# The method of solution-regions applied to existence and multiplicity problems for systems of first order differential equations with nonlinear boundary conditions

Marlène Frigon

Université de Montréal

International Meetings on Differential Equations and Their Applications
Institute of Mathematics, Lodz
May 4, 202.

# Systems of first order differential equations

We consider the following system of differential equations:

$$\text{(P)} \qquad \begin{cases} u'(t) = f(t, u(t)) & \text{a.e. } t \in [0, T], \\ u \in \mathcal{B}; \end{cases}$$

where $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function and $\mathcal{B}$ denotes the initial value or the periodic boundary value conditions:

$$\text{(IVC)} \qquad u(0) = r;$$

$$\text{(PC)} \qquad u(0) = u(T).$$

# Case $N = 1$

The method of upper and lower solutions was widely applied to (P) in the case where $N = 1$.

## Case $N = 1$

The method of upper and lower solutions was widely applied to (P) in the case where $N = 1$.

### Definition

We say that $\beta \in W^{1,1}(I, \mathbb{R})$ is an upper solution of (P) if

(i) $f(t, \beta(t)) \leq \beta'(t)$ for a.e. $t \in I$;

(ii) - if $\mathcal{B}$ denotes (IVC), $\beta(0) \geq r$;
- if $\mathcal{B}$ denotes (PC), $\beta(0) \geq \beta(T)$.

$\alpha \in W^{1,1}(I, \mathbb{R})$ is a lower solution of (P) if it satisfies (i) and (ii) with the reversed inequalities.

# Case $N = 1$

The method of upper and lower solutions was widely applied to (P) in the case where $N = 1$.

**Definition**

We say that $\beta \in W^{1,1}(I, \mathbb{R})$ is an upper solution of (P) if

(i) $f(t, \beta(t)) \leq \beta'(t)$ for a.e. $t \in I$;

(ii) - if $\mathcal{B}$ denotes (IVC), $\beta(0) \geq r$;
- if $\mathcal{B}$ denotes (PC), $\beta(0) \geq \beta(T)$.

$\alpha \in W^{1,1}(I, \mathbb{R})$ is a lower solution of (P) if it satisfies (i) and (ii) with the reversed inequalities.

Assuming the existence of $\alpha \leq \beta$ respectively lower and upper solutions of (P), a solution $u$ is obtained such that $\alpha \leq u \leq \beta$.

# Case $N > 1$: A first generalization of the method of upper and lower solutions to systems of differential equations

## Case $N > 1$: A first generalization of the method of upper and lower solutions to systems of differential equations

For the periodic problem, Knobloch used the following component-wise lower and upper solutions:
$\forall i = 1, \dots, N, \exists \alpha_i \leq \beta_i$ with $\alpha_i(0) = \alpha_i(T)$, $\beta_i(0) = \beta_i(T)$,

$$\alpha_i'(t) \leq f_i(t, x_1, \dots, x_{i-1}, \alpha_i(t), x_{i+1}, \dots, x_N),$$
$$\beta_i'(t) \geq f_i(t, x_1, \dots, x_{i-1}, \beta_i(t), x_{i+1}, \dots, x_N),$$
$$\forall x_j \in [\alpha_j(t), \beta_j(t)], \quad \forall j \neq i.$$

# Case $N > 1$: A first generalization of the method of upper and lower solutions to systems of differential equations

For the periodic problem, Knobloch used the following component-wise lower and upper solutions:
$\forall i = 1, \ldots, N, \exists \alpha_i \leq \beta_i$ with $\alpha_i(0) = \alpha_i(T)$, $\beta_i(0) = \beta_i(T)$,

$$\alpha_i'(t) \leq f_i(t, x_1, \ldots, x_{i-1}, \alpha_i(t), x_{i+1}, \ldots, x_N),$$
$$\beta_i'(t) \geq f_i(t, x_1, \ldots, x_{i-1}, \beta_i(t), x_{i+1}, \ldots, x_N),$$
$$\forall x_j \in [\alpha_j(t), \beta_j(t)], \quad \forall j \neq i.$$

- This assumption was used to deduce $\exists u = (u_1, \ldots, u_N)$ a solution of (P) such that
$\alpha_i(t) \leq u_i(t) \leq \beta_i(t) \quad \forall t \in I, \quad \forall i = 1, \ldots, N.$

# Case $N > 1$: Solution-tubes of systems of DEs

The notion of ordered lower and upper solutions was generalized by the notion of solution-tubes of systems of first order differential equations.

# Case $N > 1$: Solution-tubes of systems of DEs

The notion of ordered lower and upper solutions was generalized by the notion of solution-tubes of systems of first order differential equations.

## Definition

We say that $(v, M) \in W^{1,1}(I, \mathbb{R}^N) \times W^{1,1}(I, [0, \infty[)$ is a solution-tube of (P) if

(i) $\langle x - v(t), f(t, x) - v'(t) \rangle \le M(t)M'(t)$ for a.e. $t \in I$ and $\forall x$ such that $\|x - v(t)\| = M(t)$;

(ii) $v'(t) = f(t, v(t))$ a.e. $t \in \{t \in I : M(t) = 0\}$;

(iii) - if $\mathcal{B}$ denotes (IVC), $\|v(0) - r\| \le M(0)$;
- if $\mathcal{B}$ denotes (PC), $\|v(0) - v(T)\| \le M(0) - M(T)$.

# Case $N > 1$: Solution-tubes of systems of DEs

The notion of ordered lower and upper solutions was generalized by the notion of solution-tubes of systems of first order differential equations.

## Definition

We say that $(v, M) \in W^{1,1}(I, \mathbb{R}^N) \times W^{1,1}(I, [0, \infty[)$ is a solution-tube of (P) if

(i) $\langle x - v(t), f(t, x) - v'(t) \rangle \leq M(t)M'(t)$ for a.e. $t \in I$ and $\forall x$ such that $\|x - v(t)\| = M(t)$;

(ii) $v'(t) = f(t, v(t))$ a.e. $t \in \{t \in I : M(t) = 0\}$;

(iii) - if $\mathcal{B}$ denotes (IVC), $\|v(0) - r\| \leq M(0)$;
- if $\mathcal{B}$ denotes (PC), $\|v(0) - v(T)\| \leq M(0) - M(T)$.

- $\exists (v, M)$ a solution-tube of (P) $\implies$ (P) has a solution $u$ s.t. $u \in T(v, M) = \{u \in W^{1,1}(I, \mathbb{R}^N) : \|u(t) - v(t)\| \leq M(t) \quad \forall t \in I\}$.

## Case $N > 1$: Solution-tubes of systems of DEs

The notion of ordered lower and upper solutions was generalized by the notion of solution-tubes of systems of first order differential equations.

**Definition**

We say that $(v, M) \in W^{1,1}(I, \mathbb{R}^N) \times W^{1,1}(I, [0, \infty[)$ is a solution-tube of (P) if

(i) $\langle x - v(t), f(t, x) - v'(t)\rangle \leq M(t)M'(t)$ for a.e. $t \in I$ and $\forall x$ such that $\|x - v(t)\| = M(t)$;

(ii) $v'(t) = f(t, v(t))$ a.e. $t \in \{t \in I : M(t) = 0\}$;

(iii) - if $\mathcal{B}$ denotes (IVC), $\|v(0) - r\| \leq M(0)$;
- if $\mathcal{B}$ denotes (PC), $\|v(0) - v(T)\| \leq M(0) - M(T)$.

