Gluing methods in the water wave problem

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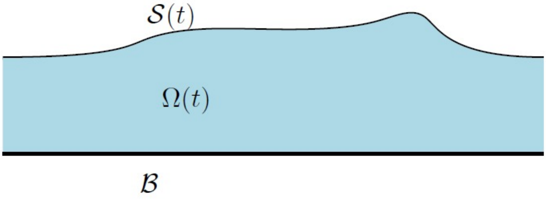
The water wave problem

Incompressible fluid occupying a region $\Omega(t)$ with constant pressure $= P_{atm}$ on the free boundary $S(t)$ under the action of gravity $-g\hat{\jmath}$

$$\partial\Omega(t) = \mathcal{B} \cup \mathcal{S}(t)$$



$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -g\hat{\jmath} - \nabla P & \text{in } \Omega(t), \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega(t), \\ P = p_{atm} & \text{on } \mathcal{S}(t), \end{cases}$$

A classical problem: travelling water waves. For a $c > 0$,

$$\begin{cases} \mathbf{u}(x,y,t) = & \vec{u}(x-ct,y) \\ P(x,y,t) = & p(x-ct,y) \\ \mathcal{S}(t) = & \mathcal{S} + \{ct\hat{\imath}\} \\ \Omega(t) = & \Omega + \{ct\hat{\imath}\} \end{cases}$$

Euler equation for $\vec{u}(x, y) = u(x, y)\hat{\imath} + v(x, y)\hat{\jmath}$ becomes

$$\begin{cases} (\vec{u} - c\hat{\imath}) \cdot \nabla \vec{u} = -g\hat{\jmath} - \nabla p \quad \text{in } \Omega, \\ \operatorname{div} \vec{u} = 0 \quad \text{in } \Omega, \\ p = p_{atm} \quad \text{on } \mathcal{S}. \end{cases}$$

$\vec{u} - c\hat{\imath}$ is the steady velocity of the fluid relative to the moving frame.

- Bernoulli's law: $(\vec{u} - c\hat{\imath}) \cdot \nabla(\frac{1}{2}|\vec{u} - c\hat{\imath}|^2 + p + gy) = 0.$

This means energy of fluid particles is preserved along stream lines.

Incompressibility implies existence of *stream function*.

$$\operatorname{div} \vec{u} = 0 \implies \exists\, \psi : \quad \nabla^{\perp}\psi = \vec{u} - c\hat{\imath}.$$

Streamlines: level curves of stream function $\psi(x, y)$.

- No flux boundary conditions

$$\nabla^{\perp}\psi \cdot \nu = 0 \quad \text{on } \mathcal{S} \cup \mathcal{B}$$

This implies that for constants A, B

$$\psi = A \quad \text{on } \mathcal{S},$$
$$\psi = B \quad \text{on } \mathcal{B}.$$

We normalize $A = 0$ and choose B later.

$$(\vec{u} - c\hat{\imath}) \cdot \nabla \vec{u} = -g\hat{\jmath} - \nabla p$$

$$\vec{u} - c\hat{\imath} = (u - c)\hat{\imath} + v\hat{\jmath} = \psi_y \hat{\imath} - \psi_x \hat{\jmath}.$$

- Vorticity is, by definition, the scalar

$$\omega = \nabla \times \vec{u} = v_x - u_y.$$

We have that

$$\omega = -(\psi_{xx} + \psi_{yy}) = -\Delta\psi.$$

$$(\nabla^{\perp}\psi \cdot \nabla)\vec{u} = -\nabla p - gy\hat{\jmath} \implies \nabla^{\perp}\psi \cdot \nabla\omega = 0 \quad \text{in } \Omega$$

This equation is satisfied if $-\Delta\psi(x, y) = \gamma(\psi(x, y))$.

We consider the **constant vorticity case**. We fix $\omega = \gamma$ with $\gamma > 0$ a given constant.

