Impact of noise in some MEMS models

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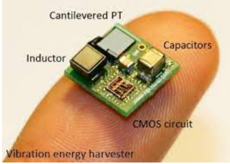
Overview

MEMS devices



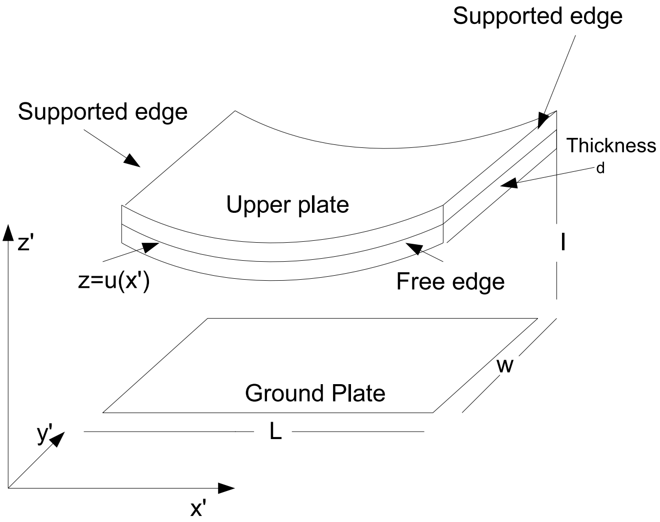
(a)

Automotive	Electronics	Medical	Communications	Defence
Internal navigation sensors	Disk drive heads	Blood pressure sensor	Fibre-optic network components	Munitions guidance
Air conditioning compressor sensor	Inkjet printer heads	Muscle stimulators & drug delivery systems	RF Relays, switches and filters	Surveillance
Brake force sensors & suspension control accelerometers	Projection screen televisions	Implanted pressure sensors	Projection displays in portable communications devices and instrumentation	Arming systems
Fuel level and vapour pressure sensors	Earthquake sensors	Prosthetics	Voltage controlled oscillators (VCOs)	Embedded sensors
Airbag sensors	Avionics pressure sensors	Miniature analytical instruments	Splitters and couplers	Data storage
"Intelligent" tyres	Mass data storage systems	Pacemakers	Tuneable lasers	Aircraft control

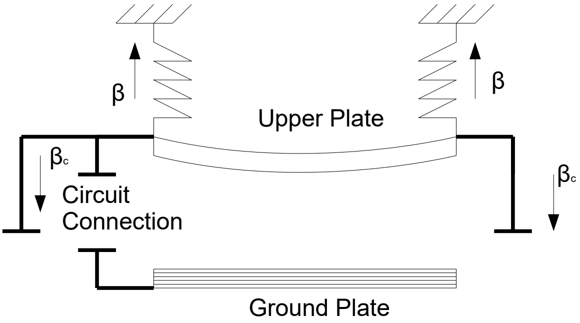
(b)

- MEMS (Micro-Electro-Mechanical Systems) : precision devices combine both mechanical processes with electrical circuits

Electrically actuated MEMS device - Configuration 1



Electrically actuated MEMS device - Configuration 2



- β : magnitude of spring force
- β_c : magnitude of external force (e.g gravity force).

Local model

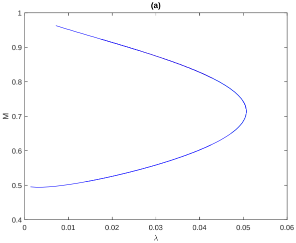
$$u_t = \Delta u + \frac{\lambda h(x)}{(1-u)^2}, \quad x \in \Omega,\ t > 0 \tag{1}$$

$$\frac{\partial u}{\partial \nu} + \beta u = \beta_c, \quad x \in \partial\Omega,\ t > 0 \tag{2}$$

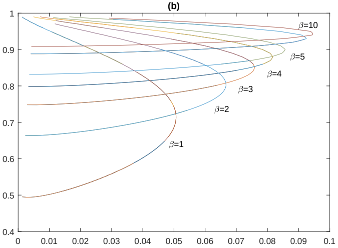
$$0 \leq u(x,0) = u_0(x) < 1, \quad x \in \Omega \tag{3}$$

- u =the displacement of the elastic membrane toward the ground plate
- $\lambda \propto V^2$, where V is the applied voltage
- $h(x)$ =dielectric profile of the elastic membrane, eg. $h(x) \equiv 1$
- Finite-time quenching: $\|u(\cdot,t)\|_\infty \to 1$ as $t \to T_q < \infty$.
- Quenching $\implies$ Touching down (possible destruction of MEMS device)