- $\exists(v, M)$ a solution-tube of (P) $\implies$ (P) has a solution $u$ s.t. $u \in T(v, M) = \{u \in W^{1,1}(I, \mathbb{R}^N) : \|u(t) - v(t)\| \leq M(t) \quad \forall t \in I\}$.
- If $N = 1$, $(v, M)$ is a solution-tube of (P) $\iff$ $v + M$ and $v - M$ are respectively upper and lower solutions of (P).

## $N \geq 1$: An other approach: viability theory

Let $K \subset \mathbb{R}^N$ be closed. A solution of (P) remaining in $K$ is called a viable solution (i.e. $u(t) \in K$ for all $t$).

## $N \geq 1$: An other approach: viability theory

Let $K \subset \mathbb{R}^N$ be closed. A solution of (P) remaining in $K$ is called a viable solution (i.e. $u(t) \in K$ for all $t$).

- Nagumo, 1942: first viability result (to my knowledge).
- Many generalizations: Aubin and the references therein, ...

## $N \geq 1$: An other approach: viability theory

Let $K \subset \mathbb{R}^N$ be closed. A solution of (P) remaining in $K$ is called a viable solution (i.e. $u(t) \in K$ for all $t$).

- Nagumo, 1942: first viability result (to my knowledge).
- Many generalizations: Aubin and the references therein, ...
- Plaskacz, 1990 and Bielawski, Górniewicz and Plaskacz, 1992: viability results for differential inclusions with
  $K$ a proximate retract
  (i.e. $K$ is compact and $\exists U$ an open neighborhood of $K$ and
  $\exists \rho : U \to K$ continuous such that
  $\|\rho(x) - x\| = \text{dist}(x, K)\ \forall x \in U$).

## $N \geq 1$: An other approach: viability theory

Let $K \subset \mathbb{R}^N$ be closed. A solution of (P) remaining in $K$ is called a viable solution (i.e. $u(t) \in K$ for all $t$).

- Nagumo, 1942: first viability result (to my knowledge).
- Many generalizations: Aubin and the references therein, ...
- Plaskacz, 1990 and Bielawski, Górniewicz and Plaskacz, 1992: viability results for differential inclusions with
  $K$ a proximate retract
  (i.e. $K$ is compact and $\exists U$ an open neighborhood of $K$ and
  $\exists \rho : U \to K$ continuous such that
  $\|\rho(x) - x\| = \mathrm{dist}(x, K)\ \forall x \in U$).

- A tangential condition was imposed:
  $f(t, x) \in T_K(x)\ \ \forall (t, x) \in I \times K$, where $T_K(x)$ is the Bouligand tangent cone of $K$ at $x$, i.e.
  $$T_K(x) = \left\{ y \in \mathbb{R}^N : \liminf_{t \to 0^+} \frac{1}{t} \mathrm{dist}(x + ty), K) = 0 \right\}.$$

**Remarks:**

- There are few generalizations of viability results to sets depending on $t$, where it is shown that $u(t) \in K(t)\ \forall t \in I$.

**Remarks:**

- There are few generalizations of viability results to sets depending on $t$, where it is shown that $u(t) \in K(t)\ \forall t \in I$.

- Let $\alpha \leq \beta$ be respectively lower and upper solutions of (P) and $K(t) = \{x \in \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$. Looking for a solution $u$ such that $\alpha \leq u \leq \beta$ can be seen as looking for a viable solution.

Generalization of the method of upper and lower solutions and the method of solution-tubes

# Case $N \geq 1$: Admissible regions

**Definition**

A set $R \subset I \times \mathbb{R}^N$ is an admissible region

# Case $N \geq 1$: Admissible regions

## Definition

A set $R \subset I \times \mathbb{R}^N$ is an admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t, x) : h(t, x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t, x) \in R\} \neq \emptyset$;

# Case $N \geq 1$: Admissible regions

## Definition

A set $R \subset I \times \mathbb{R}^N$ is an admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t,x) : h(t,x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t,x) \in R\} \neq \emptyset$;

(ii) $h$ has partial derivatives at $(t,x)$ for a.e. $t$ and $\forall x$ with $(t,x) \in R^c = \left(I \times \mathbb{R}^N\right) \setminus R$, and $\frac{\partial h}{\partial t}$, $\nabla_x h$ are locally Carathéodory maps on $R^c$;

(iii) $p$ is bounded and such that $p(t,x) = (t,x)$ $\forall (t,x) \in R$ and

$$\left\langle \nabla_x h(t,x), p_2(t,x) - x \right\rangle < 0 \quad \text{a.e. } t \text{ and } \forall x \text{ with } (t,x) \in R^c.$$

# Case $N \geq 1$: Admissible regions

## Definition

A set $R \subset I \times \mathbb{R}^N$ is an admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t,x) : h(t,x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t,x) \in R\} \neq \emptyset$;

(ii) $h$ has partial derivatives at $(t,x)$ for a.e. $t$ and $\forall x$ with $(t,x) \in R^c = \left(I \times \mathbb{R}^N\right) \backslash R$, and $\frac{\partial h}{\partial t}$, $\nabla_x h$ are locally Carathéodory maps on $R^c$;

(iii) $p$ is bounded and such that $p(t,x) = (t,x)$ $\forall (t,x) \in R$ and

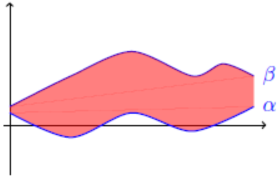
$$\left\langle \nabla_x h(t,x), p_2(t,x) - x \right\rangle < 0 \quad \text{a.e. } t \text{ and } \forall x \text{ with } (t,x) \in R^c.$$

We call $(h,p)$ an admissible pair associated to $R$.

# Case $N = 1$: Examples of admissible regions

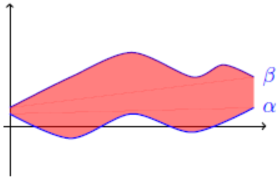
# Case $N = 1$: Examples of admissible regions

- $R = \{(t,x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$ is an admissible region, where $\alpha \leq \beta$ in $W^{1,1}(I,\mathbb{R})$



# Case $N = 1$: Examples of admissible regions

- $R = \{(t,x) \in I \times \mathbb{R} : \alpha(t) \le x \le \beta(t)\}$ is an admissible region, where $\alpha \le \beta$ in $W^{1,1}(I,\mathbb{R})$




$$h(t,x) = \left| x - \frac{\alpha(t)+\beta(t)}{2} \right| - \frac{\beta(t)-\alpha(t)}{2},$$
$$p(t,x) = (t, p_2(t,x))$$

with $p_2(t,x)$ the projection of $x$ on $[\alpha(t),\beta(t)]$,

$$\langle \frac{\partial h}{\partial x}(t,x), p_2(t,x) - x \rangle = -\mathrm{dist}(x, [\alpha(t),\beta(t)]) < 0 \quad \forall (t,x) \in R^c.$$

