# Macroscopic and microscopic behavior of the solutions of a transmission problem for the Helmholtz equation in a domain with a small inclusion.

Massimo Lanza de Cristoforis
Università degli Studi di Padova, Italy
joint work with Tuğba Akyel,
Maltepe University, Istanbul, Turkey.

International Meetings
on Differential Equations and Their Applications
Lodz University of Technology
University of Rzeszów
11 May, 2022

First of all I would like to thank the organizers for giving me the opportunity to talk today. Today's talk is in part based on the papers

• T. Akyel and mldc, Mathematical Methods in the Applied Sciences, 2022.

https://doi.org/10.1002/mma.8111

• T. Akyel and mldc, Studia Universitatis Babeş-Bolyai Mathematica (to appear).

We plan to consider a linear transmission problem for the Helmholtz equation in a domain with a small inclusion, that is motivated by the analysis of time-harmonic Maxwell's Equations. Here we refer to

• M.S. Vogelius and D. Volkov, Mathematical Modelling and Numerical Analysis, **34**, (2000), 723–748.

We first introduce a Neumann problem for the Helmholtz equation (and no transmission) in an

'unperturbed domain' $\Omega^o \subseteq \mathbb{R}^n$ (with no hole).

We fix (throughout the talk) a number $\alpha \in ]0,1[$.

The 'unperturbed' domain $\Omega^o$ satisfies the following assumption:

$(DOM)$ It is a bounded open connected subset of $\mathbb{R}^n$, it has a connected exterior (and thus no holes), it contains $0$, it is of class $C^{1,\alpha}$.

Then we introduce a wave number

$$k_o \in \mathbb{C}\setminus]-\infty,0]\,, \qquad \Im k_o \geq 0\,.$$

We also assume that $k_o^2$ is not a Neumann eigenvalue for $-\Delta$ in $\Omega^o$. Let

$$g^o \in C^{0,\alpha}(\partial\Omega^o)\,.$$

Then we introduce the Neumann problem

$$(P) \qquad \begin{cases} \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o\,, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o\,. \end{cases}$$

Here $\nu_{\Omega^o}$ denotes the outward unit normal to $\partial\Omega^o$.

Problem (P) is known to have a unique solution

$\tilde{u}^o$ in $C^{1,\alpha}(\overline{\Omega}^o)$.

Next we perturb singularly our problem. Let

$$\Omega^i \subseteq \mathbb{R}^n$$

be a domain as in $(DOM)$. Let $\epsilon_0 \in ]0,1[$ be small enough so that

$$\epsilon\overline{\Omega^i} \subseteq \Omega^o \qquad \forall \epsilon \in [-\epsilon_0, \epsilon_0]\,,$$

