Fourth order problems with functional perturbed clamped beam boundary conditions

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Joint work with A. Cabada and R. Jebari

Introduction

This talk is devoted to ensure the existence of positive solutions of the following nonlinear fourth order problem coupled to the clamped beam perturbed functional boundary conditions:

$$\begin{cases} u^{(4)}(t) + M u(t) = f(t, u(t), u'(t)), \quad t \in I := [0,1], \\ u(0) = \lambda_1 \, L_1(u), \quad u(1) = \lambda_2 \, L_2(u), \\ u'(0) = \lambda_3 \, L_3(u), \quad u'(1) = -\lambda_4 \, L_4(u), \end{cases}$$

where $M, \ \lambda_i \in \mathbb{R}$ and $L_i : C(I) \to \mathbb{R}$ are **linear and continuous functionals**, $i = 1, \ldots, 4$.

Cabada, A., Jebari, R., L-S, L., 'Existence Results for a Fourth Order Problem with Functional Perturbed Clamped Beam Boundary Conditions', *Mathematica Slovaca*, 74(4), 895–916, 2024.

Introduction

It is very well-known that the beam is mostly used in many structures such as aircrafts, buildings, ships, and bridges.

Gazzola, F., *Mathematical models for suspension bridges: Nonlinear structural instability, Modeling, Simulations and Applications*, Vol. 15, Springer, 2015.

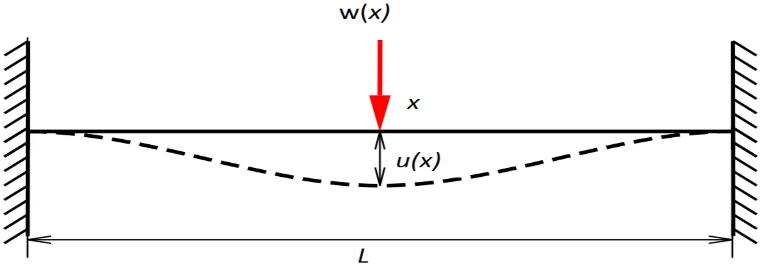
Many scientists focused on the study of deformations of the beam by deriving differential equations governing the relationship between the beam's deflection and the applied load as given by

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u(x)}{\partial x^2}\right) = w(x).$$

Introduction

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u(x)}{\partial x^2}\right) = w(x).$$

- The curve $u(x)$ describes the deflection of the one dimensional beam of length L at some position x.
- w is a distributed load (a force per unit length). It may be a function of x, u, or other variables.
- E represents the elastic modulus and I is the second moment of area.



Introduction

Many authors studied the existence of positive solutions of unidimensional equation of the beam deflection coupled with different kinds of boundary conditions. See for example:

- Lazer, A. C.; McKenna, P. J., *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev. 32 (1990), **4**, 537–578.
- Drábek, P., Holubová, G., Matas, A., Nečesal, P., *Nonlinear models of suspension bridges: discussion of the results*, Appl. Math. 48 (2003), **6**, 497–514.

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The techniques were based on fixed point theorems on cones, critical point theory, lower and upper solutions method or spectral theory.

Preliminaries: Green's Function

Differential problem

Given X a Banach space and L a differential operator, find solutions of

$$Lu(t) = f(t, u(t)), \quad t \in I, \quad u \in X.$$

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Differential problem

Given X a Banach space and L a differential operator, find solutions of

$$Lu(t) = f(t, u(t)), \quad t \in I, \quad u \in X.$$

- L is nonresonant on X if and only if

$$L\, u = 0, \quad t \in I, \quad u \in X$$

has only the trivial solution.

Preliminaries: Green's Function

The following conditions are equivalent:

- L is nonresonant on X.
- There exists a unique **Green's function** G related to problem

$$L\,u = 0, \quad t \in I, \quad u \in X.$$

- For each $\sigma \in \mathrm{L}^1(I)$, problem

$$L\,u(t) = \sigma(t), \quad t \in I, \quad u \in X,$$

  has a unique solution

$$u(t) = \int_I G(t,s)\,\sigma(s)\,\mathrm{d}\,s, \quad \forall\, t \in I.$$

Properties of the Green's Function

Definition

Given L a n-th order differential operator, G is a Green's function for problem

$$L\,u(t)=0, \quad t\in I, \quad u\in X,$$

if it satisfies the following properties:

- G is continuous on $I\times I$.
- $\frac{\partial^k G}{\partial t^k}$ exist and are continuous on $I\times I$, $k=1,\dots,n-2$.
- The partial derivatives $\frac{\partial^{n-1} G}{\partial t^{n-1}}$ and $\frac{\partial^{n} G}{\partial t^{n}}$ exist and are continuous on $I\times I\setminus\{(t,t),\ t\in I\}$.
- $\frac{\partial^{n-1} G}{\partial t^{n-1}}(t^+,t)-\frac{\partial^{n-1} G}{\partial t^{n-1}}(t^-,t)=\frac{\partial^{n-1} G}{\partial t^{n-1}}(t,t^-)-\frac{\partial^{n-1} G}{\partial t^{n-1}}(t,t^+)=1.$
- For each $s\in \mathring{I}$, $G(\cdot,s)$ is a solution of $L\,y(t)=0$, $t\in I\setminus\{s\}$.
- For each $s\in \mathring{I}$, $G(\cdot,s)$ satisfies the corresponding initial or boundary conditions.