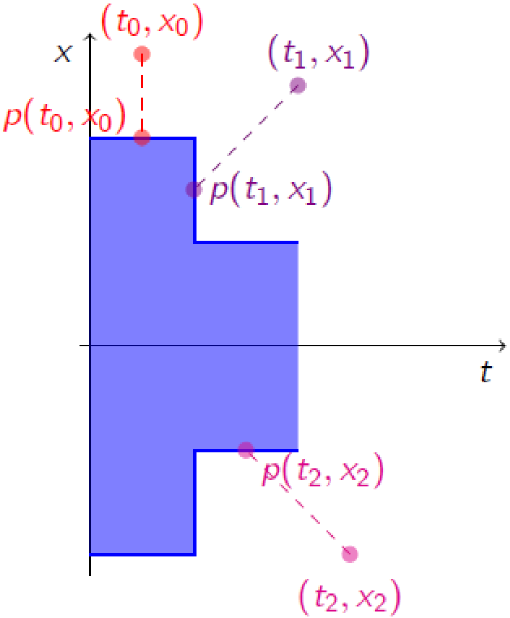
- $R = [0,1] \times [-2,2] \cup [1,2] \times [-1,1]$ is an admissible region.

- $R = [0,1]\times[-2,2]\cup[1,2]\times[-1,1]$ is an admissible region.

Indeed, $(h,p)$ is an admissible pair, where

$$h(t,x) = \begin{cases} 0 & \text{if } (t,x)\in R,\\ \frac{3}{2}(|x|-2)^2 & \text{if } 0\le t\le 1,\ 2<|x|<\infty,\\ (|x|+t-2)(|x|-1) & \text{if } 1\le t\le 2,\ 1<|x|\le t,\\ (t-1)(3|x|-t-2) & \text{if } 1\le t\le 2,\ t<|x|\le 1+t,\\ \frac{1}{2}\big(6(t-1)+(t-1)^2+3(|x|-2)^2\big) & \text{if } 1\le t\le 2,\ 1+t<|x|; \end{cases}$$

$$p(t,x) = \begin{cases} (t,x) & \text{if } (t,x)\in R,\\ \big(t,\frac{2x}{|x|}\big) & \text{if } 0\le t\le 1,\ 2<|x|,\\ \big(1+t-|x|,\frac{x}{|x|}\big) & \text{if } 1\le t\le 2,\ 1<|x|\le t,\\ \big(1,x+\frac{x(1-t)}{|x|}\big) & \text{if } 1\le t\le 2,\ t<|x|\le 1+t,\\ \big(1,\frac{2x}{|x|}\big) & \text{if } 1\le t\le 2,\ 1+t<|x|. \end{cases}$$

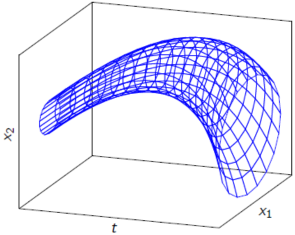


(t0, x0)
x
(t1, x1)
p(t0, x0)
p(t1, x1)
t
p(t2, x2)
(t2, x2)

## Case $N > 1$: Examples of admissible regions

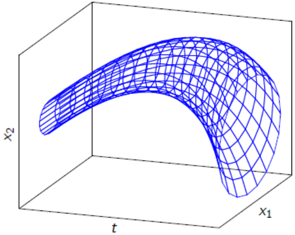
# Case $N > 1$: Examples of admissible regions

- $R = \left\{(t,x) \in I \times \mathbb{R}^N : \sum_{i=1}^N a_i(t)\big(x_i - v_i(t)\big)^2 \leq M^2(t)\right\}$ is an admissible region.

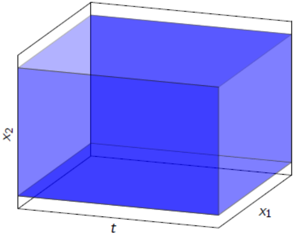


# Case $N > 1$: Examples of admissible regions

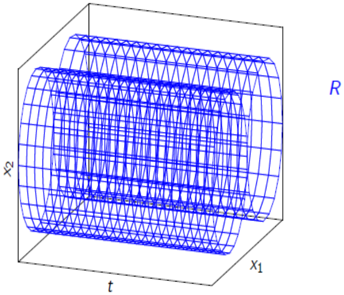
- $R = \left\{(t,x) \in I \times \mathbb{R}^N : \sum_{i=1}^{N} a_i(t)\big(x_i - v_i(t)\big)^2 \leq M^2(t)\right\}$ is an admissible region.

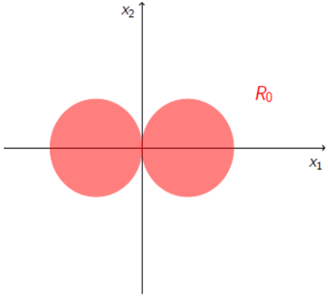



- $R = I \times [-a_1, a_1] \times \cdots \times [-a_N, a_N]$ is an admissible region.



- $R = I \times R_0 \subset I \times \mathbb{R}^2$
  is admissible, where $R_0 = \{(x_1, x_2) \in \mathbb{R}^2 : (1-|x_1|)^2 + x_2^2 \leq 1\}$.




Notice that $R$ and $R_0$ are not proximate retracts.

# Solution-regions

To insure the existence of a solution of (P) in an admissible region $R$, it is necessary to impose additional conditions involving $R$, the right-member $f$ and the boundary condition $\mathcal{B}$.

# Solution-regions

To insure the existence of a solution of (P) in an admissible region $R$, it is necessary to impose additional conditions involving $R$, the right-member $f$ and the boundary condition $\mathcal{B}$.

**Definition**

A set $R \subset I \times \mathbb{R}^N$ is a solution-region of (P) if it is an admissible region with an associated admissible pair $(h, p)$ satisfying the following conditions:

# Solution-regions

To insure the existence of a solution of (P) in an admissible region $R$, it is necessary to impose additional conditions involving $R$, the right-member $f$ and the boundary condition $\mathcal{B}$.

**Definition**

A set $R \subset I \times \mathbb{R}^N$ is a solution-region of (P) if it is an admissible region with an associated admissible pair $(h, p)$ satisfying the following conditions:

(i) $\dfrac{\partial h}{\partial t}(t,x) + \langle \nabla_x h(t,x), f(p(t,x)) \rangle \leq 0$
for a.e. $t$ and $\forall x$ with $(t,x) \notin R$;

(ii) - if $\mathcal{B}$ denotes (IVC), $h(0,r) \leq 0$;
- if $\mathcal{B}$ denotes (PC), $h(0,x) \leq h(T,x) \quad \forall x$ s.t. $(0,x) \notin R$.

# Examples

- If $\alpha \leq \beta$ are respectively lower and upper solutions of (P), then $R = \{(t, x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$ is a solution-region.

# Examples

- If $\alpha \leq \beta$ are respectively lower and upper solutions of (P), then $R = \{(t, x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$ is a solution-region.
- If $(v, M)$ is a solution-tube of (P), then $R = \{(t, x) \in I \times \mathbb{R}^N : \|x - v(t)\| \leq M(t)\}$ is a solution-region.

