Homoclinics of Hamiltonian Systems

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Let H be a real separable Hilbert space of infinite dimension and $f : I \times H \to \mathbb{R}$ a continuous map such that each $f_\lambda := f(\lambda, \cdot) : H \to \mathbb{R}$ is C^2 with derivatives depending continuously on the parameter $\lambda \in I := [0, 1]$. Let 0 be a critical point of all f_λ and consider the family of equations

$$(\nabla f_\lambda)(u) = 0,$$

which now has $u = 0$ as solution for all $\lambda \in I$.

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which now has $u = 0$ as solution for all $\lambda \in I$.

A parameter value $\lambda_0 \in I$ is called a bifurcation point (of critical points) if in every neighbourhood $U \subset I \times H$ of $(\lambda_0, 0)$ there is some (λ, u) such that $(\nabla f_\lambda)(u) = 0$ and $u \neq 0$.

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By the Riesz representation theorem it uniquely determines a selfadjoint operator L_λ on H such that

$$\langle L_\lambda u, v\rangle_H = (D_0^2 f_\lambda)(u, v), \quad u, v \in H,$$

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Let us now assume that the selfadjoint operators L_λ are Fredholm, i.e., they have a finite dimensional kernel and a closed range.

The spectral flow is an integer-valued homotopy invariant that is defined for any path $L = \{L_\lambda\}_{\lambda \in I}$ of selfadjoint Fredholm operators that was introduced by Atiyah, Patodi and Singer in 1976 in connection with the Atiyah-Singer Index Theorem.

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Theorem (Fitzpatrick, Pejsachowicz, Recht (1999))

If L_0, L_1 are invertible and

$$sf(L) \neq 0 \in \mathbb{Z},$$

then there is a bifurcation point of critical points of f in $(0, 1)$.

Assume that G is a compact Lie group that acts orthogonally on H and that each functional f_λ is invariant under the action of G, i.e.,

$$f_\lambda(gu) = f_\lambda(u)$$

for all $g \in G$ and $u \in H$. Then the Hessians L_λ are readily seen to be G-equivariant, i.e.,

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I.J.W. introduced in 2021 the G-equivariant spectral flow $sf_G(L)$ for paths of selfadjoint Fredholm operators $L = \{L_\lambda\}_{\lambda \in I}$ that are equivariant under the orthogonal action of a compact Lie group. This novel homotopy invariant is an element of the representation ring $RO(G)$.

Theorem (IJSW(2024))

If $L_\lambda \in \mathcal{FS}(H)^G$, $\lambda \in I$, L_0, L_1 are invertible and

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If

$$F : RO(G) \to \mathbb{Z}, \quad [U] - [V] \mapsto \dim(U) - \dim(V)$$

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Thus even if the spectral flow $sf(L) \in \mathbb{Z}$ vanishes, $sf_G(L)$ can be non-trivial in $RO(G)$.

If the operators L_λ are of the type $L_\lambda = T + K_\lambda$ for a fixed $T \in \mathcal{FS}(H)^G$ and compact operators K_λ then the spectral flow of $L = L_\lambda$ actually only depends on the endpoints L_0 and L_1.

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We will construct a $G = \mathbb{Z}_2$-invariant family of functionals f such that the Hessians $L = L_\lambda$ are a loop in $\mathcal{FS}(H)^G$ having a non-vanishing G-equivariant spectral flow.

Let $\mathcal{H} : I \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth map and consider the Hamiltonian systems

$$(1) \qquad \begin{cases} Ju'(t) + \nabla_u \mathcal{H}_\lambda(t, u(t)) = 0, & t \in \mathbb{R} \\ \lim\limits_{t \to \pm\infty} u(t) = 0, & \end{cases}$$

where $\lambda \in I$ and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is the standard symplectic matrix.

We assume that $\mathcal{H}$ is of the form

$$\mathcal{H}_\lambda(t,u) = \frac{1}{2}\langle A(\lambda,t)u,u\rangle + R(\lambda,t,u), \tag{2}$$

where $A : I \times \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n})$ is a family of symmetric matrices, $R(\lambda,t,u)$ vanishes up to second order at $u = 0$, and there are $p > 0$, $C \geq 0$ and $r \in H^1(\mathbb{R},\mathbb{R})$ such that

$$|D_u^2 R(\lambda,t,u)| \leq r(t) + C|u|^p.$$

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Moreover, we suppose that $A_\lambda := A(\lambda,\cdot) : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n})$ converges uniformly in λ to families

$$A_\lambda(+\infty) := \lim_{t\to\infty} A_\lambda(t), \quad A_\lambda(-\infty) := \lim_{t\to-\infty} A_\lambda(t), \quad \lambda \in I,$$

and that the matrices $JA_\lambda(\pm\infty)$ are hyperbolic, i.e. they have no eigenvalues on the imaginary axis.