From differential problems to topological methods

Differential problem

Given X a Banach space and L a differential operator, find solutions of

$$Lu(t) = f(t, u(t)), \quad t \in I, \quad u \in X.$$

An equivalent problem:

Integral problem

Find fixed points of an integral operator $T \colon X \to X$ given by

$$Tu(t) = \int_I G(t,s)\, f(s, u(s))\, ds.$$

We look for fixed points of operator T using topological methods.

Fixed points of integral operators

Basic Conditions:

- $(X, \|\cdot\|)$ is a Banach space;
- $T\colon (X, \|\cdot\|) \to (X, \|\cdot\|)$ a compact operator.

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Let $E \subset X$ be bounded, closed and convex. If T is a compact operator such that $T(E) \subset E$, then T has a fixed point in E.

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Definition (Cone)

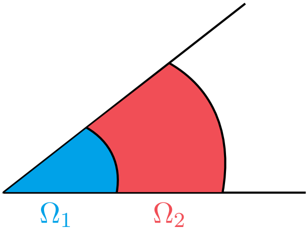
Given a Banach space X, we say that $K \subset X$ is a cone if it is a closed and convex subset of X satisfying the two following properties:

- If $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$.
- $K \cap (-K) = \{0\}$.

Fixed points of integral operators

Basic Conditions

- $(X, \|\cdot\|)$ is a Banach space;
- $T\colon (X, \|\cdot\|) \to (X, \|\cdot\|)$;
- T compact;
- K a cone, $T(K) \subset K$.



Fixed Point Index in Cones

Let $\Omega_1, \Omega_2 \subset K$.

- If $i_K(T, \Omega_1 \cup \Omega_2) = 1$ and $i_K(T, \Omega_1) = 0$, then $i_K(T, \Omega_2) = 1$.
- If $i_K(T, \Omega_1 \cup \Omega_2) = 0$ and $i_K(T, \Omega_1) = 1$, then $i_K(T, \Omega_2) = -1$.

In any of the cases, there exists a fixed point of operator T in Ω_2.

The Clampled Beam Equation with Perturbed Functional Boundary Conditions

We will prove the existence of solutions of the following problem:

$$\begin{cases} u^{(4)}(t) + Mu(t) = f(t, u(t), u'(t)), \quad t \in I := [0,1], \\ u(0) = \lambda_1 L_1(u), \quad u(1) = \lambda_2 L_2(u), \\ u'(0) = \lambda_3 L_3(u), \quad u'(1) = -\lambda_4 L_4(u), \end{cases}$$

where M, $\lambda_i \in \mathbb{R}$ and $L_i : C(I) \to \mathbb{R}$ are **linear and continuous functionals**, $i = 1, \ldots, 4$.

Existence results

The existence results follow from two different approaches:

- In the first one, we construct an integral operator whose kernel g_M is the Green's function of the problem with homogeneous boundary conditions ($L_i = 0, \ i = 1, \dots, 4$):

$$T_1 u(t) := \int_0^1 g_M(t,s)\, f(s, u(s), u'(s))\, ds + \sum_{i=1}^{4} \lambda_i w_i(t) L_i(u), \ t \in I.$$

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  The existence of positive solutions follows from fixed point theory in cones.

- On the second approach we obtain the explicit expression of the Green's function related to the functional problem:

$$T_2 u(t) := \int_0^1 G_M(t,s) f(s, u(s), u'(s))\, ds.$$

  The existence of nontrivial solutions is based on Schauder's fixed point theorem.

Green's Function of the Homogeneous Problem

Cabada, A., Enguiça, R. R., *Positive solutions of fourth order problems with clamped beam boundary conditions*, Nonlinear Anal. **74** (2011), 3112–3122.

The authors considered the fourth order **linear operator** coupled to the homogeneous clamped beam conditions:

$$\begin{cases} u^{(4)} + Mu = \sigma(t), \quad t \in [0,1], \\ u(0) = 0, \quad u(1) = 0, \quad u'(0) = 0, \quad u'(1) = 0. \end{cases} \tag{LP}$$

- They obtained the values of M for which (LP) has a unique solution.
- They obtained the values of M for which the related Green's function is strictly positive in $(0,1) \times (0,1)$.