Local model(Bifurcation 1−dimension)



(c)



(d)

- Finite-time quenching occurs only for “big” values of λ. **(Guo 1991)**
- Finite-time quenching occurs only for “large” enough initial values $u_0(x)$. **(Guo 1991)**

Nonlocal model

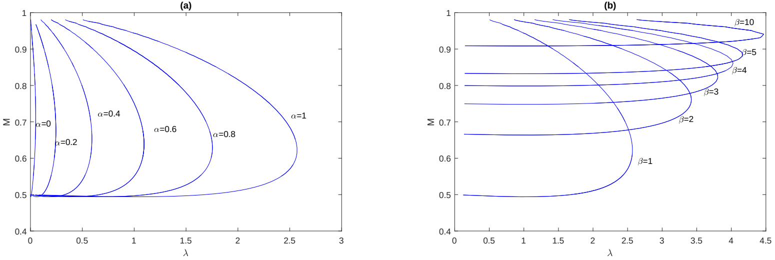
$$u_t = \Delta u + \frac{\lambda}{(1-u)^2\left(1+\alpha\int_\Omega \frac{1}{1-u}\,dx\right)^2}, \quad x \in \Omega,\ t > 0 \tag{4}$$

$$\frac{\partial u}{\partial \nu} + \beta u = \beta_c, \quad x \in \partial\Omega,\ t > 0 \tag{5}$$

$$0 \leq u(x,0) = u_0(x) < 1, \quad x \in \Omega \tag{6}$$

- Purpose: extend the stable operation of MEMS device
- Implementation: addition of a series capacitance C_f
- $\alpha = \frac{C_0}{C_f}$, where C_0 denotes the capacitance of the undeflected device.

Nonlocal model(Bifurcation 1−dimension)



(e) (f)

- Finite-time quenching occurs only for “big” values of λ. **(Drosinou, K., Nikolopoulos 2021)**
- Finite-time quenching occurs only for “large” enough initial values $u_0(x)$. **(Drosinou, K., Nikolopoulos 2021)**

Local stochastic model

- $\lambda \to \lambda +$noise then model (4)-(6) can take the form, **(K. 2016)**,

$$u_t = \Delta u + \frac{\lambda}{(1-u)^2} + \sigma(u)\partial_t W, \quad x \in \Omega,\ t > 0 \qquad (7)$$

$$\frac{\partial u}{\partial \nu} + \beta u = \beta_c, \quad x \in \partial\Omega,\ t > 0 \qquad (8)$$

$$0 \leq u(x,0) = u_0(x) < 1, \quad x \in \Omega \qquad (9)$$

- $W(x,t) = \sum_{i=1}^{\infty} \mu_i^{1/2} \chi_i(x) B_i(t)$: Wiener process
- $\sigma(\cdot)$ locally Lipschitz with linear growth
- $\sigma(u)\partial_t W(x,t)$: multiplicative spatial-temporal noise

Quenching for model (7)-(9)

- Finite-time quenching: $\limsup_{t\to T_q} \mathbb{E}\left[\|u(\cdot,t)\|_\infty\right] = 1$

Theorem 1(K. 2016)

Given that $\mathbb{E}\left[\|u_0(\cdot)\|_\infty\right] < 1$ then the solution of (7)-(9) quenches in finite time for "big" values of λ.

Theorem 2(K. 2016)

Take $\lambda > 0$ and $\mathbb{E}\left[\|u_0(\cdot)\|_\infty\right] < 1$ then the solution of (7)-(9) quenches in finite time provided that $\mathbb{E}\left[\int_\Omega u_0(x)\phi_1(x)\,dx\right] > \zeta$ where ζ is the largest root of the equation $g(\tau) = \frac{\lambda}{(1-\tau)^2} - \mu_1\tau = 0$ and (μ_1, ϕ_1) first eigenpair of $-\Delta_R$.

- If finite-time quenching occurs then all solution moments quench.

MEMS model with Brownian noise

$$z_t = \Delta z - \frac{\lambda}{z^2} - \kappa z dB_t, \quad x \in \Omega, \; t > 0 \tag{10}$$

$$\frac{\partial z}{\partial \nu} + \beta z = 0, \quad x \in \partial\Omega, \; t > 0 \tag{11}$$

$$0 < z(x,0) = z_0(x) \leq 1, \quad x \in \Omega \tag{12}$$

- $z := 1 - u$
- B_t one-dimensional Brownian motion
- κ postive constant
- $\beta = \beta_c$
- $\kappa z dB_t$: multiplicative linear temporal noise