Under the assumption (2), the map $f_\lambda : H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n}) \to \mathbb{R}$ given by

$$f_\lambda(u) = \frac{1}{2} b(u,u) + \frac{1}{2}\int_{-\infty}^{\infty} \langle A(\lambda,t)u(t), u(t)\rangle \, dt + \int_{-\infty}^{\infty} R(\lambda, t, u(t))\, dt$$

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Here

$$b(u,v) = \langle Ju', v\rangle_{L^2(\mathbb{R},\mathbb{R}^{2n})},$$

$u, v \in H^1(\mathbb{R}, \mathbb{R}^{2n})$ is the bilinear form that extends to a bounded form on the fractional Sobolev space $H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n})$.

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The critical points are the classical solutions of (1) and each sequence of critical points that converges to a bifurcation point actually converges in $C^1(\mathbb{R}, \mathbb{R}^{2n})$.

The second derivative of f_λ at the critical point $0 \in H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n})$ is given by

$$D_0^2 f_\lambda(u,v) = b(u,v) + \int_{-\infty}^{\infty} \langle A(\lambda,t)u(t), v(t)\rangle \, dt$$

and, by the hyperbolicity of $JA_\lambda(\pm\infty)$, the corresponding Riesz representations $L_\lambda : H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n}) \to H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n})$ are Fredholm.

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Consequently, the operators L_λ are selfadjoint Fredholm operators, and it follows by elliptic regularity that the kernel of L_λ consists of the classical solutions of the linear differential equation

$$\begin{cases} Ju'(t) + A(\lambda,t)u(t) = 0, & t \in \mathbb{R} \\ \lim_{t\to\pm\infty} u(t) = 0. \end{cases} \tag{3}$$

The stable and the unstable subspaces of (3) are

$$E^s(\lambda,0) = \{u(0)\in\mathbb{R}^{2n} : Ju'(t)+A(\lambda,t)u(t)=0,\ t\in\mathbb{R}; u(t)\to 0, t\to\infty\},$$

$$E^u(\lambda,0) = \{u(0)\in\mathbb{R}^{2n} : Ju'(t)+A(\lambda,t)u(t)=0,\ t\in\mathbb{R}; u(t)\to 0, t\to -\infty\},$$

and it is clear that (3) has a non-trivial solution if and only if $E^s(\lambda,0)$ and $E^u(\lambda,0)$ intersect non-trivially.

Denote by γ the non-trivial element of $G = \mathbb{Z}_2$. We set

$$\rho(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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$$\rho(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and consider Hamitonian systems in $\mathbb{R}^4$, where

$$A(\lambda, t) = \begin{pmatrix} a_\lambda(t) & 0 & c_\lambda(t) & 0 \\ 0 & b_\lambda(t) & 0 & d_\lambda(t) \\ c_\lambda(t) & 0 & e_\lambda(t) & 0 \\ 0 & d_\lambda(t) & 0 & h_\lambda(t) \end{pmatrix} \tag{4}$$

is equivariant under the action of G for any functions $a, b, c, d, e, h : I \times \mathbb{R} \to \mathbb{R}$.

The fixed point space of our action is

$$H^G = \{(u_1, u_2, u_3, u_4) \in H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^4) : \; u_2 = u_4 = 0\}$$

and it follows from (3) that the kernel of $L_\lambda \mid_{H^G}$ is made of the solutions of the Hamiltonian systems

$$(5) \qquad \begin{cases} J\begin{pmatrix} u_1' \\ u_3' \end{pmatrix} + \begin{pmatrix} a_\lambda(t) & c_\lambda(t) \\ c_\lambda(t) & e_\lambda(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = 0, \quad t \in \mathbb{R} \\ \lim_{t \to \pm\infty} u(t) = 0, \end{cases}$$

in $\mathbb{R}^2$.