- Bernoulli's law becomes

$$\nabla^{\perp}\psi \cdot \nabla(\frac{1}{2}|\nabla\psi|^2 + p + gy) = 0.$$

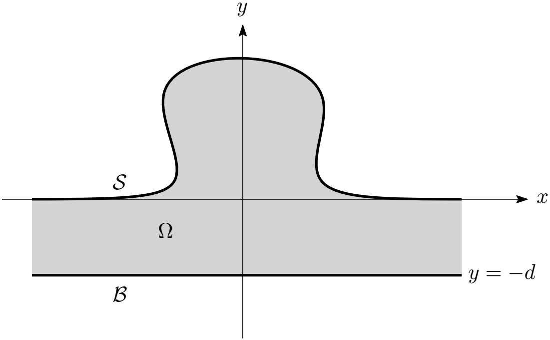
Since $p = p_A$ on $\mathcal{S}$ we get

$$\frac{1}{2}|\nabla\psi|^2 + gy = C = \textit{constant} \quad \text{on } \mathcal{S}.$$

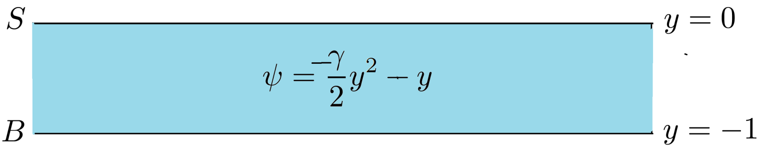
Complete formulation of the water wave problem

$$\begin{cases} -\Delta\psi = \gamma & \text{in } \Omega \\ \psi = 0 & \text{on } \mathcal{S} \\ \frac{1}{2}|\nabla\psi|^2 = C - gy & \\ \psi = B & \text{on } \mathcal{B} \end{cases}$$

A constant boundary condition B at the bottom $\mathcal{B} = \{y = -d\}$



A trivial solution: Strip $S = \{(x, y) \ / \ -d < y < 0\}$



$$\psi^S(y) = -\gamma\frac{y^2}{2} - y, \quad B = -\gamma\frac{d^2}{2} - d.$$

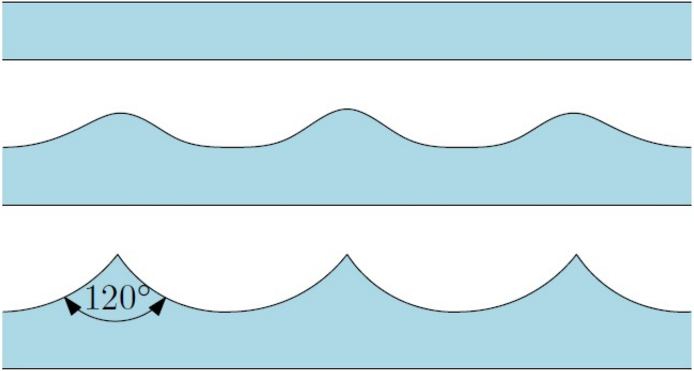
$$\begin{cases} -\Delta\psi^S = \gamma & \text{in } \Omega \\ \psi^S = 0 & \text{on } \mathcal{S} = \partial S \\ |\nabla\psi^S|^2 = 1 & \text{on } \mathcal{S} = \partial S \\ \psi^S = B & \text{on } \mathcal{B} \end{cases}$$

The irrotational case $\gamma = 0$: ψ is harmonic and the problem can be formulated in terms of holomorphic functions.

• Studied for more that 2 centuries. Including Lagrange, Cauchy, Poisson, Stokes.

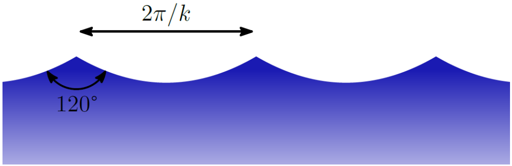
$$\begin{cases} -\Delta\psi = 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \mathcal{S} \\ \frac{1}{2}|\nabla\psi|^2 = C - gy & \text{on } \mathcal{S} \\ \psi = B & \text{on } \mathcal{B} \end{cases}$$

• It is known that a continuum of periodic solutions in the x variables emerges from the strip. (Stokes waves) $\mathcal{S}$ is a curve, it must typically be a graph



• Formulation integral equation on curves Nekrasov 1921, Solutions with large amplitude: Krasovskii 1961, Levi-Civita (1925) Keady-Norbury 1978.

• Maximal height wave, conjectured by Stokes (1847) with a cusp 120-degree angle. Construction of wave with greatest height Toland (1978): Complete proof Amick-Fraenkel-Toland, Acta Math. 1982.