and we consider the perforated domain

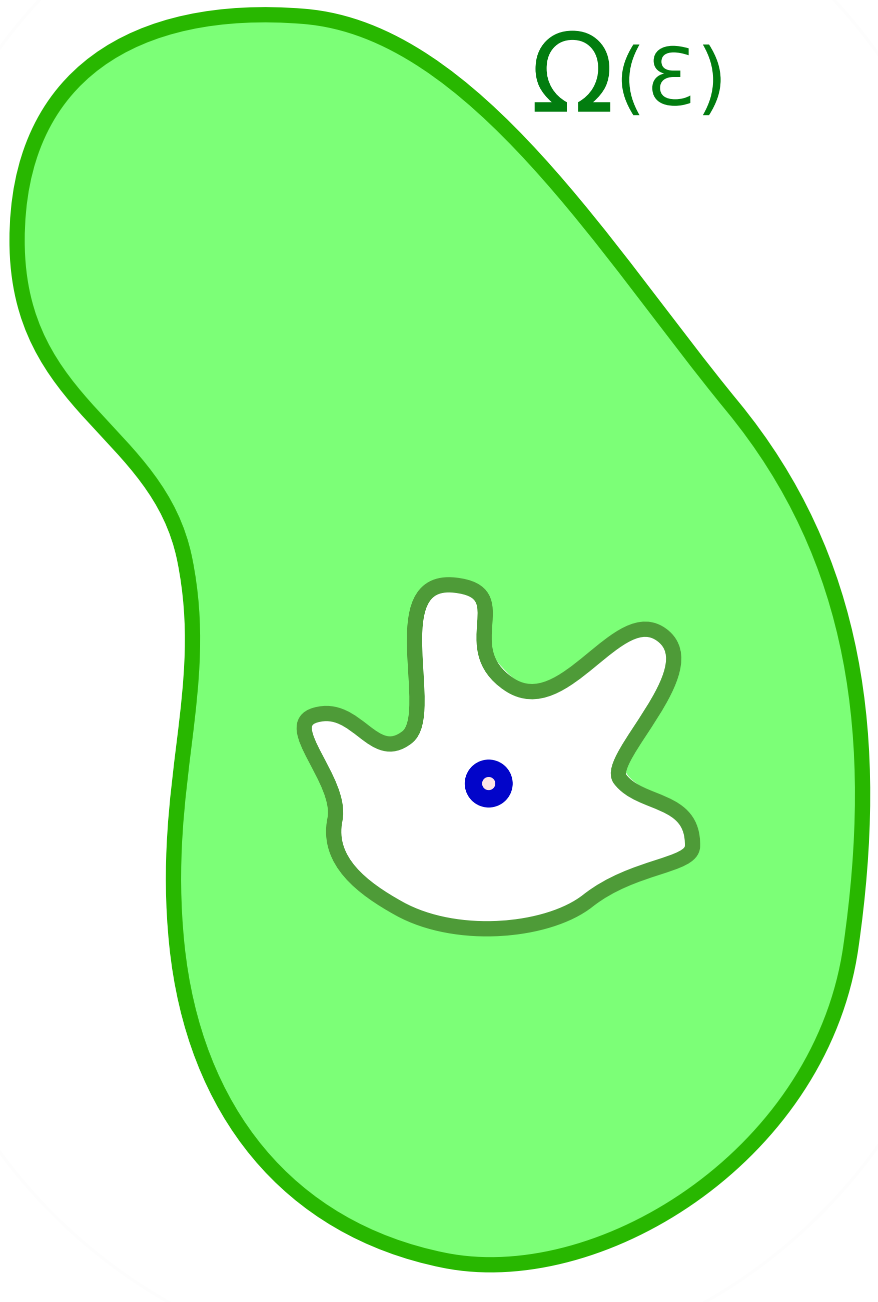
$$\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}\,,$$

for $|\epsilon| \leq \epsilon_0$. Obviously,

$$\partial\Omega(\epsilon) = \epsilon\partial\Omega^i \cup \partial\Omega^o\,.$$

Also, if $\epsilon$ shrinks to 0, then

$$\underbrace{\Omega(\epsilon)}_{\text{is of class } C^{1,\alpha}} \qquad \text{degenerates to} \qquad \underbrace{\Omega^o \setminus \{0\}}_{\text{is not of class } C^{1,\alpha}} \quad .$$


Ω(ε)

Next we define a transmission problem in $(\epsilon\Omega^i, \Omega(\epsilon))$. To do so, we introduce the constants

$$m^i, m^o \in ]0, +\infty[\,, \qquad a \in ]0, +\infty[\,,\ b \in \mathbb{R}\,,$$

the wave number

$$k_i \in \mathbb{C} \setminus ]-\infty, 0]\,, \qquad \Im k_i \geq 0\,,$$

and the 'jump' datum for the normal derivatives

$$g^i \in C^{0,\alpha}(\partial\Omega^i)\,.$$

Then we consider the transmission problem

$$(P_\epsilon)\begin{cases} \Delta u^i + k_i^2 u^i = 0 & \text{in } \epsilon\Omega^i\,, \\ \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega(\epsilon)\,, \\ u^o(x) - a u^i(x) = b & \forall x \in \epsilon\partial\Omega^i\,, \\ -\frac{1}{m^i}\frac{\partial}{\partial \nu_{\epsilon\Omega^i}} u^i(x) + \frac{1}{m^o}\frac{\partial}{\partial \nu_{\epsilon\Omega^i}} u^o(x) & \\ \qquad\qquad\qquad = g^i(x/\epsilon) & \forall x \in \epsilon\partial\Omega^i\,, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o\,, \end{cases}$$

in the unknown $(u^i, u^o) \in C^{1,\alpha}(\epsilon\overline{\Omega^i}) \times C^{1,\alpha}(\overline{\Omega(\epsilon)})$ for $\epsilon \in ]0, \epsilon_0[$.

The first step is prove the following existence and uniqueness theorem for problem $(P_\epsilon)$.