Likewise the kernel of $L_\lambda \mid_{(H^G)^\perp}$ consists of the solutions of

$$\begin{cases} J\begin{pmatrix} u_2' \\ u_4' \end{pmatrix} + \begin{pmatrix} b_\lambda(t) & d_\lambda(t) \\ d_\lambda(t) & h_\lambda(t) \end{pmatrix}\begin{pmatrix} u_2 \\ u_4 \end{pmatrix} = 0, \quad t \in \mathbb{R} \\ \lim_{t\to\pm\infty} u(t) = 0. \end{cases}$$

We use instead of $I = [0,1]$ as parameter interval $[-\pi, \pi]$ and consider for $\lambda \in [-\pi, \pi]$ the matrix family

$$\widetilde{A}(\lambda, t) = \begin{pmatrix} a_\lambda(t) & c_\lambda(t) \\ c_\lambda(t) & e_\lambda(t) \end{pmatrix} = \begin{cases} (\arctan t) J S_\lambda, & t \geq 0 \\ (\arctan t) J S_0, & t < 0, \end{cases} ,$$

where

$$S_\lambda = \begin{pmatrix} \cos(\lambda) & \sin(\lambda) \\ \sin(\lambda) & -\cos(\lambda) \end{pmatrix} .$$

Note that $\tilde{A}(-\pi, t) = \tilde{A}(\pi, t)$ for all $t \in \mathbb{R}$.

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To find non-trivial solutions of (5), we now consider $E^u(\lambda, 0) \cap E^s(\lambda, 0) \neq \{0\}$.

By a direct computation it can be checked that

$$u_-(t) = \sqrt{t^2+1}\, e^{-t \arctan(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; t \leq 0,$$

$$u_+(t) = \sqrt{t^2+1}\, e^{-t \arctan(t)} \begin{pmatrix} \cos\left(\frac{\lambda}{2}\right) \\ \sin\left(\frac{\lambda}{2}\right) \end{pmatrix}, \; t \geq 0,$$

are solutions of (5) on the negative and positive half-line, respectively.

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are solutions of (5) on the negative and positive half-line, respectively.

As $u_+(0)$ and $u_-(0)$ are linearly dependent if and only if $\lambda = 0$,
(5) has a non-trivial solution if and only if $\lambda = 0$,
and the kernel of $L_0\,|_{H^G}$ is the span of

$$u_*(t) = \sqrt{t^2+1}\, e^{-t\arctan(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Next we compute the spectral flow of $L\mid_{H^G}$ by a crossing form. We need to consider

$$\Gamma(L\mid_{H^G},0)[u_*]=\int_{-\infty}^{\infty}\left\langle \dot{\tilde{A}}(0,t)u_*(t),u_*(t)\right\rangle dt,$$

where

$$\dot{\tilde{A}}(0,t)=\begin{cases}(\arctan t)J\dot{S}_0, & t\geq 0\\ 0, & t<0,\end{cases}$$

and

$$\dot{S}_0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Consequently,

$$\begin{aligned}\Gamma(L\mid_{H_G},0)[u_*] &= \int_0^\infty \left\langle \dot{\tilde{A}}(0,t)u_*(t),u_*(t)\right\rangle\,dt \\ &+ \int_{-\infty}^0 \left\langle \dot{\tilde{A}}(0,t)u_*(t),u_*(t)\right\rangle\,dt \\ &= \int_0^\infty \arctan(t)\langle J\dot{S}_0u_*(t),u_*(t)\rangle\,dt \\ &= -\int_0^\infty \arctan(t)(t^2+1)e^{-2t\arctan(t)}\,dt<0,\end{aligned}$$

which shows that $\Gamma(L\mid_{H_G},0)$ is non-degenerate and of signature -1 as quadratic form on the one-dimensional kernel of $L_0\mid_{H^G}$.

Therefore,

$$sf(L\,|_{H^G}) = -1$$

and so $sf_G(L)$ is non-trivial in $RO(\mathbb{Z}_2)$.

Thus there is a bifurcation of critical points of f by Theorem (FPR), and consequently also a bifurcation of solutions of (1) from the trivial solution.

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Let us point out, that this bifurcation cannot be found by invariants that only depend on the endpoints of the path L.

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The spectral flow changes its sign if we reverse the orientation of the path of operators.
We set for $t \in \mathbb{R}$ and $\lambda \in [-\pi, \pi]$

$$b_\lambda(t) = a_{-\lambda}(t),\ h_\lambda(t) = e_{-\lambda}(t),\ d_\lambda(t) = c_{-\lambda}(t).$$

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Then $L_\lambda \mid_{(H^G)^\perp} = L_{-\lambda} \mid_{H^G}$ and thus $sf(L \mid_{(H^G)^\perp}) = -sf(L \mid_{H^G}) = 1$.
It follows that $sf(L) = 0$ and so our example has all the required properties.

THANK YOU