# Quasilinear reaction-diffusion equation with discontinuous diffusivity and bistable reaction term

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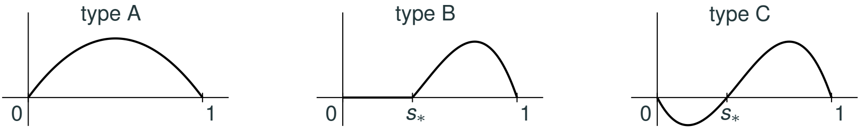
# Motivation

Semilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u), \quad (x,t) \in \mathbb{R} \times [0, +\infty)$$

constant diffusion

reaction term $g \in C^1[0,1]$, $g(0) = g(1) = 0$




Aronson, Weinberger (1975)[1]: modelling of diploid individuals

- type A + $g'(0) > 0$ (heterozygote intermediate case)
- type C + $g'(0) < 0$, $\int_0^1 g(s)\,\mathrm{d}s > 0$ (heterozygote inferiority)

[1] D. G. Aronson, H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: *Partial Differential Equations and Related Topics*, Springer Berlin Heidelberg, 1975, pp. 5–49.

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# Existence results

travelling wave solutions $u(x, t) = U(x - ct)$ connecting the stationary states 0 and 1

(unknown) speed of propagation $c \in \mathbb{R}$

$$U(-\infty) = 1, \quad U(+\infty) = 0$$

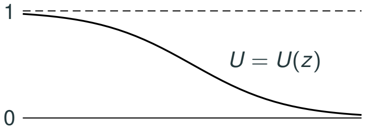
Functions of type A (Fisher-KPP type reaction)

for each $c \geq c^* > 0$ there exists a unique (up to translation) monotone decreasing t.w.s.

Functions of types B and C (combustion and bistable reaction)

there exists a unique (up to translation) monotone decreasing t.w.s. with speed $c_*$

$$\int_0^1 g(s)\, \mathrm{d}s \geq 0 \quad \Rightarrow \quad c_* \geq 0$$




$z = x - ct$

# Generalizations

Density dependent diffusion:

$$\frac{\partial^2 u}{\partial x^2} \qquad \longrightarrow \qquad \frac{\partial}{\partial x}\left(d(u)\frac{\partial u}{\partial x}\right)$$

typically $d \in C^1[0,1]$, $d > 0$ in $[0,1]$

Diffusion term driven by the *p*-Laplacian:

$$\frac{\partial^2 u}{\partial x^2} \qquad \longrightarrow \qquad \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right), \quad p > 1$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \quad (x,t) \in \mathbb{R} \times [0,+\infty), \quad p > 1$$

Functions of types B and/or C:

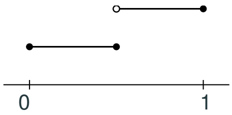
MALAGUTI, L., MARCELLI, C., AND MATUCCI, S. A unifying approach to travelling wavefronts for reaction-diffusion equations arising from genetics and combustion models *Dynamic Systems and Applications*, 12 (2003), 333–354.

ENGUIÇA, R., GAVIOLI, A., AND SANCHEZ, L. A class of singular first order differential equations with applications in reaction-diffusion. *Discrete Contin. Dyn. Syst. 33*, 1 (2013), 173–191.

DRÁBEK, P., AND TAKÁČ, P. New patterns of travelling waves in the generalized Fisher-Kolmogorov equation. *NoDEA Nonlinear Differential Equations Appl. 23*, 2 (2016), Art. 7, 19.

STRIER, D., ZANETTE, D., AND WIO, H. S. Wave fronts in a bistable reaction-diffusion system with density-dependent diffusivity. *Physica A: Statistical Mechanics and its Applications 226*, 3 (1996), 310–323.

piecewise constant diffusivity



# Problem setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left( d(u) \left| \dfrac{\partial u}{\partial x} \right|^{p-2} \dfrac{\partial u}{\partial x} \right) + g(u), \quad (x,t) \in \mathbb{R} \times [0, +\infty), \quad p > 1 \\ u(x,t) = U(x - ct), \quad c \in \mathbb{R} \text{ – speed of propagation} \end{cases}$$

reaction term $g \in C[0,1]$:

$$g(0) = g(s_*) = g(1)$$

$$g(s) \leq 0 \ \text{ if } \ s \in (0, s_*), \quad g(s) > 0 \ \text{ if } \ s \in (s_*, 1)$$

