(t)-y(t)\| \leq R\, e^{\left[\bar{\omega}+\bar{D}\left(c_1+c_2e^{-\bar{\omega}T}\right)\right](t-t_0)} e^{\sum_{t_0<t_i<t}\log\left(1+a_i\bar{D}e^{-\bar{\omega}T}\right)} \longrightarrow_{t\to+\infty} 0$$

Reformulation of the population dynamics equation in case 2.

$$\frac{\partial u}{\partial t}(t,x) \;=\; -b(t,x)u(t,x) + g\left(t, u(t,x), \int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, d\right)$$

- $y(t)(x) := u(t,x)$
- $A(t)y(t)(x) := -b(t,x)u(t,x)$
- $f(t, y(t), \int_{t_0}^{t} k(t,s)\, y(s)\, ds)(x) := g\left(t, u(t,x), \int_{t_0}^{t} \frac{e^{-(t-s)/T}}{T}\, u(s,x)\, ds\right)$
  - $k(t,s) = \frac{e^{-(t-s)/T}}{T}$, $t \geq s \geq 0$

$$y'(t) \;=\; A(t)y(t) + f\left(t, y(t), \int_{t_0}^{t} k(t,s)\, y(s)\, ds\right)$$

SEMILINEAR *INTEGRO*-DIFFERENTIAL EQUATION

The problem

$$(IDP) \begin{cases} y'(t) = A(t)y(t) + f\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right), \ t \geq t_0, \ t \neq t_m, \ m \in \mathbb{N}^+, \\ y(t_0) = \mathbf{v} \in E, \\ y(t_m^+) = y(t_m) + I_m(y(t_m)), \ m \in \mathbb{N}^+. \end{cases}$$

- $\{t_m\}_{m \in \mathbb{N}^+} \nearrow +\infty$; $I_m : E \to E, m \in \mathbb{N}^+$
- $f : [t_0, +\infty[\times E \times E to E$
- $\{A(t)\}_{t \geq 0}$ generating $\{T(t,s)\}_{t \geq s \geq 0}$

(k) $k : \Delta = \{(t,s) : t \geq s \geq 0\} \to \mathbb{R}^+$ continuous kernel

The existence theorem on $[t_0, +\infty[$

Theorem (Rubbioni P. - Mathematics, 2022)

- $\{A(t)\}_{t\geq 0}$ *generating* $\{T(t,s)\}_{t\geq s\geq 0}$ + (k)

(f) $f : [t_0, +\infty[\times E \times E \to E$

- (f1) $\forall v, w \in E$, *the map* $f(\cdot, v, w)$ *is strongly measurable*
- (f2) *for a.e.* $t \geq t_0$, *the map* $f(t, \cdot, \cdot)$ *is continuous*
- (f3) $\exists\, \alpha \in L^1_{loc}([t_0, +\infty[)$ *such that for a.e.* $t \geq t_0$ *and all* $v, w \in E$ *it is* $\|f(t,v,w)\| \leq \alpha(t)(1 + \|v\| + \|w\|)$
- (f4) $\exists\, h \in L^1_{loc}([t_0, +\infty[)$ *such that for a.e.* $t \geq t_0$ *and every* $\Omega_1, \Omega_2 \in \mathcal{P}_b(E)$ *it is* $\chi(f(t, \Omega_1, \Omega_2)) \leq h(t)\,[\chi(\Omega_1) + \chi(\Omega_2)]$

$\Longrightarrow \exists\, y$ *delayed impulsive mild solution of* (IDP)

i.e. $$y(t) = T(t,t_0)\mathbf{v} + \sum_{t_0 < t_m < t} T(t,t_m) I_m(y(t_m)) + \int_{t_0}^{t} T(t,s) f(s, y(s), \textstyle\int_{t_0}^{s} k(s,r) y(r)\,dr)\,ds$$

Idea of the proof

→ As before, to implement an "extension-with-memory" process

- ⇝ ordered iterative sequence of **non-impulsive** Cauchy problems
- ⇝ relative mild solutions generating an impulsive mild solution on the half-line

→ Which Cauchy problems?

Are
$$\begin{cases} y'(t) = A(t)y(t) + f\left(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\right), \ t \in [t_m, t_{m+1}] \\ y(t_m) = \mathbf{v}_m := \bar{y}_{m-1}(t_m) + I_m(\bar{y}_{m-1}(t_m)) \end{cases}$$
the right ones?