# Examples

- If $\alpha \leq \beta$ are respectively lower and upper solutions of (P), then $R = \{(t,x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$ is a solution-region.
- If $(v, M)$ is a solution-tube of (P), then $R = \{(t,x) \in I \times \mathbb{R}^N : \|x - v(t)\| \leq M(t)\}$ is a solution-region.
- If $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\beta = (\beta_1, \ldots, \beta_N)$ are respectively component-wise lower and upper solutions of (P) such that $\alpha_i(t) \leq \beta_i(t)\ \forall t,\ \forall i = 1, \ldots, n$, then $R = \{(t,x) \in I \times \mathbb{R}^N : \alpha_i(t) \leq x_i \leq \beta_i(t)\ \ \forall i = 1, \ldots, N\}$ is a solution-region.

# An existence result

**Theorem**

*Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. Assume that there exists $R$ a solution-region of* (P). *Then,* (P) *has a solution $u \in W^{1,1}(I, \mathbb{R}^N)$ such that $(t, u(t)) \in R$ for every $t \in I$.*

## Example 1

We consider the following problem:

$$(1) \qquad \begin{aligned} u'(t) &= f(t, u(t)) \quad \text{for a.e. } t \in [0, 3], \\ u(0) &= u(3); \end{aligned}$$

where

$$f(t, x) = e^{6x}\Big(1 - t\Big)\Big(x - \frac{1}{t^{2/3}}\Big)\Big(4x - 5\sin^2\big(\frac{t\pi}{3}\big)\Big) - 3\Big(x^5 - |x|^5\Big).$$

# Example 1

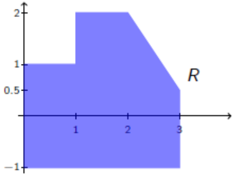
We consider the following problem:

$$(1) \qquad \begin{aligned} u'(t) &= f(t, u(t)) \quad \text{for a.e. } t \in [0,3], \\ u(0) &= u(3); \end{aligned}$$

where

$$f(t,x) = e^{6x}\Big(1-t\Big)\Big(x - \frac{1}{t^{2/3}}\Big)\Big(4x - 5\sin^2\big(\frac{t\pi}{3}\big)\Big) - 3\Big(x^5 - |x|^5\Big).$$

Let $R$ be the following set




We can show that $R$ is a solution-region of (1).

# Example 1

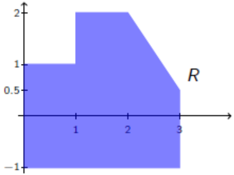
We consider the following problem:

$$(1) \qquad \begin{aligned} u'(t) &= f(t,u(t)) \quad \text{for a.e. } t \in [0,3], \\ u(0) &= u(3); \end{aligned}$$

where

$$f(t,x) = e^{6x}\Big(1-t\Big)\Big(x - \frac{1}{t^{2/3}}\Big)\Big(4x - 5\sin^2\big(\frac{t\pi}{3}\big)\Big) - 3\Big(x^5 - |x|^5\Big).$$

Let $R$ be the following set




We can show that $R$ is a solution-region of (1).
The existence theorem implies that

$$\exists u \text{ a solution of (1) s.t. } \quad (t,u(t)) \in R \quad \forall t \in [0,3].$$

## Example 2

We consider the following system:

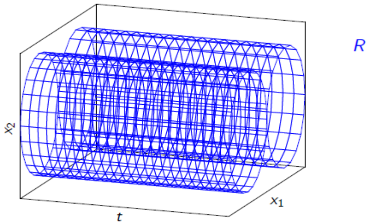
$$
(2)\quad
\begin{aligned}
&u'(t) = (u(t))^{\frac{1}{3}}(1 - u(t)^2)e^{t+5v(t)}, \\
&v'(t) = -t^{\frac{1}{2}}v(t)e^{tu(t)} && \text{for a.e. } t \in [0,1], \\
&(u(0), v(0)) = (u_0, v_0) \in R_0.
\end{aligned}
$$

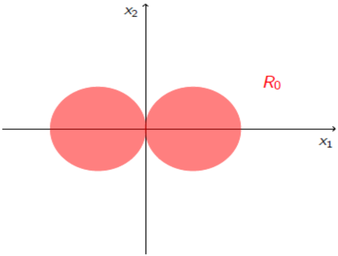
# Example 2

We consider the following system:

$$
(2) \qquad
\begin{aligned}
&u'(t) = (u(t))^{\frac{1}{3}}(1 - u(t)^2)e^{t+5v(t)}, \\
&v'(t) = -t^{\frac{1}{2}}v(t)e^{tu(t)} \qquad\qquad \text{for a.e. } t \in [0,1], \\
&(u(0), v(0)) = (u_0, v_0) \in R_0.
\end{aligned}
$$

We can verify that $R = [0,1] \times R_0$ is a solution-region of (2),



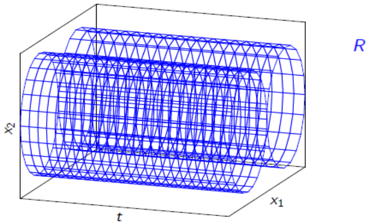



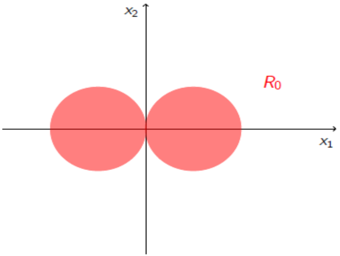
# Example 2

We consider the following system:

$$
(2)\qquad
\begin{aligned}
&u'(t) = (u(t))^{\frac{1}{3}}(1 - u(t)^2)e^{t+5v(t)}, \\
&v'(t) = -t^{\frac{1}{2}}v(t)e^{tu(t)} \qquad\qquad \text{for a.e. } t \in [0,1], \\
&(u(0), v(0)) = (u_0, v_0) \in R_0.
\end{aligned}
$$

We can verify that $R = [0,1] \times R_0$ is a solution-region of (2),




The existence theorem implies that

$$\exists (u,v) \text{ a solution of (2) s.t } \quad (t, u(t), v(t)) \in R \quad \forall t \in [0,1].$$

# A multiplicity result

# Strict solution-regions

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a strict solution-region of (P) if
$R$ is a solution-region with an associated admissible pair $(h, p)$
satisfying the following conditions:

# Strict solution-regions

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a strict solution-region of (P) if $R$ is a solution-region with an associated admissible pair $(h, p)$ satisfying the following conditions:

(i) $\text{int}(R_t) \neq \emptyset \quad \forall t \in I$, where $R_t = \{x \in \mathbb{R}^N : (t, x) \in R\}$;

# Strict solution-regions

**Definition**

A set $R \subset I \times \mathbb{R}^N$ is a strict solution-region of (P) if
$R$ is a solution-region with an associated admissible pair $(h, p)$ satisfying the following conditions:

(i) $\text{int}(R_t) \neq \emptyset \quad \forall t \in I$, where $R_t = \{x \in \mathbb{R}^N : (t, x) \in R\}$;

(ii) $\exists \varepsilon > 0$ s.t. $\dfrac{\partial h}{\partial t}(t,x) + \langle \nabla_x h(t,x), f(t,x) \rangle \leq 0$ for a.e. $t$ and $\forall x$ with $h(t,x) \in ]-\varepsilon, 0[$, and
$\frac{\partial h}{\partial t}, \nabla_x h$ are locally Carathéodory maps on $h^{-1}(]-\varepsilon, 0[)$;

(iii) -if $\mathcal{B}$ denotes (IVC), $h(0, r) < 0$;
- if $\mathcal{B}$ denotes (PC), it satisfies $h(0,x) < h(T,x) \quad \forall x$ s.t. $h(0,x) = 0$.