• The wave profile must be the graph of a function: Spielvogel (1970), Toland (1996). Similar for solitary waves: Amick, Toland (1981), Plotnikov (1982), McLeod (1997), Kozlov, Lokharu (2020).

• Similar results for *solitary waves.*

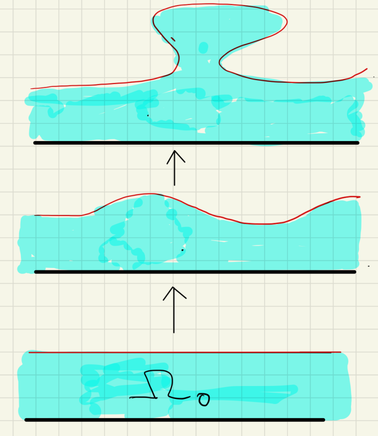
Constant vorticity $\gamma > 0$.

• Small amplitude solutions using local bifurcation Wahlen, JDE 2009, Constantin-Varvaruca ARMA 2011

• Large amplitude solutions: Global bifurcation by Constantin-Strauss-Varvaruca Acta Math. 2016, possibly giving rise to non-graphical solutions (overhanging waves).

• Explicit example of a large amplitude solitary wave Haziot-Wheeler, ARMA 2023.

Idealized bifurcation picture $\gamma = 0$ vs $\gamma \neq 0$

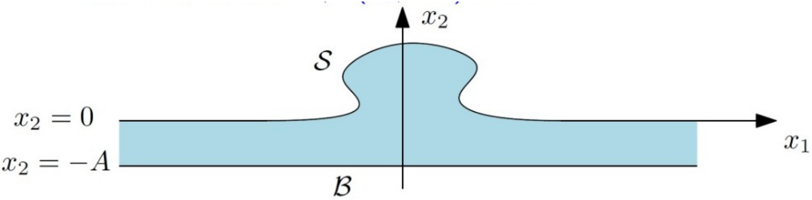


Irrotational case $\gamma = 0$ vs. constant vorticity $\gamma > 0$

The universe $\gamma > 0$ seems to be much richer but little is known

Our goal: to find **overhanging solitary waves** assuming that gravity g is a small positive parameter.

$$\begin{cases} -\Delta\psi = \gamma & \text{in } \Omega \\ \psi = 0 & \text{on } \mathcal{S} \\ \psi = B & \text{on } \mathcal{B} \\ \frac{1}{2}|\nabla\psi|^2 + gy = \frac{1}{2} & \text{on } \mathcal{S} \\ \psi(x,y) \to \psi_S(y) & \text{as } |x| \to \infty \end{cases}$$



Numerically observed overhanging waves:

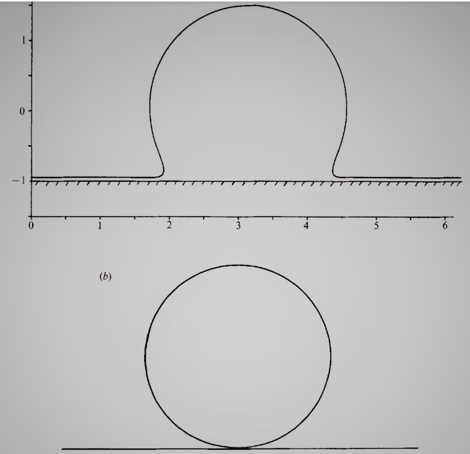
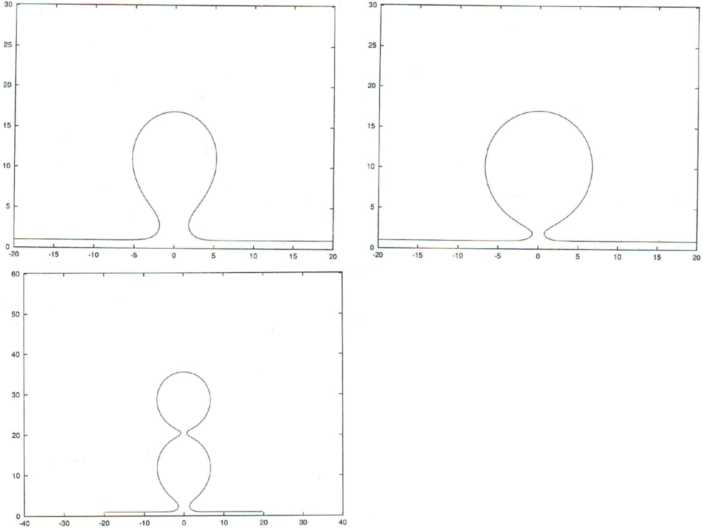