**Theorem 1 (of existence and uniqueness)** *There exists* $\epsilon' \in ]0, \epsilon_0[$ *such that if* $\epsilon \in ]0, \epsilon'[$*, then the transmission problem* $(P_\epsilon)$ *has one and only one solution*

$$(u^i(\epsilon, \cdot), u^o(\epsilon, \cdot)) \in C^{1,\alpha}(\epsilon\overline{\Omega^i}) \times C^{1,\alpha}(\overline{\Omega(\epsilon)})\,.$$

• Kress, R., Roach, G. F. J. Mathematical Phys. 19 (1978), no. 6, 1433–1437. [case of a domain and its exterior]

Then **our goal** is to understand the behavior of

$(u^i(\epsilon, \cdot), u^o(\epsilon, \cdot))$ as $\epsilon$ approaches 0

and of its rescaled version

$(u^i(\epsilon, \epsilon\cdot), u^o(\epsilon, \epsilon\cdot))$ as $\epsilon$ approaches 0

More precisely, we plan to answer the following questions

(i) Let $\xi$ be fixed in $\overline{\Omega^i}$. What can be said on the map $\epsilon \mapsto u^i(\epsilon, \epsilon\xi)$ when $\epsilon > 0$ is close to 0?

(ii) Let $\xi_m$ be fixed in $\mathbb{R}^n \setminus \Omega^i$. What can be said on the map $\epsilon \mapsto u^o(\epsilon, \epsilon\xi_m)$ when $\epsilon > 0$ is close to 0?

(iii) Let $x_M$ be fixed in $\overline{\Omega^o} \setminus \{0\}$. What can be said on the map $\epsilon \mapsto u^o(\epsilon, x_M)$ when $\epsilon > 0$ is close to 0?

In a sense, questions (i), (ii) concern the 'microscopic' behavior of $u^i(\epsilon, \cdot)$ and $u^o(\epsilon, \cdot)$, whereas question (iii) concerns the 'macroscopic' behavior of $u^o(\epsilon, \cdot)$.

Questions of this type have long been investigated with the methods of asymptotic analysis, which aim at proving complete asymptotic expansions in terms of the parameter $\epsilon$.

• A.M. Il'in, Translations of Mathematical Monographs, 102. American Mathematical Society, Providence, RI, 1992. [method of matching outer and inner asymptotic expansions]

• V.G. Mazya, S.A. Nazarov and B.A. Plamenewskii, **I, II**, Oper. Theory Adv. Appl., **111**, **112**, Birkhäuser Verlag, Basel, 2000.

[Compound Expansion Method (also known as Multi-Scale Expansion Method): a systematic approach for analyzing general Douglis and Nirenberg elliptic boundary value problems in domains with perforations and corners.]

For today's problem:

- D. Cedio-Fengya, S. Moskow and M.S. Vogelius, Inverse Problems 14 (1998) 553–595.

- M.S. Vogelius and D. Volkov, Mathematical Modelling and Numerical Analysis, **34**, (2000), 723–748.

- Hansen, D.J. , Poignard, C., Vogelius, M.S., Appl. Anal. 86 (2007), no. 4, 433–458.

Today, I have no word to add in the realm of asymptotic expansions.

What I want to say is that I and other collaborators:

M. Dalla Riva, P. Musolino, P. Luzzini, R. Pukhtaievych, S. Gryshchuk and for today's problem my co-author T. Akyel

have tried to represent the dependence of the solutions or eigenvalues of boundary value problems upon a singular perturbation parameter $\epsilon$ around the degenerate case $\epsilon = 0$, in terms of

analytic functions

or

of other known functions of $\epsilon$ (such as $\log \epsilon$, $1/\log \epsilon$, etc...)

and we are doing it today for $u^i(\epsilon, \epsilon\cdot)$, $u^o(\epsilon, \epsilon\cdot)$, $u^o(\epsilon, \cdot)$.

For an introduction to this point of view, we refer to

• Dalla Riva, Matteo; mldc; Musolino, Paolo, Springer, Cham, 2021.

Below $\kappa_n = 1$ if $n$ is even and $\kappa_n = 0$ if $n$ is odd and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geq 3$.

**Theorem 2** *Let* $x_M \in \overline{\Omega^o} \setminus \{0\}$*.*

*There exist* $\epsilon_{x_M} \in ]0, \epsilon'[$*, an open neighbourhood* $\tilde{U}$ *of* $(0,0)$ *in* $\mathbb{R}^2$

*and a real analytic map* $\mathcal{U}_{x_M}$ *from* $]-\epsilon_{x_M}, \epsilon_{x_M}[\times\tilde{U}$ *to* $\mathbb{C}$