Lemma

Let $\sigma \in \mathcal{C}(I)$ and $M \in \mathbb{R}$ such that the following condition does not hold

$$M < 0 \quad \text{and} \quad \cos\left(\sqrt[4]{-M}\right)\cosh\left(\sqrt[4]{-M}\right) = 1. \tag{SC}$$

Then problem (LP) *has a unique solution $u \in C^4(I)$, given by*

$$u(t) = \int_0^1 g_M(t,s)\,\sigma(s)\,ds,$$

where g_M is the so-called Green's function.

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where g_M *is the so-called Green's function.*

Lemma

$$g_M(t,s) > 0 \quad \forall\,(t,s) \in (0,1)\times(0,1) \Leftrightarrow M \in \left(-m_1^4, m_0^4\right].$$

- $m_1 \approxeq 4.73$ *is the smallest positive solution of* $\cos(m)\cosh(m) = 1.$
- $m_0 \approxeq 5.55$ *is the smallest positive solution of* $\tan\left(\frac{m}{\sqrt{2}}\right) = \tanh\left(\frac{m}{\sqrt{2}}\right).$

Properties of the Green's Function

Lemma

If condition (SC) does not hold, then g_M satisfies the following properties:

1. $g_M(t,s) = g_M(s,t)$ *for all* $t,\ s \in I$.
2. $g_M(0,s) = \frac{\partial g_M}{\partial t}(0,s) = g_M(1,s) = \frac{\partial g_M}{\partial t}(1,s) = 0$ *for all* $s \in I$.
3. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = g_M(t,0) = \frac{\partial g_M}{\partial s}(t,0) = 0$ *for all* $t \in I$.

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3. $g_M(t,1) = \frac{\partial g_M}{\partial s}(t,1) = g_M(t,0) = \frac{\partial g_M}{\partial s}(t,0) = 0$ *for all* $t \in I$.

Moreover, the following inequalities are fulfilled for all $M \in (-m_1^4, m_0^4)$*:*

4. $g_M(t,s) > 0$ *for all* $t, s \in (0,1)$.
5. $\frac{\partial^2 g_M}{\partial t^2}(0,s) > 0$ *and* $\frac{\partial^2 g_M}{\partial t^2}(1,s) > 0,\quad$ *for all* $s \in (0,1)$.
6. $\frac{\partial^2 g_M}{\partial s^2}(t,1) > 0$ *and* $\frac{\partial^2 g_M}{\partial s^2}(t,0) > 0,\quad$ *for all* $t \in (0,1)$.

Fundamental solutions

Lemma

The following problem:

$$\begin{cases} u^{(4)}(t) + M u(t) = 0, \ t \in I, \\ u(0) = 1, \quad u(1) = u'(0) = u'(1) = 0, \end{cases}$$

has no solution if and only if (SC) *holds.*

In any other case, it has a unique solution, denoted by w_M.

Moreover, $w_M > 0$ *on* $[0,1)$ *if and only if* $M \in (-m_1^4, 4\pi^4]$.

Remark

Note that $4\pi^4 < m_0^4$. So, the interval on M of the positiveness of the function w_M in $[0,1)$ is **smaller** than the corresponding one for the Green's function.

Fundamental solutions

Similarly, we consider the other fundamental solutions.

Lemma

The following problem:

$$\begin{cases} u^{(4)}(t) + Mu(t) = 0, \ t \in I, \\ u'(0) = 1, \quad u(0) = u(1) = u'(1) = 0, \end{cases}$$

has no solution if and only if (SC) *holds.*

In any other case, it has a unique solution, denoted by x_M.

Moreover, $x_M > 0$ *on* $(0,1)$ *if and only if* $M \in \left(-m_1^4, m_0^4\right]$.

Fundamental solutions

Using the symmetries of the problem, we have that:

Lemma

The following problem:

$$\begin{cases} u^{(4)}(t) + M u(t) = 0, \ t \in I, \\ u(1) = 1, \quad u(0) = u'(0) = u'(1) = 0, \end{cases}$$

has no solution if and only if (SC) *holds. In any other case, its unique solution is* $w_M(1-t)$.

Lemma

The following problem:

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has no solution if and only if (SC) *holds. In any other case, its unique solution is* $-x_M(1-t)$.