FIGURE 7. (a) Profile of a pure rotational wave ($g = 0$), with $\zeta = 1$, $H = 2.45$ and $c = 1.153$. (b) Sketch of a possible limiting wave, for zero gravity.

energy spectra from measurements of surface elevation. In strong winds the vorticity near the surface, may often be strong enough to make a suitable correction significant for the shorter waves. The dimensionless parameter λ defined in (4.2)

Teles da Silva-Peregrine, 1988

Vanden Broeck (1994,1995): overhanging solitary waves



Stokes waves with constant vorticity 519

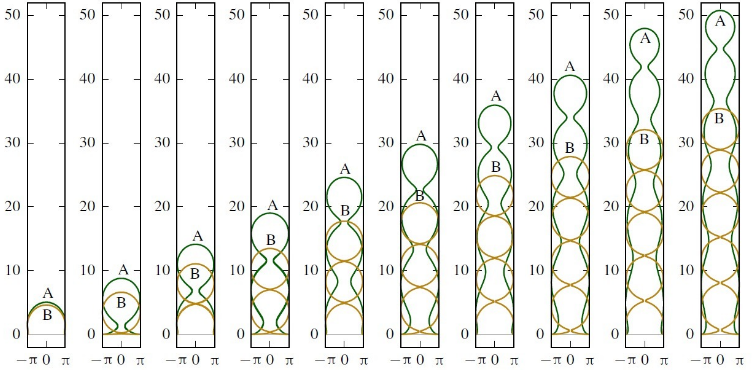


FIGURE 9. (Colour online) The profiles of almost touching waves at the beginnings and ends of the five gaps for $\omega = 4.0$ (A, green) and $\omega = 14$ (B, yellow), in the (x, y)

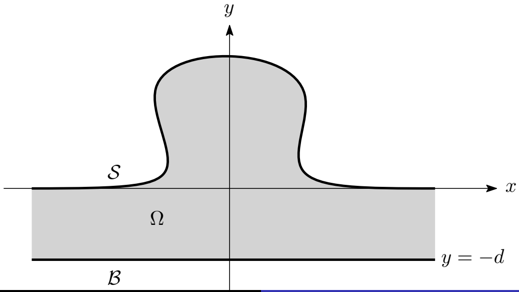
Dyachenko-Hur 2019, Vanden-Broeck 1996.

We want to find these patterns when vorticity is large

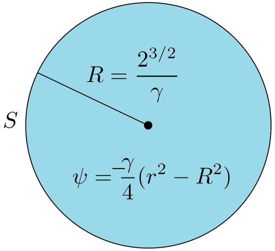
We are interested in **solitary waves**, asymptotic to the strip Ω^S. If vorticity is large and comparable to wave speed, we can scale into a regime in which $0 < g \ll 1$ is a small parameter and $c = 1$.

$$\begin{cases} -\Delta\psi = \gamma & \text{in } \Omega \\ \psi = 0 & \text{on } \mathcal{S} \\ |\nabla\psi|^2 = 1 - 2gy & \text{on } \mathcal{S}, \\ \psi(x,y) \to \psi_S(y) & \text{as } x \to \pm\infty \\ \psi = constant & \text{on } \mathcal{B} \end{cases}$$

where $\psi_S(y) = -\frac{\gamma}{2}y^2 - y$.



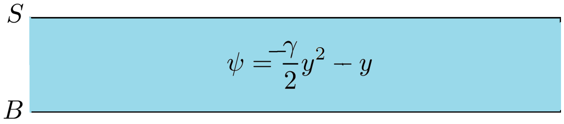
A solution with $g = 0$ and $\gamma \neq 0$: A disk Ω^D.



