Hyers-Ulam and Hyers-Ulam-Rassias Stability of First-Order Linear and Nonlinear Dynamic Equations

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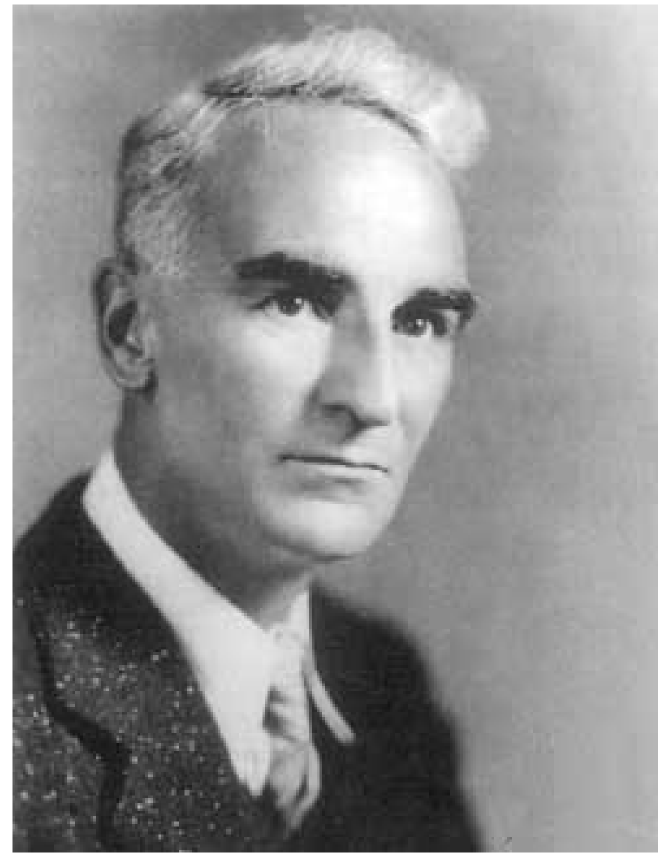
Unifying Continuous and Discrete Analysis

"A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both."

E.T. Bell, 1937



1 Time Scales

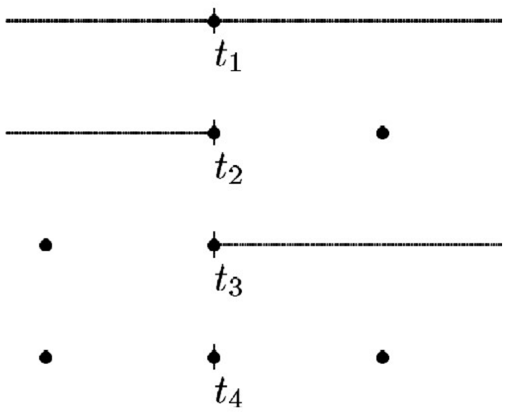
If the time scale is

- The set of all real numbers, the (delta) derivative is the usual derivative
- The set of all integers, the (delta) derivative is the usual forward difference
- The set of all nonnegative integer powers of a number $q>1$, the (delta) derivative is the usual Jackson derivative

2 Jump Operators 3 Graininess

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}$$

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\}$$



$$\mu(t) := \sigma(t) - t$$

4 Delta Derivative

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

$$f^{\sigma} = f + \mu f^{\Delta}$$

5 SUF

6 Product Rule 7 Quotient Rule

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{g f^{\Delta} - f g^{\Delta}}{g g^{\sigma}}$$

$$(t^2)^{\Delta} = (t \cdot t)^{\Delta} = t + \sigma(t)$$

$$\left(\frac{1}{t}\right)^{\Delta} = -\frac{1}{t\sigma(t)}$$

8 Integrals

$$F^{\Delta} = f$$

$$\int_r^s f(t)\Delta t = F(s) - F(r)$$

9 The Exponential Function

$1 + \mu(t)p(t) \neq 0$ $p \oplus q := p + q + \mu pq$ $p \ominus q = \dfrac{p - q}{1 + \mu q}$

$e_p(\cdot, t_0)$ $y^\Delta = p(t)y$ $y(t_0) = 1$

$e_p(t, s)e_p(s, r) = e_p(t, r)$

$e_p(\sigma(t), s) = [1 + \mu(t)p(t)]\, e_p(t, s)$

$\dfrac{e_p}{e_q} = e_{p \ominus q}$

$e_p e_q = e_{p \oplus q}$

$e^{\alpha t}$ $e_p(t, s) = \exp\left\{\int_s^t p(\tau)d\tau\right\}$ $(1 + \alpha)^t$ $e_p(t, s) = \prod_{\tau=s}^{t-1} [1 + p(\tau)]$