Decomposition of the solution

As a consequence, we obtain the following property:

Lemma

Let $\sigma \in C(I)$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, 4$, and $M \in \mathbb{R}$ be such that (SC) *does not hold. Then u_M is a solution of the problem*

$$\begin{cases} u^{(4)}(t) + M u(t) = \sigma(t), \quad t \in I, \\ u(0) = \lambda_1 L_1(u),\ u(1) = \lambda_2 L_2(u),\ u'(0) = \lambda_3 L_3(u),\ u'(1) = -\lambda_4 L_4(u), \end{cases}$$

if and only if it satisfies the following equality

$$u_M(t) = \int_0^1 g_M(t,s)\sigma(s)\,ds + \lambda_1\, w_M(t)\, L_1(u_M) + \lambda_2\, w_M(1-t)\, L_2(u_M) + \lambda_3\, x_M(t)\, L_3(u_M) + \lambda_4\, x_M(1-t)\, L_4(u_M).$$

Nonlinear Problem

We will prove the existence of positive solutions of the following problem:

$$\begin{cases} u^{(4)}(t) + Mu(t) = f(t, u(t), u'(t)), \quad t \in I := [0,1], \\ u(0) = \lambda_1 L_1(u), \quad u(1) = \lambda_2 L_2(u), \\ u'(0) = \lambda_3 L_3(u), \quad u'(1) = -\lambda_4 L_4(u), \end{cases} \tag{P}$$

where M, $\lambda_i \in \mathbb{R}$, and $L_i : C(I) \to \mathbb{R}$ are **linear and continuous operators** for all $i \in \{1, \ldots, 4\}$.

We will consider two different approaches.

Approach 1: Hypotheses

(H_0) $f : I \times [0,\infty) \times \mathbb{R} \to [0,\infty)$ is a continuous function.

(H_1) $L_i(u) \geq 0$ whenever $u \geq 0$ on I, $i = 1, \ldots, 4$.

(H_2) There exist a strictly positive function $a \in C(I)$, and two continuous and increasing functions ψ, $\varphi : [0,\infty) \to [0,\infty)$, such that

$$f(t,u,v) \leq a(t)\,(\psi(u) + \varphi(|v|)) \quad \text{for all } (t,u,v) \in I \times [0,\infty) \times \mathbb{R},$$

with $\lim_{v\to\infty} \frac{\varphi(v)}{v} = 0$.

(H_3) $\lambda_i \geq 0$ for all $i = 1, \ldots, 4$.

Approach 1: Hypotheses

By introducing the following notation

$$w_1(t) = w_M(t),\ w_2(t) = w_M(1-t),\ w_3(t) = x_M(t),\ w_4(t) = x_M(1-t).$$

We denote by $T_1 : \mathcal{C}^1(I) \to \mathcal{C}^1(I)$ the operator defined by

$$T_1 u(t) := \int_0^1 g_M(t,s)\, f(s,u(s),u'(s))\, ds + \sum_{i=1}^{4} \lambda_i\, w_i(t)\, L_i(u),\ t \in I.$$

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- Fixed points of operator T_1 coincide with the solutions of (P).
- As a direct application of the Arzelà-Ascoli Theorem we can deduce that operator T_1 is compact.

Definition of a Suitable Cone

If equality (SC) does not hold, then the following boundary value problem

$$\begin{cases} z^{(4)}(t) + M z(t) = 1, \quad t \in I, \\ z(0) = z(1) = z'(0) = z'(1) = 0, \end{cases}$$

has a unique solution z_M given by

$$z_M(t) = \int_0^1 g_M(t,s)\, ds, \quad t \in I.$$

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has a unique solution z_M given by

$$z_M(t) = \int_0^1 g_M(t,s)\, ds, \quad t \in I.$$

- If $M \in (-m_1^4, m_0^4)$, then $z_M(t) > 0$ for all $t \in (0,1)$, $z''_M(0) > 0$ and $z''_M(1) > 0$.

Definition of a Suitable Cone

Theorem

Assume that equality (SC) does not hold and $M \in (-m_1^4, m_0^4)$.

Then there exist $R_M > 0$ *and* $h_M \in \mathcal{C}(I)$, *such that* $h_M(0) = h_M(1) = 0$ *and* $h_M > 0$ *on* $(0,1)$, *for which the following inequalities are fulfilled:*

$$h_M(t)\, z_M(s) \le g_M(t,s) \le R_M\, z_M(s), \quad \text{for all } (t,s) \in I \times I.$$

Definition of a Suitable Cone

Proof.