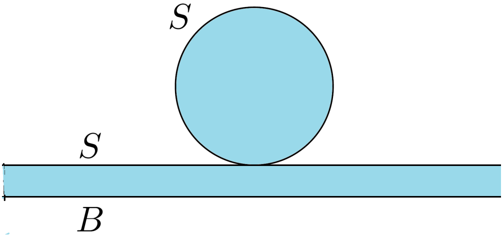
$$\begin{cases} -\Delta\psi^D = \gamma & \text{in } \Omega^D \\ \psi^D = 0 & \text{on } \mathcal{S} \\ |\nabla\psi^D|^2 = 1 & \text{on } \mathcal{S} \end{cases}$$

$$\psi^D = \frac{\gamma}{4}(R^2 - r^2), \quad R = \frac{2^{\frac{3}{2}}}{\gamma}$$

$$\psi^S = -\frac{\gamma}{2}y^2 - y.$$



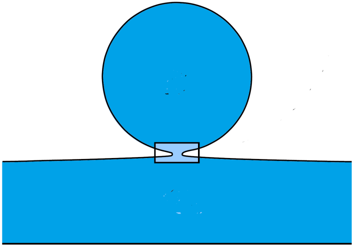
We want to find a solution that in the **singular limit** $g \to 0$ looks like

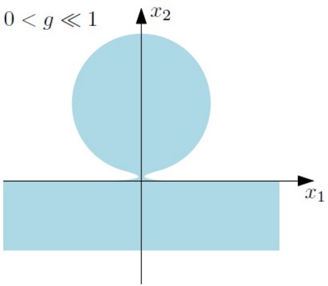


$$\psi^D = \frac{\gamma}{4}(R^2 - r^2),$$

$$\psi^S = -\frac{\gamma}{2}y^2 - y.$$

How to desingularize into a small $g > 0$? Gluing by small neck





As $g \to 0$, we look for a solution (ψ_g, Ω_g) with

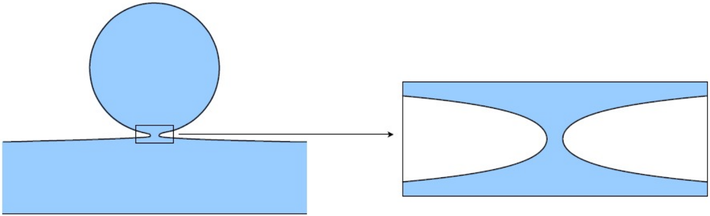
$$(\psi_g, \Omega_g) \to (\psi_S, \Omega_S) \cup (\psi_D, \Omega_D)$$

Profile near the neck?

For $\varepsilon = \varepsilon(g) \to 0$ we postulate that on the neck

$$\psi_\varepsilon(y) = \varepsilon^{-1}\psi_g(\varepsilon y) \to \psi_H(y), \quad \Omega_\varepsilon = \varepsilon^{-1}\Omega_g \to H$$

$$|\nabla\psi_\varepsilon(y)|^2 \to 1 = |\nabla\psi_H(y)|^2, \quad -\Delta_y\psi_\varepsilon(y) = \omega\varepsilon \to 0 = -\Delta_y\psi_H(y)$$

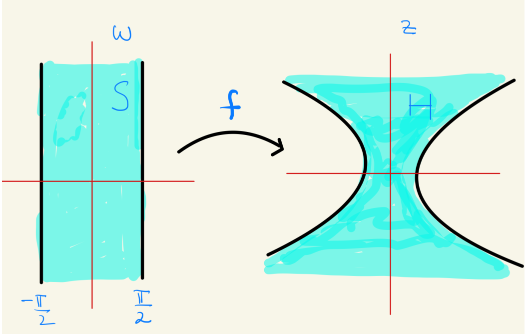


$$\boxed{\Delta\psi_H = 0 \quad \text{in } H, \quad \psi_H = 0, \ |\nabla\psi_H| = 1 \quad \text{on } \partial H.}$$

$H =$ "exceptional domain".