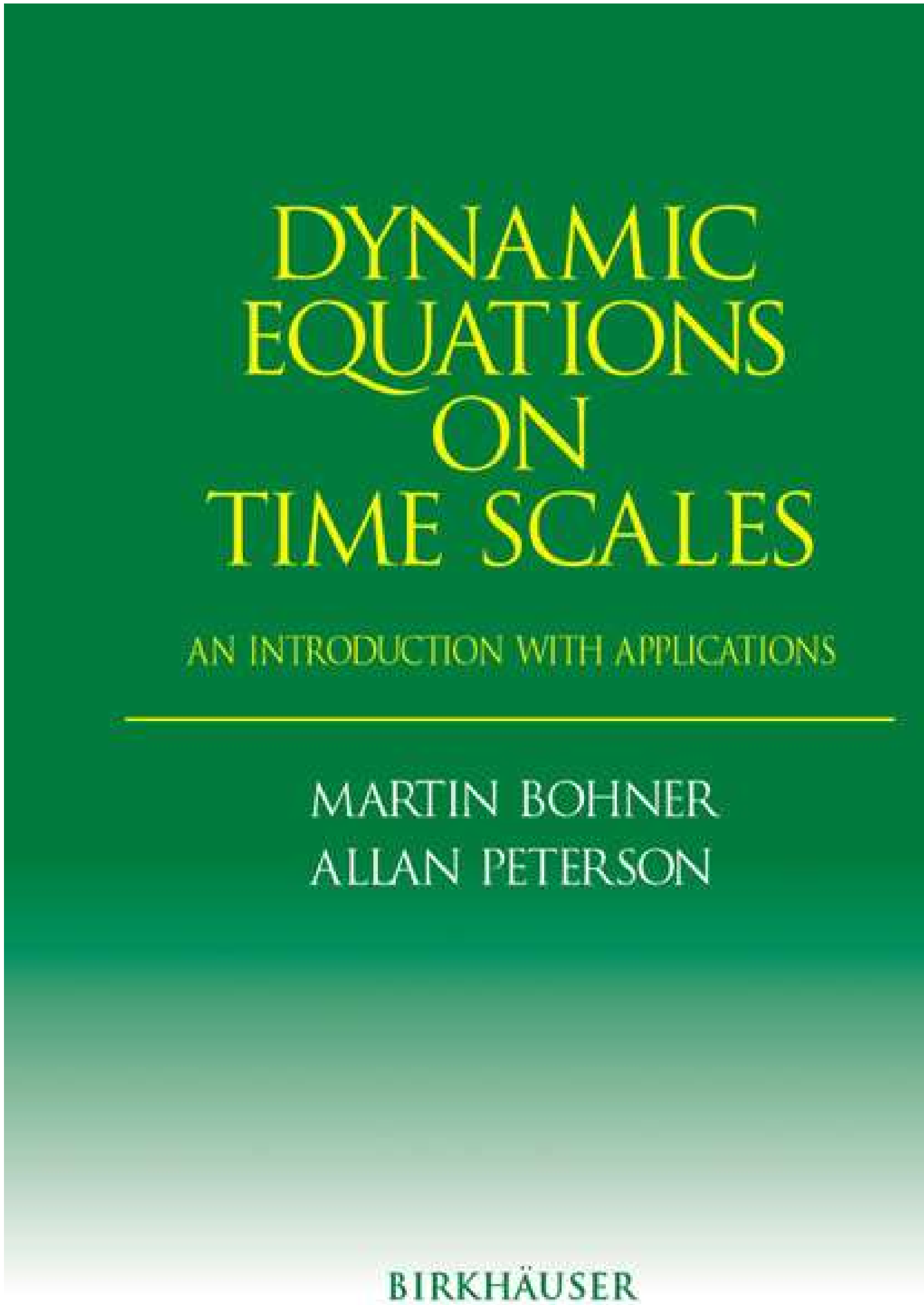
10 Variation of Parameters

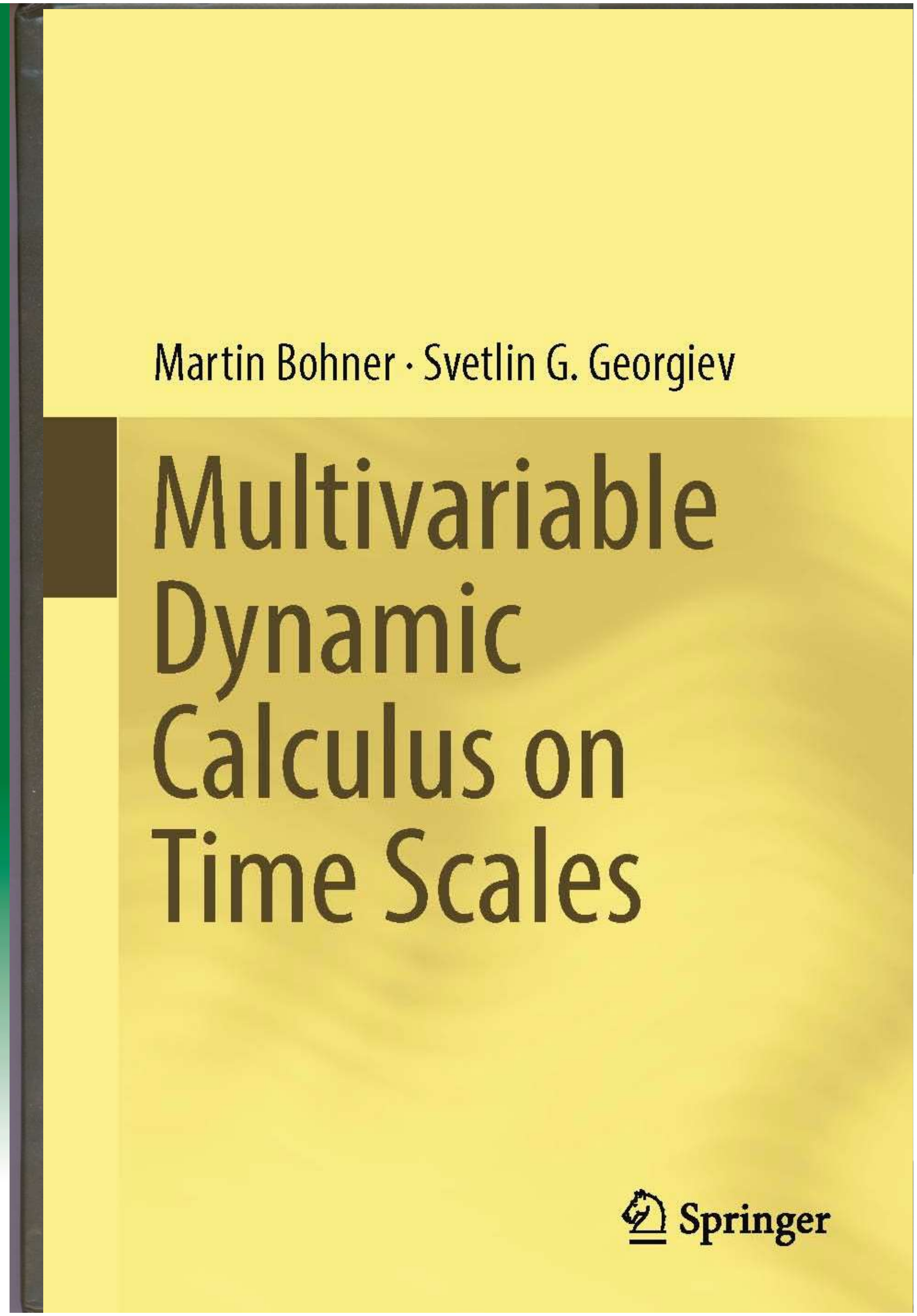
$$x^{\Delta} = -p(t)x^{\sigma} + f(t), \quad x(t_0) = x_0$$

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau$$

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0$$

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau$$


DYNAMIC EQUATIONS ON TIME SCALES
AN INTRODUCTION WITH APPLICATIONS
MARTIN BOHNER
ALLAN PETERSON
BIRKHÄUSER


Martin Bohner · Svetlin G. Georgiev
Multivariable Dynamic Calculus on Time Scales
Springer

This is joint work with

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Gronwall's Inequality

Theorem 1.1 (See [8, Theorem 6.4]). *Let $y, f \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{R})$ and $\wp \in \mathrm{C_{rd}}(\mathcal{I}, [0, \infty))$. Then*

$$y(t) \leq f(t) + \int_a^t y(s)\wp(s)\Delta s \quad \textit{for all} \quad t \in \mathcal{I}$$

implies

$$y(t) \leq f(t) + \int_a^t e_{\wp}(t, \sigma(s)) f(s)\wp(s)\Delta s \quad \textit{for all} \quad t \in \mathcal{I}.$$

Dynamic / Integral Equation

$$\psi^{\Delta}(t) + \wp(t)\psi(t) = f(t), \quad t \in \mathcal{I}^{\kappa}, \tag{1.1}$$

where $\mathcal{I} = [a,b] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $a, b \in \mathbb{T}$, $a < b$, $\wp \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$, $f \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$, and $\mathbb{X}$ is a Banach space.

Lemma 2.1. *ψ solves (1.1) if and only if ψ satisfies the integral equation*

$$\psi(t) = x_0 - \int_a^t (\wp(s)\psi(s) - f(s))\Delta s \quad \textit{for all} \quad t \in \mathcal{I} \tag{2.1}$$

for some constant $x_0 \in \mathbb{X}$.