(includes functions of types B and C)

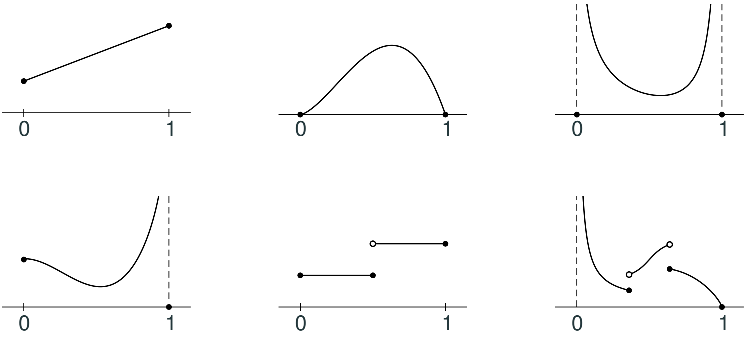
diffusion coefficient $d = d(s)$:

$d : [0,1] \to \mathbb{R}$, nonnegative lower semicontinuous with $d > 0$ in $(0,1)$

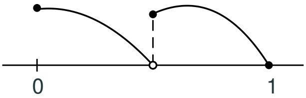
$$0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = 1$$

$$d\big|_{(s_i, s_{i+1})} \in C(s_i, s_{i+1}),\ i = 0, \ldots, n$$

jump discontinuity at $s_i$, $i = 1, 2, \ldots, n$




We do <u>not</u> allow:



$z = x - ct$ *wave variable*

$$\Big(d(U(z))|U'(z)|^{p-2}U'(z)\Big)' + cU'(z) + g(U(z)) = 0$$

We look for $c \in \mathbb{R}$ and monotone profile $U = U(z)$

$$\lim_{z\to-\infty} U(z) = 1 \qquad \text{and} \qquad \lim_{z\to+\infty} U(z) = 0$$

$U \in C(\mathbb{R})$, piecewise $C^1$

$\forall z, \hat{z} \in \mathbb{R}$:

$$v(\hat{z}) - v(z) + c\,(U(\hat{z}) - U(z)) + \int_z^{\hat{z}} g(U(\xi))\,\mathrm{d}\xi = 0$$

for any $z \in \mathbb{R} : U(z) = s_i$, $i = 1, 2, \ldots, n$ there exist finite one-sided derivatives $U'(z\pm)$ and

$$v(z) := \left|U'(z-)\right|^{p-2} U'(z-) \lim_{\xi\to z-} d(U(\xi)) = \left|U'(z+)\right|^{p-2} U'(z+) \lim_{\xi\to z+} d(U(\xi))$$

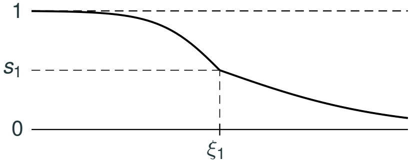
for any $z \in \mathbb{R} : U(z) \in \{0, 1\}$

$$v(z) = 0$$

$d$ has jumps at $s_i$, $i = 1, 2, \ldots, n$, $U(\xi_i) = s_i$

**transition condition**

$$v(z) := \left|U'(\xi_i-)\right|^{p-2} U'(\xi_i-) \lim_{s \to s_i+} d(s) = \left|U'(\xi_i+)\right|^{p-2} U'(\xi_i+) \lim_{s \to s_i-} d(s)$$



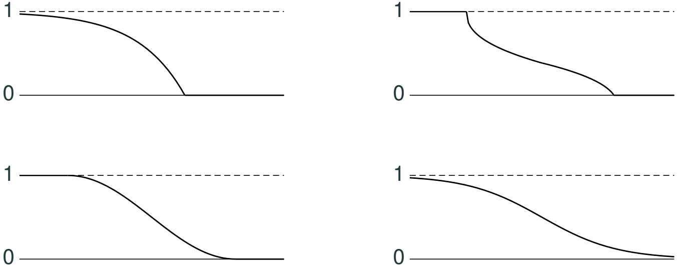

$$d(s_1+) < d(s_1-)$$

the jump of $U'$ at $\xi_1$ "compensates" the jump of $d$ at $s_1$

$$N_U := \{z \in \mathbb{R} : U(z) = 0 \text{ or } U(z) = 1\}$$

$z \in \operatorname{int} N_U$: $U'(z) = 0$

$z \in \partial N_U$: the derivative $U'(z)$ need not exist



# Reduction to a first order problem

Let $U : \mathbb{R} \to [0,1]$ be a nonincreasing solution of (8) satisfying boundary conditions

$$\lim_{z\to-\infty} U(z) = 1 \qquad \text{and} \qquad \lim_{z\to+\infty} U(z) = 0 \tag{1}$$

such that $U'(z\pm) < 0$ whenever $0 < U(z) < 1$

$\Rightarrow$ there is an open interval $(z_0, z_1)$, $-\infty \le z_0 < z_1 \le +\infty$, such that $U$ is strictly decreasing in $(z_0, z_1)$,