# Multiplicity result for the problem (P) with $N \geq 1$

**Theorem**

*Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. Assume that $\exists R^1, R^2$ two strict solution-regions and $\exists R^3$ a solution-region of (P) such that*

$$R^1 \cup R^2 \subset R^3 \quad \text{and} \quad R^1_t \cap R^2_t = \emptyset \quad \text{for some } t \in I.$$

# Multiplicity result for the problem (P) with $N \geq 1$

**Theorem**

*Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. Assume that $\exists R^1, R^2$ two strict solution-regions and $\exists R^3$ a solution-region of (P) such that*

$$R^1 \cup R^2 \subset R^3 \quad \text{and} \quad R^1_t \cap R^2_t = \emptyset \quad \text{for some } t \in I.$$

*Then, (P) has at least three distinct solutions $u_1, u_2, u_3$ such that*

$$(t, u_i(t)) \in R^i \quad \forall t \in I \text{ and } i = 1, 2, 3,$$

*and* $$\{t \in I : (t, u_3(t)) \in R^3 \backslash (R^1 \cup R^2)\} \neq \emptyset.$$

# Idea of the proof for the periodic problem.

- For $i = 1, 2, 3$, we consider the family of modified problems:

$$(\mathrm{P}^i_\lambda) \qquad \begin{cases} u'(t) = \lambda f_{R^i}(t, u(t)) + \dfrac{1-\lambda}{T} \displaystyle\int_0^T f_{R^i}(t, u(t))\, dt, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in I, \\ u(0) = u(T); \end{cases}$$

## Idea of the proof for the periodic problem.

- For $i = 1, 2, 3$, we consider the family of modified problems:

$$(\mathrm{P}^i_\lambda) \qquad \begin{cases} u'(t) = \lambda f_{R^i}(t, u(t)) + \dfrac{1-\lambda}{T} \displaystyle\int_0^T f_{R^i}(t, u(t))\, dt, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in I, \\ u(0) = u(T); \end{cases}$$

with

$$f_{R^i}(t, x) = \begin{cases} f(t, x) & \text{if } (t, x) \in R^i, \\ f(p^i(t, x)) - c^i(t)(x - p^i_2(t, x)) & \text{otherwise;} \end{cases}$$

where $(h^i, p^i)$ is an admissible pair associated to $R^i$ and $c^i \in L^1(I)$ is chosen such that

$$c^i(t) > \|f(p^i(t, x))\| \quad \text{for a.e. } t \in I \text{ and } \forall x \in \mathbb{R}^N.$$

# Idea of the proof (continued)

- We consider the associated operator $\mathcal{P}^i : [0,1] \times C(I, \mathbb{R}^N) \to C(I, \mathbb{R}^N)$ defined by

$$\mathcal{P}^i(\lambda, u)(t) = u(0) - \frac{(1+\lambda t)}{T} \int_0^T f_{R^i}(s, u(s))\, ds + \lambda \int_0^t f_{R^i}(s, u(s))\, ds.$$

# Idea of the proof (continued)

- We consider the associated operator $\mathcal{P}^i : [0,1] \times C(I,\mathbb{R}^N) \to C(I,\mathbb{R}^N)$ defined by

$$\mathcal{P}^i(\lambda, u)(t) = u(0) - \frac{(1+\lambda t)}{T} \int_0^T f_{R^i}(s, u(s))\, ds + \lambda \int_0^t f_{R^i}(s, u(s))\, ds.$$

- A fixed point of $\mathcal{P}^i(\lambda, \cdot)$ is a solution of $(\mathrm{P}^i_\lambda)$.

# Idea of the proof (continued)

- We consider the associated operator $\mathcal{P}^i : [0,1] \times C(I, \mathbb{R}^N) \to C(I, \mathbb{R}^N)$ defined by

$$\mathcal{P}^i(\lambda, u)(t) = u(0) - \frac{(1+\lambda t)}{T} \int_0^T f_{R^i}(s, u(s))\, ds + \lambda \int_0^t f_{R^i}(s, u(s))\, ds.$$

- A fixed point of $\mathcal{P}^i(\lambda, \cdot)$ is a solution of $(\mathrm{P}^i_\lambda)$.
- We show that, for $i = 1, 2, 3$,

$$\mathrm{index}(\mathcal{P}^i(\lambda, \cdot), \mathcal{U}^i) = (-1)^N \qquad \forall \lambda \in [0,1],$$

with $\mathcal{U}^i = \{u \in C(I, \mathbb{R}^N) : u(t) \in \mathrm{int}(R^i_t)\}$, for $i = 1, 2$, and $\mathcal{U}^3$ containing $\{u \in C(I, \mathbb{R}^N) : \mathrm{graph}(u) \subset R^3\}$.

# Idea of the proof (continued)

- $\mathcal{P}^3(1,\cdot) = \mathcal{P}^i(1,\cdot)$ on $\mathcal{U}^i$ for $i = 1, 2$.

# Idea of the proof (continued)

- $\mathcal{P}^3(1,\cdot) = \mathcal{P}^i(1,\cdot)$ on $\mathcal{U}^i$ for $i = 1, 2$.
- Excision property of the fixed point index implies

$$\text{index}\Big(\mathcal{P}^3(1,\cdot), \mathcal{U}^3 \backslash \big(\overline{\mathcal{U}^1 \cup \mathcal{U}^2}\big)\Big) = (-1)^{N+1}.$$

# Idea of the proof (continued)

- $\mathcal{P}^3(1,\cdot) = \mathcal{P}^i(1,\cdot)$ on $\mathcal{U}^i$ for $i = 1, 2$.
- Excision property of the fixed point index implies

$$\text{index}\Big(\mathcal{P}^3(1,\cdot), \mathcal{U}^3 \backslash \big(\overline{\mathcal{U}^1 \cup \mathcal{U}^2}\big)\Big) = (-1)^{N+1}.$$

- Let $u_i \in \mathcal{U}^i$ be a solution of $(P^i_{\lambda=1})$. We show that $(t, u_i(t)) \in R^i \quad \forall t \in I$.
- We conclude that $u_i$ is a solution of (P) since $f$ and $f_{R^i}$ coincide on $R^i$.