An equivalent RPDE model

- Finite-time quenching for (10)-(12): $\lim_{t\to\tau}\inf_{x\in\Omega}|z(x,t)|=0,$ a.s. on the event $\{\tau<\infty\}$,
- Set $v=e^{\kappa B_t}z \quad 0\le t<\tau$ then we derive the equivalent RPDE problem

$$v_t=\Delta v-\frac{\kappa^2}{2}v+\lambda e^{3\kappa B_t}v^{-2} \quad x\in\Omega,\, t>0, \tag{13}$$

$$\frac{\partial v}{\partial \nu}+\beta v=0, \quad \text{on} \quad x\in\partial\Omega,\, t>0, \tag{14}$$

$$v(x,0)=z_0(x), \quad x\in\Omega \tag{15}$$

- The solution to (13)-(15) should be understood trajectorywise
- z quenches in time τ if and only if v quenches in random time τ.

Finite-time quenching

Theorem 3(Drosinou, K. , Nikolopoulos 2022)

The (weak) solution of problem

$$
\begin{aligned}
&z_t = \Delta z - \frac{\lambda}{z^2} - \kappa z dB_t, \quad x \in \Omega,\ t > 0 \\
&\frac{\partial z}{\partial \nu} + \beta z = 0, \quad x \in \partial\Omega,\ t > 0 \\
&0 < z(x,0) = z_0(x) \leq 1, \quad x \in \Omega,
\end{aligned}
$$

quenches in finite time with probability one, i.e. almost surely, regardless the size of its initial condition as well as that of parameter λ.

- The impact of noise is vital into quenching! Note that the corresponding deterministic problem quenches only for "big" λ or "large enough" initial data $z_0(x)$.

Regularizing effect

Theorem 4(Drosinou, K. , Nikolopoulos 2022)

The (weak) solution of problem

$$z_t = \Delta z - \frac{\lambda e^{-3\gamma t}}{z^2} - \kappa z dB_t, \quad x \in \Omega, \ t > 0 \tag{16}$$

$$\frac{\partial z}{\partial \nu} + \beta z = 0, \quad x \in \partial\Omega, \ t > 0, \tag{17}$$

$$0 < z(x,0) = z_0(x) \leq 1, \quad x \in \Omega. \tag{18}$$

- quenches in finite time with positive probability $\mathbb{P}\left[\tau < +\infty\right] \geq \frac{1}{\Gamma(-\mu)} \int_0^\theta e^{-y} y^{-\mu-1}\, dy$, for given θ, when $\gamma \geq \mu_1 + \frac{\kappa^2}{2}$,
- quenches in finite time almost surely provided that $\gamma < \mu_1 + \frac{\kappa^2}{2}$.

Global existence

Theorem 5(Drosinou, K. , Nikolopoulos 2022)

Consider initial data $0 < z_0(x) \leq 1$ such that

$$\int_0^\infty e^{3\kappa B_r} \rho^{-3}(r)\, dr < \frac{1}{4\lambda},$$

where $\rho(t) := \inf_{x\in\Omega} e^{\gamma t} \mathcal{S}_t v_0(x) > 0$. Then, with probability 1, problem (16)-(18) admits a global in time solution satisfying

$$0 < e^{\gamma t - \kappa B_t} \mathcal{S}_t v_0(x) \mathcal{G}(t) \leq z(x,t) < 1, \; x \in \Omega, t \geq 0,$$

where $\mathcal{G}(t) := \left[1 - 4\lambda \int_0^t e^{3\kappa B_r} \mu^{-3}(r)\, dr\right]^{1/4} < 1$ and $\mathcal{S}_t$ stands for the semigroup generated by $-\Delta_R$.

Estimates of Quenching time

Theorem 6(Drosinou, K. , Nikolopoulos 2022)

Consider now $v_0(x) = K_1 e^{-\mu_1 \eta}\phi_1(x)$, for some $K_1, \eta > 0$ then the quenching time (stopping time) τ for problem (16)-(18) is bounded by $\tau_* < \tau < \tau^*$ where

$$\tau_* := \inf\left\{ t > 0 : \int_0^t e^{3[(\mu_1-\gamma)s+\kappa B_s]} ds \geq \frac{K_1^3 \left(\inf_{x\in\Omega} \phi_1(x)\right)^3 e^{-3\lambda_1 \eta}}{4\lambda} \right\},$$

and

$$\tau^* := \inf\left\{ t > 0 : \int_0^t e^{3[(\mu_1-\gamma)s+\kappa B_s]} ds \geq \frac{K_1^3 e^{-3\lambda_1 \eta} \left(\int_\Omega \phi_1^2(x)\, dx\right)^3}{3\lambda} \right\}.$$