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad U(z) = 1 \qquad \text{if} \quad -\infty < z \le z_0,$$

$$\lim_{z\to z_1-} U(z) = 0 \quad \text{and} \quad U(z) = 0 \qquad \text{if} \quad z_1 \le z < +\infty$$

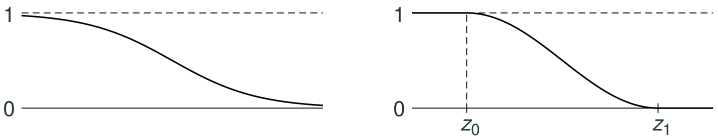
$U'(z\pm) < 0$ for all $z \in (z_0, z_1)$

$U'(z) < 0$ for all $z$ such that $U(z) \in (0,1) \setminus \bigcup_{i=1}^{n} \{s_i\}$

$U$ is continuous and

$$U\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1})$$

$$U(\xi_i) = s_i, \quad i = 1, 2, \ldots, n$$




There exists a strictly decreasing inverse function $U^{-1} : (0,1) \to (z_0, z_1)$, $z = U^{-1}(U)$

$$(\underbrace{d(U(z))|U'(z)|^{p-2}U'(z)}_{v(z)})' + cU'(z) + g(U(z)) = 0, \quad z \in (\xi_i, \xi_{i+1})$$

$$w(U) = v(U^{-1}(U)), \quad U \in (0,1)$$

$$-\frac{\mathrm{d}w}{\mathrm{d}U}\left|\frac{w(U)}{d(U)}\right|^{p'-1} - c\left|\frac{w(U)}{d(U)}\right|^{p'-1} + g(U) = 0, \quad U \in (s_i, s_{i+1})$$

this is equivalent to

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}U}|w|^{p'} = c\,|w|^{p'-1} - (d(U))^{p'-1}\,g(U)$$

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}U}\,|w|^{p'} = c\,|w|^{p'-1} - \underbrace{(d(U))^{p'-1}\,g(U)}_{f(U)}$$

$U \to t$, $y(t) = |w(t)|^{p'}$

$$y'(t) = p'\left[c\,(y(t))^{\frac{1}{p}} - f(t)\right], \quad t \in (0,1) \setminus \bigcup_{i=1}^{n}\{s_i\} \tag{2}$$

$$y(0) = y(1) = 0 \tag{3}$$

## Proposition

*A function* $U : \mathbb{R} \to [0,1]$, $U \in \widehat{C}^1(\mathbb{R})$, *is a monotone nonincreasing solution of*

$$\begin{cases} \left(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\right)' + cU'(z) + g(U(z)) = 0, \quad z \in \mathbb{R}, \\ \lim\limits_{z\to-\infty} U(z) = 1, \quad \lim\limits_{z\to+\infty} U(z) = 0, \end{cases}$$

*which is strictly decreasing on* $(z_0, z_1) = \{z \in \mathbb{R} : U(z) \in (0,1)\}$, *if and only if* $y : [0,1] \to \mathbb{R}$, $y \in C[0,1]$, *is a positive solution of* (2), (3).

# Existence results for the first order b.v.p.

We look for positive solutions of

$$\begin{cases} y'(t) = p'\left[c\left(y^+(t)\right)^{\frac{1}{p}} - f(t)\right], & t \in (0,1), \\ y(0) = y(1) = 0 \end{cases} \tag{4}$$

$$y^+(t) = \max\{y(t), 0\}, \quad p > 1, \quad p' = \frac{p}{p-1}$$

$$f \in L^1(0,1)$$

solution in the sense of Carathéodory

*two "unknowns"* :

- absolutely continuous function $y = y(t)$, $y(t) > 0$ in $(0,1)$
- $c \in \mathbb{R}$

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$f \in C[0,1]$, $f(0) = f(1) = 0$:


Enguiça, R., Gavioli, A., and Sanchez, L. A class of singular first order differential equations with applications in reaction-diffusion. *Discrete Contin. Dyn. Syst. 33*, 1 (2013), 173–191.