Since $M \in (-m_1^4, m_0^4)$ we know that $g_M(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$. Now, we denote

$$\varphi_M(t,s) = \frac{g_M(t,s)}{z_M(s)}, \quad (t,s) \in [0,1] \times (0,1).$$

- φ_M is continuous on $[0,1] \times (0,1)$.
- $\varphi_M > 0$ on $(0,1) \times (0,1)$ and $\varphi_M(0,s) = \varphi_M(1,s) = 0$, $s \in (0,1)$.
- For any $t \in (0,1)$ fixed, we have

$$\lim_{s \to 0^+} \frac{g_M(t,s)}{z_M(s)} = \lim_{s \to 0^+} \frac{\frac{\partial^2 g_M}{\partial s^2}(t,s)}{z''_M(s)} = \frac{\frac{\partial^2 g_M}{\partial s^2}(t,0)}{z''_M(0)} > 0$$

  and

$$\lim_{s \to 1^-} \frac{g_M(t,s)}{z_M(s)} = \lim_{s \to 1^-} \frac{\frac{\partial^2 g_M}{\partial s^2}(t,s)}{z''_M(s)} = \frac{\frac{\partial^2 g_M}{\partial s^2}(t,1)}{z''_M(1)} > 0.$$

Definition of a Suitable Cone

Proof.

- We can extend φ_M continuously to $\widetilde{\varphi}_M$ on $[0,1] \times [0,1]$.
- $\widetilde{\varphi}_M$ will be positive on $(0,1) \times [0,1]$ and $\widetilde{\varphi}_M(0,s) = \widetilde{\varphi}_M(1,s) = 0$ for all $s \in [0,1]$.
- We define $h(t) = \min_{s\in[0,1]} \widetilde{\varphi}_M(t,s)$ and $R_M = \max_{t,\, s\in[0,1]} \widetilde{\varphi}_M(t,s)$.

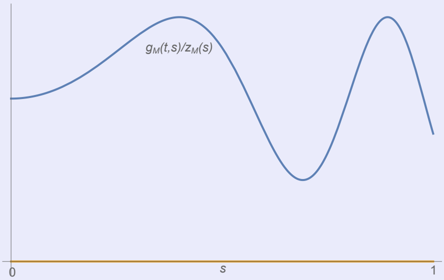

Figure: Graph of $\widetilde{\varphi}_M(t,\cdot)$, $t \in (0,1)$ fixed.

Definition of a Suitable Cone

Corollary

Assume that (SC) does not hold and $M \in (-m_1^4, m_0^4)$. *Then, for every* $[a,b] \subset (0,1)$, *there exists* $\gamma_M^{a,b} \in (0,1)$ *such that*

$$\frac{\gamma_M^{a,b}}{R_M}\, z_M(s) \leq g_M(t,s), \text{ for all } (t,s) \in [a,b] \times I.$$

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Corollary

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We look for fixed points of operator T_1 in the cone

$$K_{\delta_M}^{a,b} := \left\{ u \in \mathcal{C}^1(I) : u(t) \geq 0 \text{ on } I, \text{ and } \min_{t \in [a,b]} u(t) \geq \delta_M^{a,b} \|u\|_\infty \right\},$$

where

$$\delta_M^{a,b} := \min \left\{ \frac{\gamma_M^{a,b}}{R_M^2}, \frac{\min_{t \in [a,b]} w_i(t)}{\|w_i\|_\infty}, \ i = 1, \dots, 4 \right\}.$$

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Lemma

If $M \in (-m_1^4, 4\,\pi^4]$ *and* (H_0), (H_1) *and* (H_3) *hold, then* $T_1 \left(K_{\delta_M}^{a,b} \right) \subset K_{\delta_M}^{a,b}$.

Fixed Point Index Properties

Lemma

Let X be a Banach space, $K \subset X$ be a cone, and Ω be an open bounded set with $\Omega_K = \Omega \cap K \neq \emptyset$ and $\overline{\Omega_K} \neq K$. Assume that $T : \overline{\Omega_K} \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial\Omega_K = \partial\Omega \cap K$. Then the fixed point index $i_K(T, \Omega_K)$ has the following properties:

1. *If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \mu e$ for all $x \in \partial\Omega_K$ and all $\mu > 0$, then $i_K(T, \Omega_K) = 0$.*

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2. *If $x \neq \mu Tx$ for all $x \in \partial\Omega_K$ and for every $\mu \leq 1$, then $i_K(T, \Omega_K) = 1$.*
3. *If $i_K(T, \Omega_K) \neq 0$, then T has a fixed point in Ω_K.*

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2. *If $x \neq \mu Tx$ for all $x \in \partial\Omega_K$ and for every $\mu \leq 1$, then $i_K(T, \Omega_K) = 1$.*
3. *If $i_K(T, \Omega_K) \neq 0$, then T has a fixed point in Ω_K.*
4. *Let Ω^1 be an open set with $\overline{\Omega^1_K} \subset \Omega_K$. If $i_K(T, \Omega_K) = 1$ and $i_K(T, \Omega^1_K) = 0$, then T has a fixed point in $\Omega_K \setminus \overline{\Omega^1_K}$. The same result holds if $i_K(T, \Omega_K) = 0$ and $i_K(T, \Omega^1_K) = 1$.*