Uniqueness

Corollary 2.2. *For* $x_0 \in \mathbb{X}$, (1.1) *has at most one solution* ψ *satisfying* $\psi(a) = x_0$.

Proof. Let $x_0 \in \mathbb{X}$. Assume that ψ_1 and ψ_2 are solutions of (1.1) with

$$\psi_1(a) = \psi_2(a) = x_0.$$

Then, by Lemma 2.1, both ψ_1 and ψ_2 satisfy (2.1). This implies

$$\|\psi_1(t) - \psi_2(t)\| \leq \int_a^t |\wp(s)| \, \|\psi_1(s) - \psi_2(s)\| \, \Delta s \quad \text{for all} \quad t \in \mathcal{I}.$$

By Gronwall's inequality, Theorem 1.1,

$$\|\psi_1(t) - \psi_2(t)\| \leq 0 \quad \text{for all} \quad t \in \mathcal{I},$$

so $\psi_1 = \psi_2$. □

Existence

Theorem 2.3. *Assume that there exists* $\alpha \in (0,1)$ *such that*

$$\int_a^t |\wp(s)|\,\Delta s \le \alpha \quad \textit{for all} \quad t \in \mathcal{I}. \tag{2.2}$$

If $x_0 \in \mathbb{X}$*, then* (1.1) *has a unique solution* ψ *satisfying* $\psi(a) = x_0$*.*

Proof. Fix $x_0 \in \mathbb{X}$. Define the operator $T : \mathrm{C}(\mathcal{I}, \mathbb{X}) \to \mathrm{C}(\mathcal{I}, \mathbb{X})$ by

$$T\psi(t) := x_0 - \int_a^t (\wp(s)\psi(s) - f(s))\Delta s, \quad t \in \mathcal{I}.$$

For $\psi_1, \psi_2 \in \mathrm{C}(\mathcal{I}, \mathbb{X})$, we have

$$\|T\psi_1(t) - T\psi_2(t)\| \le \|\psi_1 - \psi_2\|_\infty \int_a^t |\wp(s)|\,\Delta s \le \alpha \|\psi_1 - \psi_2\|_\infty, \quad t \in \mathcal{I},$$

Hence, $\|T\psi_1 - T\psi_2\|_\infty \le \alpha \|\psi_1 - \psi_2\|_\infty$, so T is a contraction. Therefore, T has a unique fixed point ψ, which is the unique solution of (2.1) satisfying $\psi(a) = x_0$. Thus, by Lemma 2.1, ψ is the unique solution of (1.1) satisfying $\psi(a) = x_0$. □

Hyers-Ulam Stability

Definition 3.1 (Hyers–Ulam Stability). We say that (1.1) has Hyers–Ulam stability if there exists a constant $L > 0$, a so-called HUS constant, with the following property. For any $\varepsilon > 0$, if $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\|\psi^\Delta(t) + \wp(t)\psi(t) - f(t)\right\| \le \varepsilon \quad \text{for all} \quad t \in \mathcal{I}^\kappa,$$

then there exists a solution $\phi : \mathcal{I} \to \mathbb{X}$ of (1.1) such that

$$\|\psi(t) - \phi(t)\| \le L\varepsilon \quad \text{for all} \quad t \in \mathcal{I}.$$

HU Stability - Result

(H) For any $x_0 \in \mathbb{X}$, (1.1) has a solution ϕ satisfying $\phi(a) = x_0$.

Theorem 3.2. *If (H) holds, then (1.1) has Hyers–Ulam stability with HUS constant*

$$L := (b-a)e_{|\wp|}(b, a).$$

Proof. Note that $|\wp| \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, [0, \infty))$, and so L is well defined and $L > 0$. Let $\varepsilon > 0$. Suppose $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\|\psi^\Delta(t) + \wp(t)\psi(t) - f(t)\right\| \le \varepsilon \quad \text{for all} \quad t \in \mathcal{I}^\kappa. \tag{3.1}$$

Defining $h(t) := \psi^\Delta(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$. Moreover, ψ satisfies the equation

$$\psi^\Delta(t) + \wp(t)\psi(t) = f(t) + h(t) \quad \text{for all} \quad t \in \mathcal{I}.$$

Let $x_0 = \psi(a)$. By Lemma 2.1,

$$\psi(t) = x_0 - \int_a^t (\wp(s)\psi(s) - (f(s) + h(s)))\Delta s \quad \text{for all} \quad t \in \mathcal{I}. \tag{3.2}$$

By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1,

$$\phi(t) = x_0 - \int_a^t (\wp(s)\phi(s) - f(s))\Delta s \quad \text{for all} \quad t \in \mathcal{I}. \tag{3.3}$$

Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &= \left\|\int_a^t h(s)\Delta s+\int_a^t \wp(s)(\phi(s)-\psi(s))\Delta s\right\| \\
&\le \int_a^t \|h(s)\|\,\Delta s+\int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\overset{(3.1)}{\le} \varepsilon(t-a)+\int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\le \varepsilon(b-a)+\int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s.
\end{aligned}
$$

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \varepsilon(b-a)+\int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\varepsilon(b-a)\Delta s \\
&= \varepsilon(b-a)\left(1+\int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\Delta s\right) \\
&= \varepsilon(b-a)\left(1+e_{|\wp|}(t,a)-e_{|\wp|}(t,t)\right) \\
&= (b-a)e_{|\wp|}(t,a)\varepsilon \le (b-a)e_{|\wp|}(b,a)\varepsilon = L\varepsilon,
\end{aligned}
$$

HU Stability – More Results

Theorem 3.4. *If $\wp$ is regressive, then (1.1) has Hyers–Ulam stability.*

Theorem 3.5. *If there exists $\alpha \in (0,1)$ such that (2.2) holds, then (1.1) has Hyers–Ulam stability.*

Hyers-Ulam-Rassias Stability

Definition 4.1 (Hyers–Ulam–Rassias Stability). Let Ω be a family of positive rd-continuous functions defined on $\mathcal{I}$. We say that (1.1) has Hyers–Ulam–Rassias stability of type Ω if there exists a constant $L > 0$, a so-called HURS_Ω constant, with the following property. For any $\omega \in \Omega$, if $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\|\psi^\Delta(t) + \wp(t)\psi(t) - f(t)\right\| \leq \omega(t) \quad \text{for all} \quad t \in \mathcal{I}^\kappa,$$

then there exists a solution $\phi : \mathcal{I} \to \mathbb{X}$ of (1.1) such that

$$\|\psi(t) - \phi(t)\| \leq L\omega(t) \quad \text{for all} \quad t \in \mathcal{I}.$$

HUR Stability - Result

Theorem 4.2. *Let*

$$\Omega^* := \{\omega \in \mathrm{C_{rd}}(\mathcal{I}, (0,\infty)) : \ \omega \text{ is nondecreasing}\}.$$

If (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* Ω^* *with* $HURS_{\Omega^*}$ *constant*

$$L := (b-a)e_{|\wp|}(b,a).$$

$$
\text{(4.1)} \qquad \left\| \psi^{\Delta}(t) + \wp(t)\psi(t) - f(t) \right\| \le \omega(t) \quad \text{for all} \quad t \in \mathcal{I}^{\kappa}.
$$

$$
\begin{aligned}
\|\psi(t) - \phi(t)\| &\le \int_a^t \|h(s)\| \,\Delta s + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s \\
&\overset{(4.1)}{\le} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s \\
&\le \int_a^t \omega(t)\Delta s + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s \\
&\le (b-a)\omega(t) + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s.
\end{aligned}
$$

$$
\begin{aligned}
\|\psi(t) - \phi(t)\| &\le (b-a)\omega(t) + \int_a^t e_{|\wp|}(t,\sigma(s)) \,|\wp(s)| \,(b-a)\omega(s)\Delta s \\
&\le (b-a)\omega(t) + \int_a^t e_{|\wp|}(t,\sigma(s)) \,|\wp(s)| \,(b-a)\omega(t)\Delta s \\
&= (b-a)\omega(t) \left(1 + \int_a^t e_{|\wp|}(t,\sigma(s)) \,|\wp(s)| \,\Delta s\right) \\
&= (b-a)e_{|\wp|}(t,a)\omega(t) \le (b-a)e_{|\wp|}(b,a)\omega(t) = L\omega(t).
\end{aligned}
$$

HUR Stability – More Results

$$\Omega_p := \left\{ \omega \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, (0,\infty)) : \int_a^t \omega^p(s)\Delta s \le \omega^p(t) \text{ for all } t \in \mathcal{I} \right\}.$$

Theorem 4.3. *If* (*H*) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\Omega^* \cap \Omega_1$ *with* $HURS_{\Omega^* \cap \Omega_1}$ *constant*

$$L := e_{|\wp|}(b, a).$$

Discrete Case

Remark 4.4. Note that for $\mathbb{T} = \mathbb{N}_0$, $\Omega^* \cap \Omega_1 = \Omega_1$, since any $\omega \in \Omega_1$ satisfies

$$\omega(t+1) \geq \sum_{s=a}^{t} \omega(s) \geq \omega(t) \quad \text{for} \quad t \in \mathcal{I}^{\kappa}.$$