Numerical Results

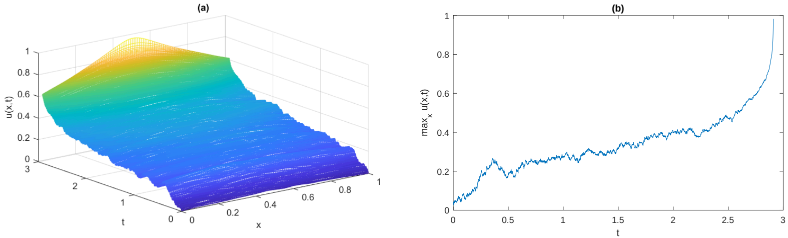


Figure 1: (a) Realisation of the numerical solution of problem (10)-(12) for $\lambda = 0.3$, $\kappa = 1$, initial condition $u(x,0) = c\,x(1-x)$ for $c = 0.1$ and with $\beta = \beta_c = 1$ in the nonhomogeneous boundary condition. (b) Plot of $\|u(\cdot,t)\|_\infty$. The quenching behaviour is apparent.

Numerical Results - Cont.

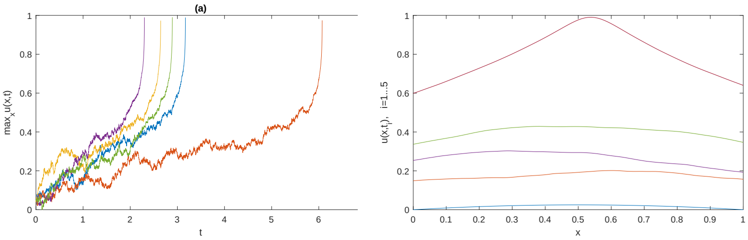


Figure 2: (a) Realisation of the $\|u(\cdot,t)\|_\infty$ of the numerical solution of problem (10)-(12) or $\lambda = 2$, $\kappa = 1$ and initial condition $u(x,0) = c\,x(1-x)$ for $c = 0.1$. (b) Plot of $u(x,t_i)$ from a different realization with the same values of the parameters at five time instants.

Fractional Brownian noise

$$z_t = \Delta z - \lambda e^{-\gamma t} z^{-2} - \kappa z dB_t^H, \quad x \in \Omega, \ \ t > 0,$$
$$\frac{\partial z}{\partial \nu} + \beta z = 0, \quad x \in \partial\Omega, \ \ t > 0,$$
$$0 < z(x,0) = z_0(x) \leq 1, \quad x \in \Omega.$$

(Drosinou, Nikolopoulos, Matzavinos, K., (2023))

- B_t^H one-dimensional fractional Brownian motion with Hurst index H.
- $H > \frac{1}{2}$ long-range effects, stochastic integral is defined as a Riemann-Stieltje integral
- Estimation of quenching probability
- Estimation of quenching time
- Estimation of probability of global existence

Nonlocal diffusion and mixed noise

$$z_t = -\frac{1}{2}k^2(t)\mathcal{L}z - g(x,z) - zdN_t, \quad x \in \Omega,\ t > 0,$$
$$\mathcal{N}z(x,t) + \beta(x)z(x,t) = 0, \quad x \in \Omega^c := \mathbb{R}^d \setminus \Omega,\ t > 0,$$
$$0 \leq z(x,0) = z_0(x) < 1, \quad x \in \Omega,$$

(K., Nikolopoulos, Yannacopoulos (2023))

- $\mathcal{L}u(x,t) := p.v. \int_{\mathbb{R}^d}(u(x,t) - u(y,t))k(x,y)dy, \quad x \in \mathbb{R}^d,$

$$\widetilde{\Lambda}^{-1}\nu(x-y) \leq k(x,y) \leq \widetilde{\Lambda}\nu(x-y), \quad x,y \in \mathbb{R}^d,\ \widetilde{\Lambda} > 0,$$

  $\nu : \mathbb{R}^d \setminus \{0\} \to [0,\infty)$ density of a symmetric Lévy measure.
- $\mathcal{N}u(y,t) := \int_\Omega (u(y,t) - u(x),t)k(x,y)dx, \quad y \in \Omega^c$
- $g(x,z) = \lambda\zeta(x)z^{-2} - \gamma z$ for $\gamma, \lambda, \zeta(x) > 0$ and $g \to \infty$ as $z \to 0$
- $N_t := \int_0^t a(s)dB_s + \int_0^t b(s)dB_s^H,\ H > \frac{1}{2}$

Key References

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Thank you for your attention!