Existence of Positive Solutions

We introduce the following notations:

- $C = \max_{t \in I} \int_0^1 \left| \frac{\partial g_M}{\partial t}(t,s) \right| |a(s)|\, ds.$
- $\phi_\rho(t) := C\,\psi(\rho) + \sum_{i=1}^{4} \lambda_i \, \|L_i\| \, \|w_i'\|_\infty \, \rho + C\,\varphi(t) - t,\ t \geq 0,\ \rho > 0.$
- Let $\alpha_\rho > 0$ be the biggest zero of ϕ_ρ (we note that such zero exists since $\phi_\rho(0) \geq 0$ and $\lim_{t \to \infty} \phi_\rho(t) = -\infty$).

Existence of Positive Solutions

$$K_{\delta_M}^{a,b} := \left\{ u \in \mathcal{C}^1(I) : \; u(t) \geq 0 \text{ on } I, \text{ and } \min_{t\in[a,b]} u(t) \geq \delta_M^{a,b} \, \|u\|_\infty \right\}$$

For any $[a,b] \subset I$, $\delta_M^{a,b}$ and $\rho > 0$, we introduce the following subsets:

$$\Omega_\rho = \left\{ u \in K_{\delta_M}^{a,b} : \min_{t\in[a,b]} u(t) < \delta_M^{a,b} \rho, \; \|u'\|_\infty < \alpha_\rho + 1 \right\},$$

$$K_\rho = \left\{ u \in K_{\delta_M}^{a,b} : \|u\|_\infty < \rho, \; \|u'\|_\infty < \alpha_\rho + 1 \right\}.$$

Lemma

The following properties hold for any $\rho > 0$:

(a) Ω_ρ *is open relative to* $K_{\delta_M}^{a,b}$.

(b) $K_{\delta_M^{a,b}\rho} \subset \Omega_\rho \subset K_\rho$.

(c) $u \in \partial\Omega_\rho$ *if and only if* $\min_{t\in[a,b]} u(t) = \delta_M^{a,b} \rho$.

(d) *If* $u \in \partial\Omega_\rho$, *then* $\delta_M^{a,b} \rho \leq u$ *on* $[a,b]$ *and* $u \leq \rho$ *on* I.

Existence of Positive Solutions

Let

$$f^\rho = \sup\left\{\frac{f(t,u,v)}{\rho} : (t,u,v) \in [0,1]\times[0,\rho]\times[-\alpha_\rho,\alpha_\rho]\right\}.$$

Lemma

Assume (H_0)–(H_3), $M \in (-m_1^4, 4\,\pi^4]$, *and* $\sum_{i=1}^{4} \lambda_i \|w_i\|_\infty \|L_i\| < 1$. *If there exists* $\rho > 0$ *such that*

$$f^\rho < D_1,$$

with

$$D_1 = \left\{\frac{R_M}{\left(1 - \sum_{i=1}^4 \lambda_i \|w_i\|_\infty \|L_i\|\right)} \int_0^1 z_M(s)\,ds\right\}^{-1},$$

then the fixed point index, $i_{K^{a,b}_{\delta_M}}(T_1, K_\rho) = 1$.

Existence of Positive Solutions

Let

$$f_\rho = \inf\left\{\frac{f(t,u,v)}{\rho} : (t,u,v) \in [a,b] \times \left[\delta_M^{a,b}\rho, \rho\right] \times [-\alpha_\rho, \alpha_\rho]\right\}.$$

Lemma

Assume that (H_0)–(H_3) *hold,* $M \in (-m_1^4, 4\,\pi^4]$, *and there exists* $\rho > 0$ *such that*

$$f_\rho > d_1,$$

with

$$d_1 = \left\{\frac{\gamma_M^{a,b}}{R_M}\int_a^b z_M(s)\,ds\right\}^{-1}.$$

Then the fixed point index, $i_{K^{a,b}_{\delta_M}}(T_1, \Omega_\rho) = 0$.

Existence of Positive Solutions

We deduce the following existence result.

Theorem

Assume (H_0)–(H_3), $M \in (-m_1^4, 4\,\pi^4]$, *and* $\sum_{i=1}^{4} \lambda_i \, \|w_i\|_\infty \, \|L_i\| < 1$.

If there exist $\rho_1,\ \rho_2 > 0$ *such that* **one of the following conditions holds***:*

- $\dfrac{\rho_1}{\delta_M^{a,b}} < \rho_2$, $f^{\rho_2} < D_1$ *and* $f_{\rho_1} > d_1$,
- $\rho_2 < \rho_1$, $f^{\rho_2} < D_1$ *and* $f_{\rho_1} > d_1$,

then problem (P) *has at least one positive solution on* I*.*

Approach 2: Non Trivial Solutions

Now, we will present an alternative approach to prove the existence of solutions of problem (P).