$f \in L^1(0,1)$

*forward initial value problem*

$$y'(t) = p' \left[ c \left( y^+(t) \right)^{\frac{1}{p}} - f(t) \right], \quad y(0) = 0$$

*backward initial value problem*

$$y'(t) = p' \left[ c \left( y^+(t) \right)^{\frac{1}{p}} - f(t) \right], \quad y(1) = 0 \tag{5}$$

## Global existence result

for any $c \in \mathbb{R}$ there exists at least one global solution $y_c = y_c(t)$ of the forward i.v.p. defined on the entire interval $[0, 1]$

the same holds for the backward i.v.p.

## Uniqueness result

$y \mapsto (y^+)^{\frac{1}{p}}$ is not Lipschitz at 0

$$y \mapsto c(y^+)^{\frac{1}{p}}, \;\; c \leq 0 \text{ – nonincreasing}, \quad c \geq 0 \text{ – nondecreasing}$$

if $c \leq 0$ then the forward i.v.p. has exactly one solution $y_c = y_c(t)$, $t \in [0, 1]$

if $c \geq 0$ then the backward i.v.p. has exactly one solution $y_c = y_c(t)$, $t \in [0, 1]$

## Continuous dependence of solutions on parameter $c$

$c_0 \geq 0$ ($c_0 \leq 0$), $y_c = y_c(t)$ solutions of the backward (forward) i.v.p.

$$c \to c_0 \quad \Rightarrow \quad \left\| y_c - y_{c_0} \right\|_{C[0,1]} \to 0$$

$$h(t, y, c) := p' \left[ c \left( y^+ \right)^{\frac{1}{p}} - f(t) \right]$$

We restrict on $c \geq 0$ and the backward i.v.p.

$$y'(t) = h(t, y(t), c), \quad y(1) = 0$$

*defect* $P_c\varphi$[2]

$$P_c\varphi := \varphi'(t) - h(t, \varphi(t), c)$$

## Lemma

*Let* $f \in L^1(0,1)$, $c \geq 0$ *and assume that the functions* $\varphi, \psi \in AC[0,1]$ *satisfy* $\varphi(1) \leq \psi(1)$, $P_c\varphi \geq P_c\psi$ *a.e. in* $[0,1]$. *Then* $\varphi \leq \psi$ *in* $[0,1]$.

## Corollary

*Let* $0 \leq c_1 < c_2$. *Let* $y_{c_1}$ *and* $y_{c_2}$ *be the solutions of* (5) *with* $c = c_1$ *and* $c = c_2$, *respectively. Then*

$$y_{c_1}(t) \geq y_{c_2}(t), \quad t \in [0,1].$$

*In particular,* $y_{c_1}(0) \geq y_{c_2}(0)$.

[2] WALTER, W. Ordinary differential equations, vol. 182 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

## Bistable unbalanced case

$$f(t) \le 0 \quad \text{if } t \in (0, s_*), \qquad f(t) > 0 \quad \text{if } t \in (s_*, 1)$$

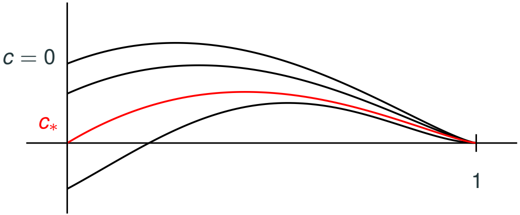
$f$ lower semicontinuous in $(s_*, 1)$

$$\int_0^1 f(t)\,\mathrm{d}t > 0$$

$\Rightarrow$ there exists a number $c_* > 0$ such that the b.v.p. (4) has a unique positive solution if and only if $c = c_*$

$$y_0(t) = p' \int_t^1 f(s)\,\mathrm{d}s > 0 \quad \text{for all } t \in [0, 1)$$

in particular: $y_0(0) > 0$



## Outline of the proof

$f$ lower semicontinuous in $(s_*, 1)$

$c \geq 0$, $y_c = y_c(t)$ solution of the backward i.v.p.

$$y_c(t) > 0 \text{ in } (s_*, 1)$$

$$y_0(t) = p' \int_t^1 f(s)\, ds > 0 \quad \text{for all} \ \ t \in [0, 1)$$

Set

$$c_* := \sup \{c > 0 : y_c(t) > 0 \ \text{for all} \ \ t \in (0, 1)\}$$

1. $0 < c_* < +\infty$
2. $y_{c_*}(0) = 0$ and $y_{c_*}(t) > 0$, $t \in (0, 1)$
3. $y_c(t)$ positive solution of the backward i.v.p. $\Rightarrow y_c(0) = 0$ if and only if $c = c_*$
4. uniqueness for the backward i.v.p. $\Rightarrow$ unique solution of the b.v.p.