Therefore, by Theorem 4.3, if $\mathbb{T} = \mathbb{N}_0$ and (H) holds, then (1.1) has Hyers–Ulam–Rassias stability of type Ω_1 with HURS$_{\Omega_1}$ constant

$$L = \prod_{s=a}^{b-1} \left(1 + \wp(s)\right).$$

HUR Stability – More Results

Theorem 4.5. *If* (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* Ω_1 *with* $HURS_{\Omega_1}$ *constant*

$$L := 1 + e_{|\wp|}(b,a)\,|\wp|_\infty\,, \qquad \textit{where} \qquad |\wp|_\infty := \sup_{t\in\mathcal{I}} |\wp(t)|\,.$$

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \int_a^t \|h(s)\|\,\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\overset{(4.1)}{\le} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\le \omega(t) + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s.
\end{aligned}$$

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \omega(t) + \int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\omega(s)\Delta s \\
&\le \omega(t) + e_{|\wp|}(b,a)\,|\wp|_\infty \int_a^t \omega(s)\Delta s \\
&\le L\omega(t),
\end{aligned}$$

HUR Stability – More Results

Theorem 4.6. *If (H) holds, then (1.1) has Hyers–Ulam–Rassias stability of type* Ω_2 *with* $HURS_{\Omega_2}$ *constant*

$$L := \sqrt{b-a} + (b-a)e_{|\wp|}(b,a)\,|\wp|_\infty\,.$$

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\leq \int_a^t \|h(s)\|\,\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\overset{(4.1)}{\leq} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\leq \sqrt{t-a}\sqrt{\int_a^t \omega^2(s)\Delta s} + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\leq \sqrt{b-a}\sqrt{\omega^2(t)} + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&= \sqrt{b-a}\omega(t) + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s,
\end{aligned}$$

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\leq \sqrt{b-a}\omega(t) + \int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\sqrt{b-a}\omega(s)\Delta s \\
&\leq \sqrt{b-a}\omega(t) + e_{|\wp|}(b,a)\,|\wp|_\infty\,\sqrt{b-a}\int_a^t \omega(s)\Delta s \\
&\leq L\omega(t),
\end{aligned}$$

HUR Stability – More Results

Theorem 4.7. *Let $p > 1$ and $q := p/(p-1)$. If (H) holds, then (1.1) has Hyers–Ulam–Rassias stability of type Ω_p with $HURS_{\Omega_p}$ constant*

$$L := \sqrt[q]{b-a} + (b-a)^{2/q} e_{|\wp|}(b,a) \, |\wp|_\infty \, .$$

Theorem 4.8. *If $\wp$ is regressive, then (1.1) has Hyers–Ulam–Rassias stability of types Ω^* and Ω_p for all $p \geq 1$.*

Theorem 4.9. *If there exists $\alpha \in (0,1)$ such that (2.2) holds, then (1.1) has Hyers–Ulam–Rassias stability of types Ω^* and Ω_p for all $p \geq 1$.*

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Dynamic / Integral Equation

$$(1.1) \quad \psi^{\Delta}(t) = \wp(t)\psi(t) + \mathcal{F}(t, \psi(t), h(\psi(t))) + f(t), \quad t \in \mathcal{I}^{\kappa}, \quad \psi(a) = a_0 \in \mathbb{X},$$

Theorem 2.1. *If $\wp \in \mathcal{R}$, then ψ solves (1.1) if and only if*

$$(2.1) \quad \psi(t) = e_{\wp}(t, a)a_0 + \int_a^t e_{\wp}(t, \sigma(s))[\mathcal{F}(s, \psi(s), h(\psi(s))) + f(s)]\Delta s, \quad t \in \mathcal{I}.$$

Existence/Uniqueness

(H$_1$) $\wp \in \mathcal{R}$ and $f \in \mathrm{C_{rd}}$.
(H$_2$) $\mathcal{F}$ and h satisfy Lipschitz conditions with constants β and γ, respectively.
(H$_3$) For any $a_0 \in \mathbb{X}$, (1.1) has a solution ϕ satisfying $\phi(a) = a_0$.
(H$_4$) $\theta := \sup_{t \in \mathcal{I}} \int_a^t |e_\wp(t, \sigma(s))| \Delta s < \dfrac{1}{\beta(1+\gamma)}$.
(H$_5$) $(b-a)e_{|\wp|}(b,a) < \dfrac{1}{\beta(1+\gamma)}$.