*such that*

$$x_M \in \overline{\Omega(\epsilon)} \qquad \forall \epsilon \in ]-\epsilon_{x_M}, \epsilon_{x_M}[\,,$$

$$\left(\epsilon, \kappa_n \epsilon \log \epsilon, \frac{\delta_{2,n}}{\log \epsilon}\right) \in ]-\epsilon_{x_M}, \epsilon_{x_M}[\times\tilde{U}, \quad \forall \epsilon \in ]0, \epsilon_{x_M}[$$

*and*

$$u^o(\epsilon, x_M) = \mathcal{U}_{x_M}\left[\epsilon, \kappa_n \epsilon \log \epsilon, \frac{\delta_{2,n}}{\log \epsilon}\right] \ \forall \epsilon \in ]0, \epsilon_{x_M}[\,.$$

*Moreover,*

$$\mathcal{U}_{x_M}[0,0,0] = \tilde{u}^o(x_M)$$

*where* $\tilde{u}^o$ *is the only solution of the 'unperturbed' Neumann problem (P).*

What does the previous theorem say?

It says that if $x_M \in \overline{\Omega^o} \setminus \{0\}$, then

• If $n = 2$, the solution $u^o(\epsilon, \cdot)$ evaluated at $x_M$ tends to $\tilde{u}^o(x_M)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$, $\epsilon \log \epsilon$, $\frac{1}{\log \epsilon}$ for $\epsilon > 0$ small enough.

• If $n \geq 3$ and $n$ is even, then the solution $u^o(\epsilon, \cdot)$ evaluated at $x_M$ tends to $\tilde{u}^o(x_M)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$, $\epsilon \log \epsilon$ for $\epsilon > 0$ small enough.

• If $n \geq 3$ and $n$ is odd, then the solution $u^o(\epsilon, \cdot)$ evaluated at $x_M$ tends to $\tilde{u}^o(x_M)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$ for $\epsilon > 0$ small enough.

How about the explicit computation of the Taylor coefficients of the function $\mathcal{U}_{x_M}$? We have proved no result on today's problem. However, for this type of computation, we mention the work of

• Dalla Riva, M., Musolino, P., Rogosin,S.V.: Asymptot. Anal. 92, 339–361 (2015)

for a singularly perturbed Dirichlet problem for the Laplace operator.

It turns out that the solution $u^o(\epsilon,\cdot)$ and the rescaled pair $(u^i(\epsilon,\epsilon\cdot),u^o(\epsilon,\epsilon\cdot))$ have a limit as $\epsilon$ tends to zero that is related to the solutions of a 'limiting boundary value problem' that we analyze in the following statement.

**Theorem 3** *The limiting boundary value problem*

$$
\begin{cases}
\Delta u_1^{i,r} = 0 & \text{in } \Omega^i\,,\\
\Delta u_1^{o,r} = 0 & \text{in } \Omega^{i-}\,,\\
\Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o\,,\\
u_1^{o,r}(x) + u^o(0) - a u_1^{i,r}(x) = b & \forall x \in \partial\Omega^i\,,\\
-\frac{1}{m^i}\frac{\partial}{\partial\nu_{\Omega^i}}u_1^{i,r}(x) + \frac{1}{m^o}\frac{\partial}{\partial\nu_{\Omega^i}}u_1^{o,r}(x) = 0 & \forall x \in \partial\Omega^i\,,\\
\frac{\partial}{\partial\nu_{\Omega^o}}u^o = g^o & \text{on } \partial\Omega^o\,,\\
\lim_{\xi\to\infty} u_1^{o,r}(\xi) = 0\,,
\end{cases}
$$

*has one and only one solution* $(\tilde{u}_1^{i,r},\tilde{u}_1^{o,r},\tilde{u}^o)$ *in*

$C^{1,\alpha}(\overline{\Omega^i}) \times C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega^{i-}}) \times C^{1,\alpha}(\overline{\Omega^o})$.

We now state our main result on the microscopic behaviour of $u^i(\epsilon,\cdot)$, *i.e.*, of $u^i(\epsilon,\epsilon\cdot)$.

**Theorem 4** *Let $\xi\in\overline{\Omega^i}$. There exist an open neighbourhood $\tilde{U}$ of $(0,0)$ in $\mathbb{R}^2$ and real analytic maps $\mathcal{U}_1^i$, $\mathcal{U}_2^i$ from $]-\epsilon',\epsilon'[\times\tilde{U}$ to $\mathbb{C}$ such that*

$$u^i(\epsilon,\epsilon\xi)=\mathcal{U}_1^i\left[\epsilon,\kappa_n\epsilon\log\epsilon,\frac{\delta_{2,n}}{\log\epsilon}\right] +(\kappa_n\epsilon\log^2\epsilon)\mathcal{U}_2^i\left[\epsilon,\kappa_n\epsilon\log\epsilon,\frac{\delta_{2,n}}{\log\epsilon}\right] \tag{5}$$

*for all $\epsilon\in]0,\epsilon'[$. Moreover,*

$$\mathcal{U}_1^i[0,0,0]=\tilde{u}_1^{i,r}(\xi)\,,\qquad \mathcal{U}_2^i[0,0,0]=0\,, \tag{6}$$

*where $\tilde{u}_1^{i,r}$ has been defined in Theorem 3.*

What does the previous theorem say?