## Bistable balanced case

$$f(t) \le 0 \quad \text{if } t \in (0, s_*), \qquad f(t) > 0 \quad \text{if } t \in (s_*, 1)$$

$$f < 0 \text{ on } (0, \delta) \text{ for some } \delta \in (0, s_*)$$

$$\int_0^1 f(t)\,dt = 0$$

$\Rightarrow$ the b.v.p. (4) has a unique positive solution if and only if $c = 0$

*Proof.* Let $y = y(t)$, $t \in [0, 1]$, be a positive solution of (4). Integrating the equation in (4) from 0 to 1 and using the boundary conditions together with (20), we obtain

$$0 = y(1) - y(0) = \int_0^1 y'(t)\,dt = p'\left[c\int_0^1 (y^+(t))^{\frac{1}{p}}\,dt - \int_0^1 f(t)\,dt\right] = p'c\int_0^1 (y^+(t))^{\frac{1}{p}}\,dt.$$

Hence $c = 0$.

On the other hand, the backward initial value problem (5) with $c = 0$ has a unique solution

$$y(t) = p'\int_t^1 f(s)\,ds.$$

It follows from our assumptions on $f$ that $y(t) > 0$ for all $t \in (0, 1)$ and $y(0) = 0$. Therefore, it is also a unique positive solution of (4).

# Existence results for the second order b.v.p.

$$\begin{cases} \left(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\right)' + cU'(z) + g(U(z)) = 0, & z \in \mathbb{R}, \\ \lim_{z\to-\infty} U(z) = 1, \quad \lim_{z\to+\infty} U(z) = 0 \end{cases}$$

bistable reaction term $g$

- Bistable unbalanced condition

$$\int_0^1 (d(t))^{\frac{1}{p-1}}\, g(t)\,\mathrm{d}t > 0$$

$\Rightarrow$ there exists a unique (except for translation) solution $U = U(z)$ with a positive wave speed $c_*$

- Bistable balanced condition

$$\int_0^1 (d(t))^{\frac{1}{p-1}}\, g(t)\,\mathrm{d}t = 0$$

$\Rightarrow$ $c = 0$

there exists a unique solution of

$$\begin{cases} \left(d(u(x))\,|u'(x)|^{p-2}\,u'(x)\right)' + g(u(x)) = 0 \\ \lim_{x\to-\infty} u(x) = 1, \quad \lim_{x\to+\infty} u(x) = 0 \end{cases}$$

# Asymptotic behaviour of t.w. profiles

$$U = U(z) \text{ as } z \to \pm\infty$$

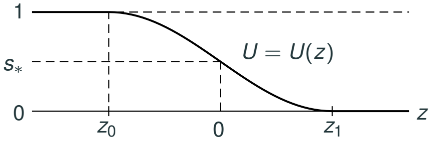
DRÁBEK, ZAHRADNÍKOVÁ, EJDE Vol. 2021, No. 76, pp.1–21

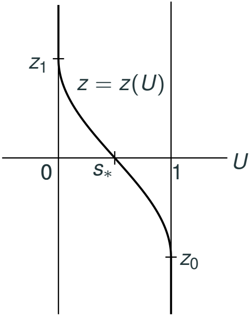
we assume **power-type** behaviour of $d$ and $g$