Theorem 2.2. *Assume (H$_1$), (H$_2$), and (H$_4$). If $a_0 \in \mathbb{X}$, then (1.1) has a unique solution ψ satisfying $\psi(a) = a_0$.*

Corollary 2.3. *Assume (H$_1$), (H$_2$), and (H$_5$). If $a_0 \in \mathbb{X}$, then (1.1) has a unique solution.*

HU Stability - Result

Theorem 3.2. *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam stability with HUS constant*

$$L := (b-a)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a). \tag{3.5}$$

$$H_\psi(t) := \mathcal{F}(t, \psi(t), h(\psi(t))) + f(t)$$

$$g_\psi(t) := \psi^\Delta(t) - \wp(t)\psi(t) - H_\psi(t)$$

Proof. Let $\varepsilon > 0$. Suppose $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that (3.3) holds. Then

$$\begin{aligned} \psi^{\Delta}(t) &= \wp(t)\psi(t) + H_{\psi}(t) + \wp^{\Delta}(t) - \wp(t)\psi(t) - H_{\psi}(t) \\ &= \wp(t)\psi(t) + H_{\psi}(t) + g_{\psi}(t). \end{aligned}$$

Set $a_0 = \psi(a)$. By Theorem 2.1,

$$\psi(t) = e_{\wp}(t,a)a_0 + \int_a^t e_{\wp}(t,\sigma(s))\left[H_{\psi}(s) + g_{\psi}(s)\right]\Delta s. \tag{3.6}$$

By (H_3), there exists a unique solution ϕ of (1.1) with $\phi(a) = a_0$, that is, by Theorem 2.1,

$$\phi(t) = e_{\wp}(t,a)a_0 + \int_a^t e_{\wp}(t,\sigma(s))H_{\phi}(s)\Delta s, \quad t \in \mathcal{I}. \tag{3.7}$$

Subtracting (3.7) from (3.6), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned} \|\psi(t) - \phi(t)\| \leq & \left\| \int_a^t e_{\wp}(t,\sigma(s))g_{\psi}(s)\Delta s \right\| \\ & + \left\| \int_a^t e_{\wp}(t,\sigma(s))\left[\mathcal{F}(s,\psi(s),h(\psi(s))) - \mathcal{F}(s,\phi(s),h(\phi(s)))\right]\Delta s \right\|. \end{aligned}$$

Since $\|g_{\psi}(t)\| \leq \varepsilon$ holds for $t \in \mathcal{I}$ and taking into account (H_2), we get

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\leq \varepsilon \int_a^t |e_\wp(t,\sigma(s))|\,\Delta s \\
&\quad + \int_a^t |e_\wp(t,\sigma(s))|\,\beta\left[\|\psi(s)-\phi(s)\| + \|h(\psi(s))-h(\phi(s))\|\right]\Delta s \\
&\leq \varepsilon \int_a^t |e_\wp(t,\sigma(s))|\,\Delta s \\
&\quad + \int_a^t |e_\wp(t,\sigma(s))|\,\beta\left[\|\psi(s)-\phi(s)\| + \gamma\,\|\psi(s)-\phi(s)\|\right]\Delta s \\
&\leq \varepsilon \int_a^t |e_\wp(t,\sigma(s))|\,\Delta s \\
&\quad + \int_a^t |e_\wp(t,\sigma(s))|\,\beta(1+\gamma)\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\leq \varepsilon(b-a)e_{|\wp|}(b,a) + \beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s.
\end{aligned}
$$

Thus, by Gronwall's inequality, Theorem 1.3, we deduce that

$$
\|\psi(t)-\phi(t)\| \leq \varepsilon(b-a)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a) = L\varepsilon.
$$

HU Stability – More Results

Corollary 3.4. *Assume (H_1) and (H_2). In addition, assume (H_4) or (H_5). Then (1.1) has Hyers–Ulam stability with constant L.*

Example 3.5. We now give an example such that (H_1), (H_2), and (H_5) are statisfied, so that, for example, Corollary 3.4 applies. Consider

$$\mathbb{T} = \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1]$$

and let

$$m \in \mathbb{N}, \quad a = 0, \quad b = 2m+1, \quad \beta \in \left(0, \frac{1}{(2m+1)(2e)^{m+1}}\right).$$

Moreover, we let $f \in \mathrm{C}_{\mathrm{rd}}$, $\wp(t) \equiv 1$, and

$$\mathcal{F}(t, x, y) = \beta(\sin x + y), \quad h(x) = \cos x.$$

Equation (1.1) then takes the form

$$\psi^{\Delta}(t) = \psi(t) + \beta\left(\sin(\psi(t)) + \cos(\psi(t))\right) + f(t).$$

We note that (H_1) is satisfied because $\wp \in \mathcal{R}$ and $f \in \mathrm{C}_{\mathrm{rd}}$. We also note that (H_2) is satisfied because $\mathcal{F}$ is Lipschitz continuous with Lipschitz constant β and h is Lipschitz continuous with Lipschitz constant $\gamma = 1$. Finally, according to [5, Example 2.58],

$$e_1(b, a) = e_1(2m+1, 0) = 2^m e^{m+1}.$$

Hence,

$$(b-a)e_{|\wp|}(b, a) = (2m+1)2^m e^{m+1} < \frac{1}{2\beta} = \frac{1}{\beta(1+\gamma)},$$

and thus (H_5) is satisfied as well.