- We will use Schauder's fixed point theorem to prove the existence of nontrivial solutions.
- We will give a sufficient condition for the solutions to be positive.

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- We will use Schauder's fixed point theorem to prove the existence of nontrivial solutions.
- We will give a sufficient condition for the solutions to be positive.

We will use a particular case of a formula given for the general Green's function related to any n-th order nonlocal linear problem and proved on the following work:

Cabada, A., L-S, L., Yousfi, M., 'Existence of solutions of nonlinear systems subject to arbitrary linear non-local boundary condition', *J. Fixed Point Theory Appl.* 25 (2023), no. 4, Paper No. 81, 24 pp.

Expression of the Green's Function

Theorem

Let $\sigma \in C(I)$, $M \in \mathbb{R}$, *be such that condition* (SC) *does not hold, and* $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, 4$, *be such that* $\det(I - A) \neq 0$, *with* I *the identity matrix of dimension four and* $A = (a_{ij})_{4\times 4}$ *given by* $a_{ij} = \lambda_j\, L_i(w_j)$.

Then the nonlocal fourth order linear problem

$$\begin{cases} u^{(4)}(t) + M u(t) = \sigma(t), \quad t \in I, \\ u(0) = \lambda_1 L_1(u),\ u(1) = \lambda_2 L_2(u),\ u'(0) = \lambda_3\, L_3(u),\ u'(1) = -\lambda_4\, L_4(u), \end{cases}$$

has a unique solution given by $u_M(t) = \int_0^1 G_M(t,s)\,\sigma(s)\,ds$, *where*

$$G_M(t,s) = g_M(t,s) + \sum_{i=1}^{4}\sum_{j=1}^{4} \lambda_i\, b_{ij}\, w_i(t)\, L_j(g_M(\cdot,s))$$

with $B = (b_{ij})_{4\times 4} = (I - A)^{-1}$.

Construction of the Operator

We define operator $T_2 : \mathcal{C}^1(I) \to \mathcal{C}^1(I)$:

$$\begin{aligned} T_2 u(t) &= \int_0^1 G_M(t,s) f(s,u(s),u'(s))\, ds \\ &= \int_0^1 g_M(t,s) f(s,u(s),u'(s))\, ds \\ &\quad + \sum_{i=1}^4 \sum_{j=1}^4 \lambda_i\, b_{ij}\, w_i(t) \int_0^1 L_j(g_M(\cdot,s))\, f(s,u(s),u'(s))\, ds. \end{aligned}$$

- Fixed points of T_2 coincide with the solutions of problem (P).

Hypotheses

We assume the following conditions:

$(\widetilde{H_0})$ $f : I \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is a continuous function.

$(\widetilde{H_1})$ There exist a strictly positive function $a \in C(I)$, and two continuous and increasing functions ψ, $\varphi : [0, \infty) \to [0, \infty)$, such that

$$f(t, u, v) \leq a(t) \left(\psi(|u|) + \varphi(|v|)\right) \text{ for all } (t, u, v) \in I \times \mathbb{R} \times \mathbb{R}.$$

Nontrivial Solutions

Notations:

- $\widetilde{C} = \max_{t \in I} \int_0^1 |g_M(t,s)|\, |a(s)|\, ds.$
- $K_1 = \widetilde{C} + \sum_{i=1}^4 \sum_{j=1}^4 |\lambda_i|\, |b_{ij}|\, \|w_i\|_\infty \int_0^1 |L_j(g_M(\cdot, s))|\, |a(s)|\, ds.$
- $K_2 = C + \sum_{i=1}^4 \sum_{j=1}^4 |\lambda_i|\, |b_{ij}|\, \|w_i'\|_\infty \int_0^1 |L_j(g_M(\cdot, s))|\, |a(s)|\, ds.$
- $K_3 = \max\{K_1,\, K_2\}.$

Theorem

Assume that $(\widetilde{H_0})$– $(\widetilde{H_1})$ *hold,* $M \in \mathbb{R}$ *such that* (SC) *does not hold,* $\det(I - A) \neq 0$, $f(t,0,0) \not\equiv 0$ *on* $[0,1]$, *and*

$$\lim_{t \to \infty} \frac{\psi(t) + \phi(t)}{t} \leq \frac{1}{K_3}.$$

Then problem (P) *has at least one non trivial solution.*

Nontrivial Solutions

Proof.