It says that if $\xi \in \overline{\Omega^i}$, then

• If $n = 2$, then $u^i(\epsilon, \epsilon\xi)$ tends to $\tilde{u}_1^{i,r}(\xi)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$, $\epsilon \log \epsilon$, $\frac{1}{\log \epsilon}$, $\epsilon \log^2 \epsilon$ for $\epsilon > 0$ small enough.

• If $n \geq 3$ and $n$ is even, then $u^i(\epsilon, \epsilon\xi)$ tends to $\tilde{u}_1^{i,r}(\xi)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$, $\epsilon \log \epsilon$, $\epsilon \log^2 \epsilon$ for $\epsilon > 0$ small enough.

• If $n \geq 3$ and $n$ is odd, then $u^i(\epsilon, \epsilon\xi)$ tends to $\tilde{u}_1^{i,r}(\xi)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$ for $\epsilon > 0$ small enough.

We now state our main result on the microscopic behaviour of $u^o(\epsilon,\cdot)$, *i.e.*, of $u^o(\epsilon,\epsilon\cdot)$.

**Theorem 7** *Let $\xi_m \in \mathbb{R}^n \setminus \overline{\Omega^i}$. Then there exist an open neighbourhood $\tilde{U}$ of $(0,0)$ in $\mathbb{R}^2$ and $\epsilon_m \in ]0,\epsilon'[$, and two real analytic maps*

$$\mathcal{U}^o_{m,1}, \mathcal{U}^o_{m,2} : ]-\epsilon_m,\epsilon_m[ \times \tilde{U} \to \mathbb{C}$$

*such that*

$$\epsilon\xi_m \in \Omega(\epsilon) \ \ \forall \epsilon \in ]-\epsilon_m,\epsilon_m[\,, \tag{8}$$

$$u^o(\epsilon,\epsilon\xi_m) = \mathcal{U}^o_{m,1}\left[\epsilon, \kappa_n \epsilon \log\epsilon, \frac{\delta_{2,n}}{\log\epsilon}\right] + (\kappa_n \epsilon \log^2\epsilon)\,\mathcal{U}^o_{m,2}\left[\epsilon, \kappa_n \epsilon \log\epsilon, \frac{\delta_{2,n}}{\log\epsilon}\right] \tag{9}$$

*for all $\epsilon \in ]0,\epsilon_m[$. Moreover,*

$$\mathcal{U}^o_{m,1}[0,0,0] = \tilde{u}^o(0) + \tilde{u}^{o,r}_1(\xi_m)\,,$$
$$\mathcal{U}^o_{m,2}[0,0,0] = 0\,,$$

*where $\tilde{u}^{o,r}_1$ has been defined in Theorem 3.*

What does the previous theorem say?

It says that if $\xi_m \in \mathbb{R}^n \setminus \overline{\Omega^i}$, then

• If $n = 2$, then $u^o(\epsilon, \epsilon\xi_m)$ tends to $\tilde{u}^o(0)+\tilde{u}_1^{o,r}(\xi_m)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$, $\epsilon \log \epsilon$, $\frac{1}{\log \epsilon}$, $\epsilon \log^2 \epsilon$ for $\epsilon > 0$ small enough.

• If $n \geq 3$ and $n$ is even, then $u^o(\epsilon, \epsilon\xi_m)$ tends to $\tilde{u}^o(0) + \tilde{u}_1^{o,r}(\xi_m)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$, $\epsilon \log \epsilon$, $\epsilon \log^2 \epsilon$ for $\epsilon > 0$ small enough.

• If $n \geq 3$ and $n$ is odd, then $u^o(\epsilon, \epsilon\xi_m)$ tends to $\tilde{u}^o(0) + \tilde{u}_1^{o,r}(\xi_m)$ as $\epsilon$ tends to 0 and can be expanded into a convergent power expansion of

$\epsilon$ for $\epsilon > 0$ small enough.

THANK YOU FOR YOUR ATTENTION!