HUR Stability - Result

$$\mathcal{M}^* := \{\omega \in \mathrm{C_{rd}}(\mathcal{I}, (0,\infty)) : \ \omega \text{ is nondecreasing}\}$$

Theorem 4.2. *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}^*$ *with* $HURS_{\mathcal{M}^*}$ *constant*

$$L := (b-a)e_{|\wp|}(b,a)\left(1 + (b-a)\beta(1+\gamma)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.3}$$

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \int_a^t |e_\wp(t,\sigma(s))|\,\|g_\psi(s)\|\,\Delta s \\
&\quad + \int_a^t |e_\wp(t,\sigma(s))|\,\|\mathcal{F}(s,\psi(s),h(\psi(s)))-\mathcal{F}(s,\phi(s),h(\phi(s)))\|\,\Delta s \\
&\le e_{|\wp|}(b,a)\int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s \\
&\le (b-a)e_{|\wp|}(b,a)\omega(t)+\beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s.
\end{aligned}
$$

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\le (b-a)e_{|\wp|}(b,a)\omega(t) \\
&\quad + \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))(b-a)e_{|\wp|}(b,a)\omega(s)\beta(1+\gamma)e_{|\wp|}(b,a)\Delta s \\
&= (b-a)e_{|\wp|}(b,a)\omega(t) \\
&\quad +(b-a)\left(e_{|\wp|}(b,a)\right)^2\beta(1+\gamma)\int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))\omega(s)\Delta s \\
&\le (b-a)e_{|\wp|}(b,a)\omega(t) \\
&\quad +(b-a)^2\left(e_{|\wp|}(b,a)\right)^2\beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\omega(t)
\end{aligned}
$$

HUR Stability – More Results

$$\mathcal{M}_p := \left\{ \omega \in \mathrm{C_{rd}}(\mathcal{I}, (0,\infty)) : \int_a^t \omega^p(s)\Delta s \le \omega^p(t) \text{ for all } t \in \mathcal{I} \right\}$$

Theorem 4.3. *If (H_1), (H_2), and (H_3) hold, then (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{M}^* \cap \mathcal{M}_1$ with $HURS_{\mathcal{M}^* \cap \mathcal{M}_1}$ constant*

$$L := e_{|\wp|}(b,a)\left(1 + (b-a)\beta(1+\gamma)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.4}$$

HUR Stability – More Results

Theorem 4.4. *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}_1$ *with* $HURS_{\mathcal{M}_1}$ *constant*

$$L := e_{|\wp|}(b,a)\left(1+\beta(1+\gamma)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.5}$$

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \int_a^t |e_{\wp}(t,\sigma(s))|\,\|g_{\psi}(s)\|\,\Delta s \\
&\quad + \int_a^t |e_{\wp}(t,\sigma(s))|\,\|\mathcal{F}(s,\psi(s),h(\psi(s)))-\mathcal{F}(s,\phi(s),h(\phi(s)))\|\,\Delta s \\
&\le e_{|\wp|}(b,a)\int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t)+\beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s.
\end{aligned}$$

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le e_{|\wp|}(b,a)\omega(t) \\
&\quad + \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))e_{|\wp|}(b,a)\omega(s)\beta(1+\gamma)e_{|\wp|}(b,a)\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t) \\
&\quad + \left(e_{|\wp|}(b,a)\right)^2\beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\int_a^t \omega(s)\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t)+\left(e_{|\wp|}(b,a)\right)^2\beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\omega(t) \\
&= L\omega(t).
\end{aligned}$$

HUR Stability – More Results

Theorem 4.5. *Let $p > 1$ and $q := p/(p-1)$. If (H_1), (H_2), and (H_3) hold, then (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{M}_p$ with $HURS_{\mathcal{M}_p}$ constant*

$$L := e_{|\wp|}(b,a)\sqrt[q]{b-a}\left(1+\beta(1+\gamma)e_{|\wp|}(b,a)\sqrt[q]{b-a}\,e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.6}$$

$$\mathcal{M}_p^r := \left\{\omega \in \mathrm{C_{rd}}(\mathcal{I},(0,\infty)) : \int_a^t \omega^p(s)\Delta s \le r\omega^p(t) \text{ for all } t \in \mathcal{I}\right\}$$

Theorem 4.6. *If (H_1), (H_2), and (H_3) hold, then (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{M}_p^r$ with $HURS_{\mathcal{M}_p^r}$ constant*

$$L := e_{|\wp|}(b,a)\sqrt[q]{b-a}\,\sqrt[p]{r}\left(1+\beta(1+\gamma)\sqrt[q]{b-a}\,\sqrt[p]{r}\,e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.7}$$

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