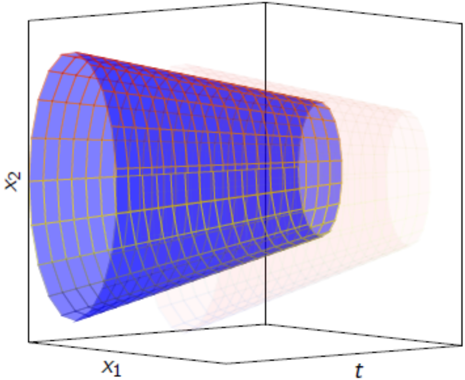
# Example

We consider the following system of differential equations with the periodic boundary value conditions:

$$(3) \quad \begin{cases} u'(t) = \big(u^2(t) - t^2 + 10t - 25\big)\big(\sin(t + v(t)) - 2u(t)\big), \\ v'(t) = t^3 - v(t)e^{1+u^2(t)} & \text{a.e. } t \in [0,1], \\ u(0) = u(1), \qquad v(0) = v(1). \end{cases}$$

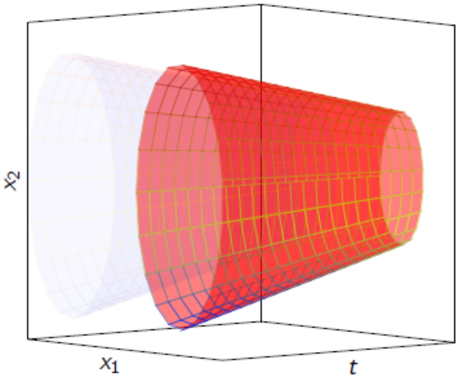
We consider the following regions in $[0,1]\times\mathbb{R}^2$:

$$R^1 = \left\{(t,x_1,x_2)\in[0,1]\times\mathbb{R}^2 : (x_1+t-5)^2+x_2^2\le(4-2t)^2\right\}.$$



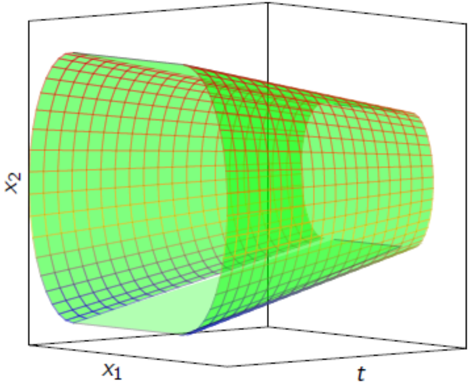

$R^1$ is a strict solution-region of (3).

$$R^2 = \left\{(t, x_1, x_2) \in [0,1] \times \mathbb{R}^2 : (x_1 - t + 5)^2 + x_2^2 \leq (4 - 2t)^2\right\}.$$



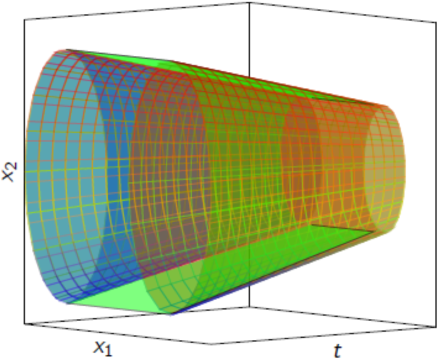

$R^2$ is a strict solution-region of (3).

$$R^3 = \Big\{(t, x_1, x_2) \in [0,1] \times \mathbb{R}^2 : t - 5 \le x_1 \le 5 - t,\ |x_2| \le (4 - 2t)\Big\}$$
$$\cup \Big\{(t, x_1, x_2) \in [0,1] \times \mathbb{R}^2 : x_1 > 5 - t,\ (x_1 + t - 5)^2 + x_2^2 \le (4 - 2t)^2\Big\}$$
$$\cup \Big\{(t, x_1, x_2) \in [0,1] \times \mathbb{R}^2 : x_1 < t - 5,\ (x_1 + t - 5)^2 + x_2^2 \le (4 - 2t)^2\Big\}.$$




$R^3$ is a solution-region of (3).

$$R^1 \cup R^2 \subset R^3 \quad \text{and} \quad R^1_t \cap R^2_t = \emptyset \quad \forall t \in I.$$




Our multiplicity theorem $\Rightarrow$

- $\exists u_1$ a solution of (3) whose graph is in $R^1$;
- $\exists u_2$ a solution of (3) whose graph is in $R^2$;
- $\exists u_3$ a solution of (3) whose graph is in $R^3$ and intersects $R^3 \backslash (R^1 \cup R^2)$.

# Nonlinear Boundary Conditions

# Systems of differential equations with nonlinear boundary conditions

We consider systems of differential equations with more general boundary conditions:

$$(\mathrm{P}_L) \qquad \begin{cases} u'(t) = f(t, u(t)) & \text{for a.e. } t \in I := [0, T], \\ u \in \mathcal{B}_L; \end{cases}$$

# Systems of differential equations with nonlinear boundary conditions

We consider systems of differential equations with more general boundary conditions:

$$(\mathrm{P}_L) \qquad \begin{cases} u'(t) = f(t, u(t)) & \text{for a.e. } t \in I := [0, T], \\ u \in \mathcal{B}_L; \end{cases}$$

where $\mathcal{B}_L$ denotes one of the boundary value conditions:

$$(\mathrm{L1}) \qquad L(u(0), u(T), u) = 0;$$

$$(\mathrm{L2}) \qquad L(u(0), u(T), u) = u(0) - u(T);$$

with $L : \mathbb{R}^{2N} \times C(I, \mathbb{R}^N) \to \mathbb{R}^N$ continuous.
$L$ does not need to be linear.

# Systems of differential equations with nonlinear boundary conditions

We consider systems of differential equations with more general boundary conditions:

$$(\mathrm{P}_L) \qquad \begin{cases} u'(t) = f(t, u(t)) & \text{for a.e. } t \in I := [0, T], \\ u \in \mathcal{B}_L; \end{cases}$$

where $\mathcal{B}_L$ denotes one of the boundary value conditions:

$$(\mathrm{L1}) \qquad L(u(0), u(T), u) = 0;$$

$$(\mathrm{L2}) \qquad L(u(0), u(T), u) = u(0) - u(T);$$

with $L : \mathbb{R}^{2N} \times C(I, \mathbb{R}^N) \to \mathbb{R}^N$ continuous.
$L$ does not need to be linear.
As before, $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

In 2019, Tojo introduced the following weaker notion.

In 2019, Tojo introduced the following weaker notion.

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a weak admissible region

In 2019, Tojo introduced the following weaker notion.

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a weak admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t, x) : h(t, x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t, x) \in R\} \neq \emptyset$;

(ii) $h$ has partial derivatives at $(t, x)$ for a.e. $t$ and $\forall x$ with $(t, x) \in R^c = \left(I \times \mathbb{R}^N\right) \setminus R$, and $\frac{\partial h}{\partial t}$, $\nabla_x h$ are locally Carathéodory maps on $R^c$;

In 2019, Tojo introduced the following weaker notion.

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a weak admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t, x) : h(t, x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t, x) \in R\} \neq \emptyset$;

(ii) $h$ has partial derivatives at $(t, x)$ for a.e. $t$ and $\forall x$ with $(t, x) \in R^c = \left(I \times \mathbb{R}^N\right) \setminus R$, and $\frac{\partial h}{\partial t}$, $\nabla_x h$ are locally Carathéodory maps on $R^c$;

(iii) $p$ is bounded and such that $p(t, x) = (t, x) \quad \forall (t, x) \in R$ and

$$\text{(WA)} \quad \langle \nabla_x h(t, x), p_2(t, x) - x \rangle \leq 0 \quad \text{a.e. } t \text{ and } \forall x \text{ with } (t, x) \in R^c.$$

In 2019, Tojo introduced the following weaker notion.