Asymptotic behaviour of the inverse function

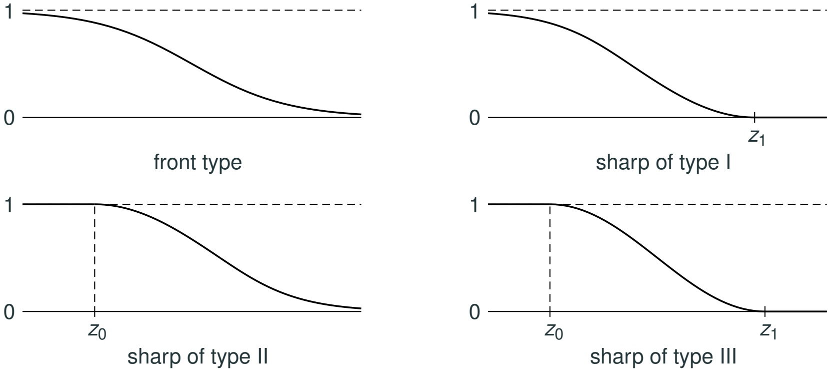
$$z(U) = -\int_{s_*}^{U} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t, \quad U \in (0,1)$$

as $U \to 1-$, $U \to 0+$



$$z_0 = -\int_{s_*}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t, \qquad z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t$$




classification due to MALAGUTI, MARCELLI (2005)[3]

[3] MALAGUTI, L., AND MARCELLI, C.Finite speed of propagation in monostable degenerate reaction-diffusion-convection equations. *Adv. Nonlinear Stud. 5*, 2 (2005), 223–252

## Asymptotics near 0

$$g(t) \sim -t^{\alpha}, \quad d(t) \sim t^{\beta} \qquad \Rightarrow \qquad f(t) = (d(t))^{\frac{1}{p-1}} g(t) \sim -t^{\alpha + \frac{\beta}{p-1}} \text{ as } t \to 0+$$

$$f \in L^1(0,1) \quad \Rightarrow \quad \boxed{\alpha + \frac{\beta}{p-1} > -1}$$

$$g(0) = 0 \quad \Rightarrow \quad \boxed{\alpha > 0}$$

lack of uniqueness for the forward i.v.p. if $c > 0$

BUT: special form of our equation allows us to derive uniqueness result for this problem if we restrict on the set of positive solutions

we can use comparison argument "near" 0:

$$\left.\begin{aligned} \varphi'(t) &\leq p'[c_* \left(\varphi^+(t)\right)^{\frac{1}{p}} - f(t)] \\ \psi'(t) &\geq p'[c_* \left(\psi^+(t)\right)^{\frac{1}{p}} - f(t)] \end{aligned}\right\} \quad \Rightarrow \quad \varphi(t) \leq y_{c_*}(t) \leq \psi(t)$$

$$t \in (0, \theta), \quad 0 < \theta < s_*$$

$\exists\,\theta > 0$ sufficiently small:

$$f(t) = -\eta(t) t^{\alpha + \frac{\beta}{p-1}}, \quad t \in (0, \theta)$$

$$\eta = \eta(t) \text{ continuous, } \lim_{t \to 0+} \eta(t) \in (0, +\infty)$$

$$\alpha + \frac{\beta}{p-1} \le \frac{1}{p-1} \quad \Rightarrow \quad y_\kappa(t) = \kappa t^{\alpha + \frac{\beta}{p-1} + 1},\ t \in (0,\theta),\ \kappa > 0$$

For $\underline{\kappa} \ll 1$:

$$y'_{\underline{\kappa}}(t) \le p' \left[ c_*(y_{\underline{\kappa}}(t)^{\frac{1}{p}}) - f(t) \right], \quad t \in (0,\theta)$$

For $\bar{\kappa} \gg 1$:

$$y'_{\bar{\kappa}}(t) \ge p' \left[ c_*(y_{\bar{\kappa}}(t)^{\frac{1}{p}}) - f(t) \right], \quad t \in (0,\theta)$$

$$\Downarrow$$

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\bar{\kappa}}(t)$$

subsolution $\nearrow$ $\qquad$ $\nwarrow$ supersolution

Let $-1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}$. Then we distinguish

Case 1: $\frac{\alpha-\beta+1}{p} < 1$

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t = \int_0^{\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t + \int_{\theta}^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t$$
$$\leq c_1 + \int_0^{\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \leq c_1 + c_2 \int_0^{\theta} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}}\,\mathrm{d}t = c_1 + c_2 \int_0^{\theta} \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} < +\infty$$

Case 2: $\frac{\alpha-\beta+1}{p} \geq 1$

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t = \int_0^{\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t + \int_{\theta}^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t$$
$$\geq c_3 + \int_0^{\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \geq c_3 + c_4 \int_0^{\theta} \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} = +\infty$$

Similarly for

$$\alpha + \frac{\beta}{p-1} > \frac{1}{p-1} \begin{cases} \nearrow \ \beta > 2-p \quad \Rightarrow \quad z_1 < +\infty \\ \searrow \ \beta \le 2-p \quad \Rightarrow \quad z_1 = +\infty \end{cases}$$