The double hairpin domain (Hauswirth-Pacard, 1995)

$$\Delta\psi_H = 0 \quad \text{in } \Omega_H, \quad \psi_H = 0,\ |\nabla\psi_H| = 1 \quad \text{on } \partial\Omega_H.$$

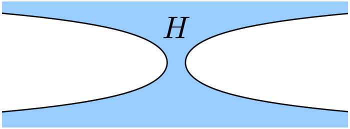


$H = f(S)$,

$$z = f(w) = w + \sin(w)$$

$$H = \{(z_1, z_2)\ /\ |z_1| < \frac{\pi}{2} + \cosh(z_2)\}$$

$$\psi^H(z) = Re\,(\cos w) = \cosh w_2 \cos w_1, \quad z = w + \sin w$$



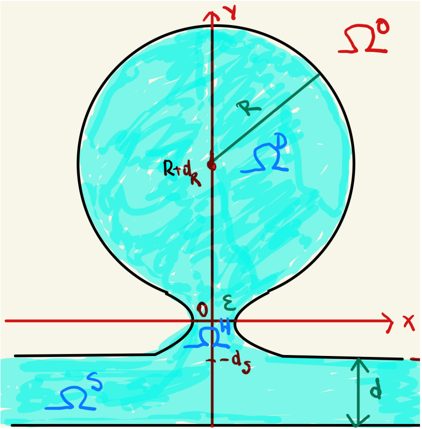
$$\psi^H(z) = Re\,(\cos w) = \cosh w_2 \cos w_1, \quad z = w + \sin w$$

$$\begin{cases} -\Delta\psi^H = 0 & \text{in } H \\ \psi = 0 & \text{on } \partial H \\ |\nabla\psi^H|^2 = 1 & \text{on } \partial H \end{cases}$$

Given $\varepsilon > 0$ we join the disk and the strip by a tiny neck

$$\Omega^H = \varepsilon H, \quad \psi^H_\varepsilon(z) = \varepsilon\psi^H(z/\varepsilon)$$

$$\begin{cases} -\Delta\psi^H_\varepsilon = 0 & \text{in } \Omega^H \\ \psi = 0 & \text{on } \partial\Omega^H \\ |\nabla\psi^H_\varepsilon|^2 = 1 & \text{on } \partial\Omega^H \end{cases}$$



• We construct a first approximation Ω_0, ψ_0 by interpolations & small modifications of the domains and stream functions ψ_ε^H, ψ^D, ψ^S. Shifts $d_R, d_S \approx \varepsilon|\log \varepsilon|$ are needed.

- An important subtle fact:

$$\psi_\varepsilon^H(x,y) = |y| + \frac{\varepsilon}{2}\log(x^2+y^2) - \varepsilon\log\varepsilon + o(\varepsilon), \quad |y| \gg \varepsilon$$

That brings a term $\varepsilon\pi\delta_0$ to the linearized equation of the disk that has to be taken care of by a first adjustment of g in the linearized equation around ψ^D. The term $\varepsilon\log\varepsilon$ needs to be matched with the shifts d_S and d_R.

Theorem (Dávila, del Pino, Musso, Wheeler (2023))

For any $g > 0$ sufficiently small there exists $\varepsilon > 0$ such that there exists a solution Ω, ψ of the problem

$$\begin{cases} -\Delta\psi = \gamma & \text{in } \Omega \\ \psi = 0 & \text{on } \mathcal{S} \\ |\nabla\psi|^2 = 1 - 2gy & \text{on } \mathcal{S}, \\ \psi(x,y) \to \psi_S(y) & \text{as } x \to \pm\infty \\ \psi = \text{constant} & \text{on } \mathcal{B} \end{cases}$$

where Ω is a g-small perturbation of Ω_0

Linearization around Ω_0, ψ_0, $\mathcal{S}_0 = \partial\Omega_0$.

- $\mathcal{S}_0$ parametrized by arclength by $\Gamma(s)$.