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a weak admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t,x) : h(t,x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t,x) \in R\} \neq \emptyset$;

(ii) $h$ has partial derivatives at $(t,x)$ for a.e. $t$ and $\forall x$ with $(t,x) \in R^c = \left(I \times \mathbb{R}^N\right) \setminus R$, and $\frac{\partial h}{\partial t}$, $\nabla_x h$ are locally Carathéodory maps on $R^c$;

(iii) $p$ is bounded and such that $p(t,x) = (t,x) \quad \forall (t,x) \in R$ and

$$\text{(WA)} \quad \langle \nabla_x h(t,x), p_2(t,x) - x \rangle \leq 0 \qquad \text{a.e. } t \text{ and } \forall x \text{ with } (t,x) \in R^c.$$

$(h,p)$ is called a weak admissible pair associated to $R$.

In 2019, Tojo introduced the following weaker notion.

**Definition**

A set $R \subset I \times \mathbb{R}^N$ is a weak admissible region if $\exists h : I \times \mathbb{R}^N \to \mathbb{R}$ and $\exists p = (p_1, p_2) : I \times \mathbb{R}^N \to I \times \mathbb{R}^N$ continuous maps such that:

(i) $R = \{(t,x) : h(t,x) \leq 0\}$ is bounded and, $\forall t \in I$, $R_t = \{x \in \mathbb{R}^N : (t,x) \in R\} \neq \emptyset$;

(ii) $h$ has partial derivatives at $(t,x)$ for a.e. $t$ and $\forall x$ with $(t,x) \in R^c = \left(I \times \mathbb{R}^N\right) \setminus R$, and $\frac{\partial h}{\partial t}$, $\nabla_x h$ are locally Carathéodory maps on $R^c$;

(iii) $p$ is bounded and such that $p(t,x) = (t,x) \quad \forall (t,x) \in R$ and

$$\text{(WA)} \quad \langle \nabla_x h(t,x), p_2(t,x) - x \rangle \leq 0 \qquad \text{a.e. } t \text{ and } \forall x \text{ with } (t,x) \in R^c.$$

$(h,p)$ is called a weak admissible pair associated to $R$.

Tojo showed that any compact set $R$ such that $R_t \neq \emptyset$ for every $t \in I$ is a weak admissible region.

# Solution-regions with nonlinear boundary conditions

We extend the notion of solution-regions to $(\mathrm{P}_L)$.

# Solution-regions with nonlinear boundary conditions

We extend the notion of solution-regions to $(P_L)$.

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a solution-region of $(P_L)$ if it is a weak admissible region with an associated weak admissible pair $(h, p)$ satisfying the following conditions:

# Solution-regions with nonlinear boundary conditions

We extend the notion of solution-regions to $(\mathrm{P}_L)$.

## Definition

A set $R \subset I \times \mathbb{R}^N$ is a solution-region of $(\mathrm{P}_L)$ if it is a weak admissible region with an associated weak admissible pair $(h, p)$ satisfying the following conditions:

(i) $\dfrac{\partial h}{\partial t}(t,x) + \langle \nabla_x h(t,x), f(p(t,x)) \rangle \leq 0$
for a.e. $t$ and $\forall x$ with $(t,x) \notin R$;

## Definition (continued)

(ii) *if* $\mathcal{B}_L$ *denotes* (L1), $\forall u \in W^{1,1}(I, \mathbb{R}^n)$ *such that* $u(0) = p_2\big(0, u(0) - L(u(0), u(T), u)\big)$,

(a) $h(0, u(0)) \leq 0$;

(b) $\big(0, u(0) - L(u(0), u(T), u)\big) \in R$ *if* $(t, u(t)) \in R \;\; \forall t \in I$.

## Definition (continued)

(ii) *if* $\mathcal{B}_L$ *denotes* (L1), $\forall u \in W^{1,1}(I, \mathbb{R}^n)$ *such that* $u(0) = p_2\big(0, u(0) - L(u(0), u(T), u)\big)$,

  (a) $h(0, u(0)) \leq 0$;

  (b) $\big(0, u(0) - L(u(0), u(T), u)\big) \in R$ *if* $(t, u(t)) \in R \ \ \forall t \in I$.

(ii)' *If* $\mathcal{B}_L$ *denotes* (L2),

  (a) $h(0, u(0)) \leq h(T, u(T)) \ \ \forall u \in W^{1,1}(I, \mathbb{R}^n)$ *such that* $(0, u(0)) \notin R$ *and* $u(T) - u(0) = tL(u(0), u(T), u)$ *for some* $t \in [0, 1]$.

  (b) *The inequality in (ii)'(a) is strict or (i) or (WA) is strict on some* $S \subset I$ *of positive measure.*

## An existence result for the problem $(\mathrm{P}_L)$

**Theorem**

*Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. Assume that there exists $R$ a solution-region of $(\mathrm{P}_L)$. Then, $(\mathrm{P}_L)$ has a solution $u \in W^{1,1}(I, \mathbb{R}^N)$ such that $(t, u(t)) \in R$ for every $t \in I$.*

# Example

We consider the following problem with nonlinear boundary conditions:

$$(P_2) \qquad \begin{cases} u_1'(t) = (t-3)u_1(t)u_2^2(t) + \dfrac{t^2}{4}, \\ u_2'(t) = -u_2(t)e^{t+|u_1(t)|} + \dfrac{1-t^2}{4}, \qquad \text{a.e. } t \in [0,1], \\ 12u_1(0) + \displaystyle\int_0^1 (6-5t)u_1(t)u_2(t)\,dt = 0, \\ 3u_2(0) - u_2(1) = 0. \end{cases}$$

We look for a solution to

$$\begin{aligned} &u'(t) = f(t, u(t)) \quad \text{a.e. } t \in [0,1], \\ &L(u(0), u(1), u) = 0. \end{aligned}$$

We look for a solution to

$$
\begin{aligned}
&u'(t) = f(t, u(t)) \quad \text{a.e. } t \in [0,1],\\
&L(u(0), u(1), u) = 0.
\end{aligned}
$$

with $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ and $L : \mathbb{R}^4 \times C([0,1], \mathbb{R}^2) \to \mathbb{R}^2$ defined by

$$
f(t, x_1, x_2) = \left( (t-3)x_1 x_2^2 + \frac{t^2}{4}, -x_2 e^{t+|x_1|} + \frac{1-t^2}{4} \right),
$$

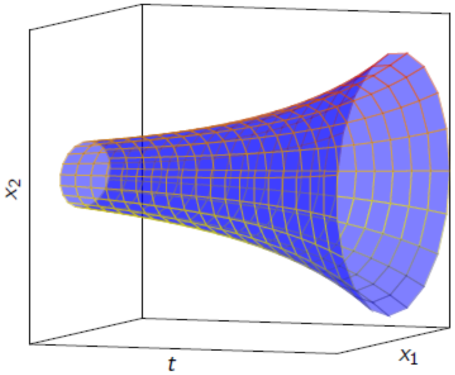
$$
L(x_1, x_2, y_1, y_2, u_1, u_2) = \left( x_1 + \frac{1}{12} \int_0^1 (6-5t) u_1(t) u_2(t)\, dt, \frac{3x_2 - y_2}{4} \right).
$$