To summarize these conditions, we denote

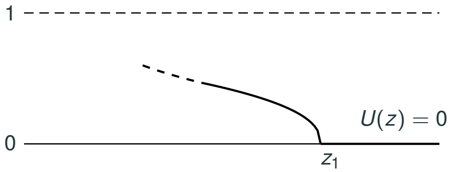
$$\mathcal{M}_0^1 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}, \alpha - \beta + 1 < p\}$$
$$\mathcal{M}_0^2 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}, \alpha - \beta + 1 \ge p\}$$
$$\mathcal{M}_0^3 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta > 2-p\}$$
$$\mathcal{M}_0^4 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta \le 2-p\}$$

$$(\alpha,\beta) \in \mathcal{M}_0^2 \cup \mathcal{M}_0^4 \quad \Rightarrow \quad z_1 = +\infty$$

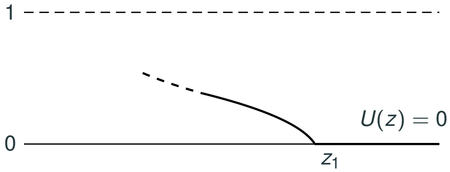
$$(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3 \quad \Rightarrow \quad z_1 < +\infty$$
$$U'(z_1+) = 0, \ -\infty \le U'(z_1-) \le 0$$

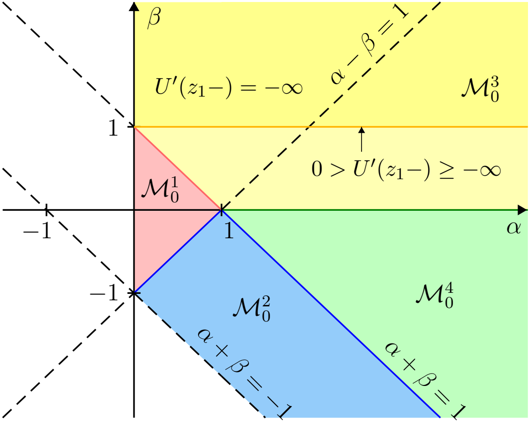
In the case $(\alpha, \beta) \in \mathcal{M}_0^3$, we are able to obtain more precise information about $U'(z-)$:

If $\beta > 1$ then $U'(z_1-) = -\infty$:




If $\beta = 1$ then $-\infty \leq U'(z_1-) < 0$:





β
α − β = 1
U′(z₁−) = −∞
$\mathcal{M}_0^3$
1
0 > U′(z₁−) ≥ −∞
$\mathcal{M}_0^1$
−1
1
α
−1
$\mathcal{M}_0^4$
$\mathcal{M}_0^2$
α + β = −1
α + β = 1


$p = 2$

## Asymptotics near 1

$$g(t) \sim -(1-t)^\gamma, \quad d(t) \sim (1-t)^\delta$$

$$f \in L^1(0,1) \quad \Rightarrow \quad \boxed{\gamma + \frac{\delta}{p-1} > -1}$$

$$g(0) = 0 \quad \Rightarrow \quad \boxed{\gamma > 0}$$

$$\mathcal{M}_1^1 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}, \gamma - \delta + 1 < p\}$$

$$\mathcal{M}_1^2 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}, \gamma - \delta + 1 \geq p\}$$

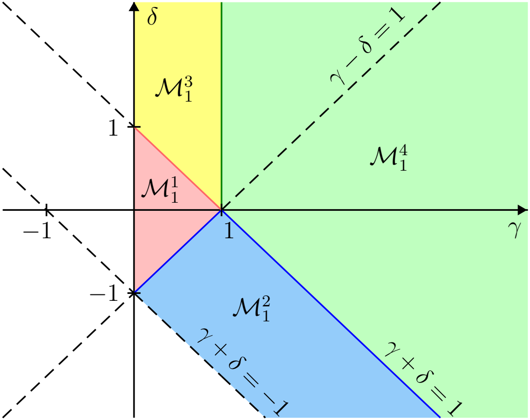
$$\mathcal{M}_1^3 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma < 1\}$$

$$\mathcal{M}_1^4 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma \geq 1\}$$

$$(\gamma,\delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4 \quad \Rightarrow \quad z_0 = -\infty$$

$$(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3 \quad \Rightarrow \quad z_0 > -\infty$$

$$U'(z_0-) = 0, \; -\infty \leq U'(z_0+) \leq 0$$


δ
$\mathcal{M}_1^3$
$\gamma - \delta = 1$
1
$\mathcal{M}_1^4$
$\mathcal{M}_1^1$
−1
1
γ
−1
$\mathcal{M}_1^2$
$\gamma + \delta = -1$
$\gamma + \delta = 1$