- $\mathcal{S}$ parametrized as $\Gamma(s) + h(s,\delta)\nu(s)$, $\psi = \psi(x, y, \delta)$, $g = g(\delta)$.

for a small parameter δ • $\psi(\cdot, 0) = \psi_0$, $h(\cdot, 0) = 0$, $g(0) = 0$

Write

$$\psi_1 = \frac{\partial \psi}{\partial \delta}\Big|_{\delta=0}, \quad h_1 = \frac{\partial h}{\partial \delta}\Big|_{\delta=0}, g_1 = \frac{\partial g}{\partial \delta}\Big|_{\delta=0}$$

Differentiating in δ

$$\begin{cases} -\Delta\psi = \gamma + f_a(\cdot, \delta) & \text{in } \Omega_h \\ \psi = 0 & \text{on } \mathcal{S}_h \\ |\nabla\psi|^2 = 1 - 2gy + f_b(\cdot, \delta) & \text{on } \mathcal{S}_h, \end{cases}$$

From $\psi(\Gamma(s) + h(s)\nu(s), \delta) = 0$ we get

$$\nabla\psi(\Gamma(s)) \cdot h_1\nu + \psi_1 = 0 \implies h_1 = \psi_1 \quad \text{on } \partial\Omega_0.$$

Now, from $|\nabla\psi(\Gamma(s)+h\nu(s))|^2 = 1-2gy+f_b$ we get

$$\nabla\psi_0(\Gamma(s))\cdot(\nabla\psi_1 + h_1 D^2\psi_0(\Gamma(s))\nu) =$$

$$\partial_\nu\psi_1 + h_1 D^2\psi_0\nu\cdot\nu = -g_1 y + f_{b,1}(\cdot,0)$$

$$D^2\psi_0\nu\cdot\nu + D^2\psi_0 T\cdot T = \Delta\psi_0 = -\gamma$$

Finally,

$$\psi_0(\Gamma(s)) = 0 \implies \nabla\psi_0(\Gamma(s))\cdot\dot{\Gamma} = 0 \implies$$

$$D^2\psi_0 T\cdot T + \nabla\psi_0\cdot\ddot{\Gamma} = 0 \implies D^2\psi_0(\gamma(s))[T]^2 = \kappa(s)$$

$$\dot{\Gamma}(s) = T(s), \quad \ddot{\Gamma}(s) = -\kappa(s)\nu(s).$$

Hence the boundary condition on ψ_1 becomes

$$\partial_\nu\psi_1 + (\kappa(s)-\gamma)\psi_1 = f_{b,1}$$

The linearized equation is the Robin problem

$$\begin{cases} -\Delta\psi_1 = f_{a,1} & \text{in } \Omega_0 \\ \dfrac{\partial\psi_1}{\partial\nu} + (\kappa-\gamma)\psi_1 = -g_1 y + f_{b,1} & \text{on } \mathcal{S}_0, \\ \psi_1 = 0 & \text{on } \mathcal{B}_0. \end{cases}$$

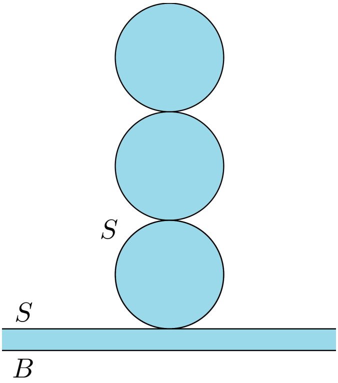
• We devise an invertibility theory for this problem, by gluing a coupled system made out of the three individual linearized equations.

• Inverting the problem in the hairpin Ω^H is obtained using a suitable complex formula. Inverting on the strip Ω^S needs $\gamma d < 1$.

• Since the linearized problem in Ω_D has a nontrivial kernel $\partial_y \psi^D$, one needs to adjust the parameter g_1 to obtain solvability.

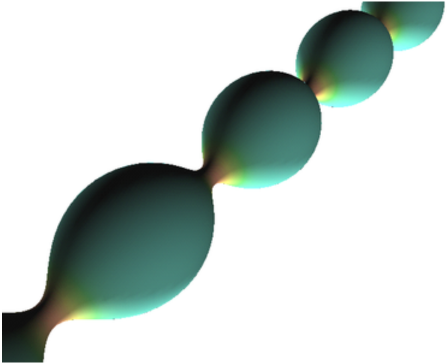
• The full nonlinear problem is solved by a fixed point of the inverse of the linearized equation.

• Similar arguments yield a desingularization of



• We need slightly different ε's for different necks.

• The method resembles gluing by desingularization of tangent spheres by small catenoidal necks in various constructions of CMC or minimal surfaces: Kapouleas 1990's, Mazzeo-Pacard 2001.



Thanks for your attention