We consider the closed and bounded set

$$R = \left\{ (t, x_1, x_2) \in [0,1] \times \mathbb{R}^2 : \left( \left( 1 - \frac{t}{3} \right) x_1 \right)^2 + \left( \left( 1 - \frac{t}{2} \right) x_2 \right)^2 \leq 1 \right\}.$$

We consider the closed and bounded set

$$R = \left\{ (t, x_1, x_2) \in [0,1] \times \mathbb{R}^2 : \left( \left(1 - \frac{t}{3}\right) x_1 \right)^2 + \left( \left(1 - \frac{t}{2}\right) x_2 \right)^2 \leq 1 \right\}.$$



$R$ is an admissible region with the associated admissible pair $(h, p)$,

$R$ is an admissible region with the associated admissible pair $(h, p)$, where $h : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ and $p : [0, 1] \times \mathbb{R}^2 \to [0, 1] \times \mathbb{R}^2$ are defined by

$$h(t, x_1, x_2) = \left( \left( \left(1 - \frac{t}{3}\right) x_1 \right)^2 + \left( \left(1 - \frac{t}{2}\right) x_2 \right)^2 \right)^{\frac{1}{2}} - 1,$$

$$p(t, x) = \begin{cases} (t, x) & \text{if } (t, x) \in R, \\ \left(t, \frac{x}{h(t,x)+1}\right) & \text{otherwise.} \end{cases}$$

$R$ is a solution-region of $(P_2)$,

$R$ is a solution-region of $(P_2)$,
since, for $(t, x) \notin R$, one has

$$\frac{\partial h}{\partial t}(t, x) + \langle \nabla_x h(t, x), f(p(t, x)) \rangle \leq 0;$$

$R$ is a solution-region of $(P_2)$,
since, for $(t,x) \notin R$, one has

$$\frac{\partial h}{\partial t}(t,x) + \langle \nabla_x h(t,x), f(p(t,x)) \rangle \leq 0;$$

and, if $u \in W^{1,1}([0,1],\mathbb{R}^2)$ is such that
$u(0) = p_2\,(0, u(0) - L(u(0), u(1), u))$, then

$$h(0,u(0)) \leq 0,$$

and

$$(0, u(0) - L(u(0), u(1), u)) \in R \quad \text{if } (t, u(t)) \in R \quad \forall t \in [0,1].$$

The existence theorem for problems with nonlinear boundary conditions insures that there exists $u$ a solution of

$$\begin{cases} u_1'(t) = (t-3)u_1(t)u_2^2(t) + \dfrac{t^2}{4}, \\ u_2'(t) = -u_2(t)e^{t+|u_1(t)|} + \dfrac{1-t^2}{4}, \qquad \text{a.e. } t \in [0,1], \\ 12u_1(0) + \displaystyle\int_0^1 (6-5t)u_1(t)u_2(t)\,dt = 0, \\ 3u_2(0) - u_2(1) = 0. \end{cases}$$

such that

$$\big(t, u(t)\big) \in R \quad \forall t \in [0,1].$$

## Case $N = 1$

Our theorem generalizes results which can be found in the literature in the case where $N = 1$.

## Case $N = 1$

Our theorem generalizes results which can be found in the literature in the case where $N = 1$.

### Corollary

*Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $L : \mathbb{R}^2 \times C(I, \mathbb{R}) \to \mathbb{R}$ be continuous and $\alpha, \beta \in W^{1,1}(I, \mathbb{R})$ such that $\alpha(t) \leq \beta(t) \quad \forall t \in I,$ and*

(i) $f(t, \beta(t)) \leq \beta'(t)$ *for a.e.* $t \in I$, *and* $L(\beta(0), \beta(T), \beta) \geq 0$;

(ii) $f(t, \alpha(t)) \geq \alpha'(t)$ *for a.e.* $t \in I$, *and* $L(\alpha(0), \alpha(T), \alpha) \leq 0$;

(iii) $L(\alpha(0), u(t), u) \leq L(\alpha(0), \alpha(T), \alpha) \quad \forall u \in C(I, \mathbb{R})$ *s.t.* $u(0) = \alpha(0)$ *and* $\alpha(t) \leq u(t) \leq \beta(t) \ \ \forall t \in I$;

(iv) $L(\beta(0), u(t), u) \geq L(\beta(0), \beta(T), \beta) \quad \forall u \in C(I, \mathbb{R})$ *such that* $u(0) = \beta(0)$ *and* $\alpha(t) \leq u(t) \leq \beta(t) \ \ \forall t \in I$.

*Then,* $(\mathrm{P}_L)$ *where* $\mathcal{B}_L$ *denotes* (L1) *has a solution* $u \in W^{1,1}(I, \mathbb{R})$ *s.t.* $(t, u(t)) \in R = \{(t, x) \in I \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\} \quad \forall t \in I$.

Thank you!

# References


[1] J.-P. Aubin, *Viability theory. Systems and Control: Foundations and Applications*. Birkhäuser, Boston, 1991.

[2] R. Bielawski, L. Górniewicz, S. Plaskacz, *Topological approach to differential inclusions on closed subset of* $\mathbb{R}^n$ . Dynamics reported: expositions in dynamical systems, Dynam. Report. Expositions Dynam. Systems (N.S.), vol. 1, Springer, Berlin, 1992, pp.225–250.

[3] M. Frigon, *Boundary and periodic value problems for systems of nonlinear second order differential equations*. Topol. Methods Nonlinear Anal. **1** (1993), 259–274.

[4] M. Frigon, *Existence and multiplicity results for systems of first-order differential equations via the method of solution-regions*. Adv. Nonlinear Stud. **18**(3) (2018), 469–485.

[5] M. Frigon, M. Lotfipour, *Multiplicity results for systems of first order differential inclusions*. J. Nonlinear Convex Anal. **16** (2015), 1025–1040.

[6] M. Frigon, M. Tella, F.A.F. Tojo *First order differential systemss with a nonlinear boundary conditions via the method of solution-regions*. Boundary Value Problems **2021:19** (2021), 18pp.

[7] H.-W. Knobloch, *An existence theorem for periodic solutions of nonlinear ordinary differential equations*. Michigan Math. J. **9** (1962), 303–309.

[8] J. Mawhin, *First order ordinary differential equations with several periodic solutions*. Z. Angew. Math. Phys. **38** (1987), 257–265.

[9] B. Mirandette, *Résultats d'existence pour des systèmes d'équations différentilles du premier ordre avec tube-solution*. Mémoire de maîtrise, Université de Montréal, Montréal, 1996.

[10] M. Nagumo, *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*. Proc. Phys.-Math. Soc. Japan (3) **24**, (1942), 551–559.

[11] M.N. Nkashama, *A generalized upper and lower solutions method and multiplicity results for nonlinear first-order ordinary differential equations*. J. Math. Anal. Appl. **140** (1989), 381–395.

[12] S. Plaskacz, *Periodic solutions of differential inclusions on compact subsets of* $\mathbb{R}^n$. J. Math. Anal. Appl. **148** (1990), 202–212.

[13] F.A.F. Tojo, *A constructive approach towards the method of solution-regions*. J. Math. Anal. Appl. **472**(2) (2019), 1803–1819.