→ No: the Volterra integral inside f starts from t_0 and not from t_m

Sketch of the proof

(1) (P_0) $\begin{cases} y'(t) = A(t)y(t) + f_0(t, y(t), \int_{t_0}^t k(t,s)y(s)ds), \ t \in [t_0, t_1] \\ y(t_0) = \mathbf{v}_0 \end{cases}$

- $f_0 := f$ and $\mathbf{v}_0 = \mathbf{v}$

$\rightsquigarrow$ mild solution:

$$\bar{y}_0(t) = T(t,t_0)\mathbf{v}_0 + \int_{t_0}^t T(t,s) f_0(s, \bar{y}_0(s), \int_{t_0}^s k(s,r)\bar{y}_0(r)dr)\, ds, \quad t \in [t_0, t_1]$$

---

(2) (P_1) $\begin{cases} y'(t) = A(t)y(t) + f_1(t, y(t), \int_{t_1}^t k(t,s)y(s)ds), \ t \in [t_1, t_2], \\ y(t_1) = \mathbf{v}_1 \end{cases}$

- $f_1(t, v, \underset{\leftrightarrow}{w}) := f_0(t, v, \underbrace{w + \int_{t_0}^{t_1} k(t,r)\bar{y}_0(r)dr}_{\leftrightarrow}), \ t \in [t_1, t_2]$

  $\Rightarrow f_1$ satisfies the hypotheses' set (f)

- $\mathbf{v}_1 := \bar{y}_0(t_1) + I_1(\bar{y}_0(t_1))$

$\rightsquigarrow$ mild solution:

$$\bar{y}_1(t) = T(t,t_1)\mathbf{v}_1 + \int_{t_1}^t T(t,s) f_1(s, \bar{y}_1(s), \int_{t_1}^s k(s,r)\bar{y}_1(r)dr)\, ds, \quad t \in [t_1, t_2]$$

(3) Iterative process:

$$(P_m) \quad \begin{cases} y'(t) = A(t)y(t) + f_m(t, y(t), \int_{t_m}^t k(t,s)y(s)ds), \ t \in [t_m, t_{m+1}], \\ y(t_m) = \mathbf{v}_m \end{cases}$$

- $f_m(t, v, \underleftrightarrow{w}) := f_{m-1}(t, v, \underleftrightarrow{w + \int_{t_{m-1}}^{t_m} k(t,r)\bar{y}_{m-1}(r)dr})$

$$= \dots$$

$$= f(t, v, \underleftrightarrow{w + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} k(t,r)\bar{y}_i(r)dr}), \ t \in [t_m, t_{m+1}]$$

$\Rightarrow f_m$ satisfies the hypotheses' set (f)

- $\mathbf{v}_m := \bar{y}_{m-1}(t_m) + I_m(\bar{y}_{m-1}(t_m))$

$\rightsquigarrow$ mild solution:
$\bar{y}_m(t) = T(t, t_m)\mathbf{v}_m + \int_{t_m}^t T(t,s) f_m(s, \bar{y}_m(s), \int_{t_m}^s k(s,r)\bar{y}_m(r)dr)\, ds, \ t \in [t_m, t_{m+1}]$

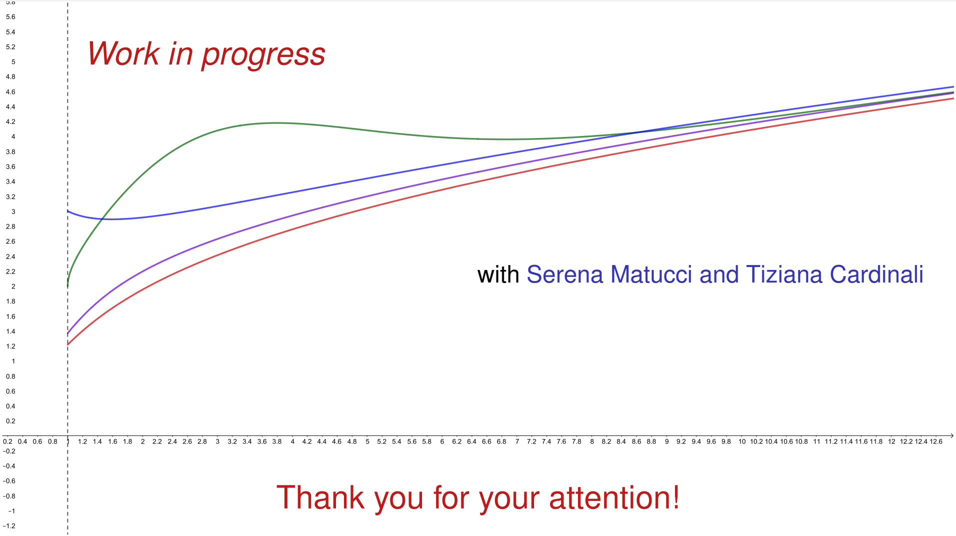
---

(4) $\bar{y} : [t_0, +\infty[\to E$ defined by

$$\bar{y}(t) := \begin{cases} \bar{y}_0(t), & t \in [t_0, t_1] \\ \bar{y}_m(t), & t \in]t_m, t_{m+1}], m > 0 \end{cases}$$

is a mild solution of (IDP) $\square$

The asymptotic stability



Work in progress

with Serena Matucci and Tiziana Cardinali

Thank you for your attention!