After obtaining suitable inequalities, we deduce that for all $\rho > 0$, if $\|u\|_\infty \leq \rho$ and $\|u'\|_\infty \leq \rho$, then

$$\|T_2 u\|_\infty \leq K_1 \left(\psi(\rho) + \varphi(\rho)\right), \quad \text{and} \quad \|(T_2 u)'\|_\infty \leq K_2 \left(\psi(\rho) + \varphi(\rho)\right).$$

Moreover, we have that there exists some $\rho_0 > 0$ such that for all $\rho > \rho_0$,

$$\frac{\psi(\rho) + \varphi(\rho)}{\rho} < \frac{1}{K_3}.$$

As a consequence, if $\rho > \rho_0$, then $T_2(B[0, \rho]) \subset B[0, \rho]$.

Therefore, from **Schauder's fixed point theorem**, taking into account that $f(t, 0, 0) \not\equiv 0$ on $[0, 1]$, we deduce that T_2 has a nontrivial fixed point, that is, problem (P) has at least one nontrivial solution.

Positive Solutions

As a consequence, we obtain the following result to ensure the existence of a positive solution.

Corollary

Under the hypotheses of previous theorem, if $M \in (-m_1^4, 4\,\pi^4]$ *and, moreover,*

- $\lambda_i \geq 0$, $i = 1, \ldots, 4$,
- $L_i(g_M(\cdot, s)) \geq 0$ *for a. e.* $s \in [0, 1]$, $i = 1, \ldots, 4$,
- $\rho(A) < 1$,

then problem (P) *has at least one* **positive solution** *on* $(0, 1)$.

Examples

Example

$$\begin{cases} u^{(4)}(t) = t\left((u(t))^2 + 0.5\sqrt{|u'(t)|}\right), \quad t \in I, \\ u(0) = 0.25\int_0^1 u(s)ds, \ u(1) = 0.5\int_0^1 u(s)ds, \\ u'(0) = 0.25\int_0^1 u(s)ds, \ u'(1) = -0.5\int_0^1 u(s)ds. \end{cases}$$

- $M = 0, \ \lambda_1 = 0.25, \ \lambda_2 = 0.5, \ \lambda_3 = 0.25, \ \lambda_4 = 0.5.$
- $L_i(u) = \int_0^1 u(s)\, ds$ for $i = 1, \ldots, 4$.
- $f(t,u,v) = t\left(u^2 + 0.5\sqrt{|v|}\right)$.

Examples

Example

$$C_1 = 142.248 \qquad c_1 = 2540.36,$$

$$f_{\rho_1} = 0.00494385\,\rho_1,$$

$$f^{\rho_2} = \rho_2 + \frac{\sqrt{\alpha_{\rho_2}}}{2\,\rho_2}.$$

If $\rho_2 = 40$ and $\rho_1 = 600000$, then $\alpha_{\rho_2} = 41.2919$.

Thus, $\rho_2 < \rho_1$, $f^{\rho_2} = 40.0803 < C_1$, and $f_{\rho_1} = 2966.31 > c_1$.

As a consequence **this problem has at least one positive solution**.

Examples

Example

$$\begin{cases} u^{(4)}(t) = (1+t)^2 \left(\sqrt{|u(t)|} + \dfrac{4\,(u'(t)+1)^2}{|u'(t)|+1} \right), \quad t \in I, \\ u(0) = 0.25 \displaystyle\int_0^1 u(s)\, s ds,\ u(1) = 0.5 \displaystyle\int_0^1 u(s)\, s^2 ds, \\ u'(0) = 0.75 \displaystyle\int_0^1 u(s)\, s^3 ds,\ u'(1) = -\displaystyle\int_0^1 u(s)\, s^4 ds. \end{cases}$$

- $M = 0,\ \lambda_1 = 0.25,\ \lambda_2 = 0.5,\ \lambda_3 = 0.75,\ \lambda_4 = 1.$
- $L_i(u) = \int_0^1 u(s)\, s^i\, ds$ for $i = 1, \ldots, 4.$
- $f(t,u,v) = (1+t)^2 \left(\sqrt{|u|} + \frac{4\,(v+1)^2}{|v|+1} \right).$

Examples

Example

By simple calculus, we have

$$\widetilde{C} \approx 0.0248669,\ C \approx 0.20846,\ K_1 \approx 0.0260751,\ K_2 \approx 0.211064,\ K_3 \approx 0.211064.$$

By defining $\psi(t) = \sqrt{|t|}$ and $\phi(t) = \frac{4\,(t+1)^2}{|t|+1}$, we have

$$\lim_{t\to\infty} \frac{\psi(t)+\phi(t)}{t} = 4 \le \frac{1}{K_3} \approx 4.7378.$$

Then the considered problem **has at least one non trivial solution**.

THANKS FOR YOUR ATTENTION