$p = 2$

# Standing waves

bistable balanced case

DRÁBEK, ZAHRADNÍKOVÁ, *Electron. J. Qual. Theory. Differ. Equ.*, Vol. 2021, No. 61, 1–16

- detailed discussion for *nondecreasing* solution of

$$\begin{cases} \left(d(u(x))\,|u'(x)|^{p-2}\,u'(x)\right)' + g(u(x)) = 0 \\ \lim\limits_{x\to-\infty} u(x) = 0, \quad \lim\limits_{x\to+\infty} u(x) = 1 \end{cases}$$

$$x(u) = \int_{s_*}^{u} \left(\frac{d(t)}{w(t)}\right)^{\frac{1}{p-1}} \mathrm{d}t = \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int\limits_{s_*}^{u} \frac{(d(t))^{\frac{1}{p-1}}}{\left(-\int\limits_{0}^{t} (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s\right)^{\frac{1}{p}}} \mathrm{d}t$$

$$g(t) \sim (-t^{\alpha}),\ d(t) \sim t^{\beta} \quad \text{for } t \to 0+$$

$$\alpha > 0, \beta \in \mathbb{R}$$

$$\boxed{\alpha + \frac{\beta}{p-1} > -1}$$

(I) If $\alpha - \beta \geq p - 1$ then $x_0 = -\infty$. Moreover, for $\alpha - \beta = p - 1$ we have

$$u(x) \sim e^x \to 0+ \qquad \text{for } x \to -\infty$$

and for $\alpha - \beta > p - 1$ we have

$$u(x) \sim |x|^{\frac{p}{p-1-(\alpha-\beta)}} \to 0+ \qquad \text{for } x \to -\infty.$$

(II) If $\alpha - \beta < p - 1$ then $x_0 > -\infty$ and for $x \to x_0+$ we have

$$u(x) \sim (x - x_0)^{\frac{p}{p-1-(\alpha-\beta)}}.$$

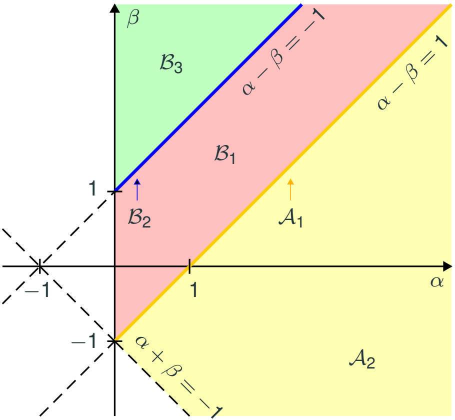
As for the derivatives, we then have

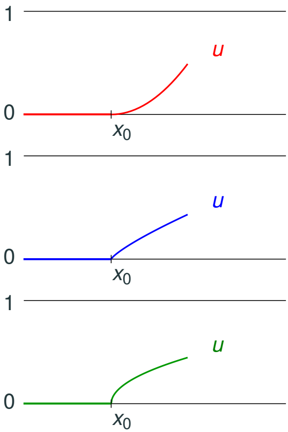
(a) $$\left.\frac{du}{dx}\right|_{x=x_0+} \sim (x - x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to 0 \quad \text{for } x \to x_0+ \qquad \text{if } \alpha - \beta > -1,$$

(b) $$\left.\frac{du}{dx}\right|_{x=x_0+} \sim (x - x_0)^0 \to k > 0 \quad \text{for } x \to x_0+ \qquad \text{if } \alpha - \beta = -1,$$

(c) $$\left.\frac{du}{dx}\right|_{x=x_0+} \sim (x - x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to +\infty \quad \text{for } x \to x_0+ \qquad \text{if } \alpha - \beta < -1.$$

$p = 2$:




$(\alpha, \beta) \in \mathcal{A}_1 \cup \mathcal{A}_2 \;\Rightarrow\; x_0 = -\infty$

Case (I)

$(\alpha, \beta) \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \;\Rightarrow\; x_0 > -\infty$

Case (II)

Thank you for your attention!