Fractional harmonic maps in homotopy classes

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Joint work with

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University of Pittsburgh

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Lemma

$u \in W^{s,\frac{n}{s}}(\Sigma,\mathcal{N})$, $\exists\, \varepsilon = \varepsilon(u) > 0$ such that if $g_0, g_1 \in C^0 \cap W^{s,\frac{n}{s}}(\Sigma,\mathcal{N})$ with

$\|u - g_i\|_{L^1(\Sigma)} + [u - g_i]_{W^{s,\frac{n}{s}}(\Sigma)} \le \varepsilon$ then $g_1 \sim g_2$

Main Result

Theorem (Sacks, Uhlenbeck) Assume $\dim \Sigma = 2$

- $\pi_2(\mathcal{N}) = \{0\} \Rightarrow \exists$ minimizing harmonic map in every homotopy class $C^0(\Sigma, \mathcal{N})$
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This is a technical assumption, used only in the regularity theory

It should be possible to extend to $n = 1$, $s \in (0,1)$

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Let $\theta \in \mathbb{S}^n$ and take $u_\lambda(\theta) := u\left(\tau(\lambda\tau^{-1}(\theta))\right), \quad u_\lambda$ belongs to the same homotopy class as u

What is the problem?

Loss of compactness:

Take $u \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$ – minimizer in its *nontrivial* homotopy class

The energy $E_{s,\frac{n}{s}}$ is conformally and scaling invariant

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Uniform regularity

Theorem

Assume

- $u \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ is a minimizing $W^{t,\frac{n}{s}}$-harmonic map in $B(R)$, i.e.,

$$E_{t,\frac{n}{s}}(u, \Sigma) \leq E_{t,\frac{n}{s}}(v, \Sigma) \quad \forall v \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N}), \quad \text{such that:}$$

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Geodesic ball in Σ

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Then

$\exists \varepsilon > 0,\ s_0 > s$ such that $\forall t \in [s, s_0]$ we have

$$u \in W^{s_0,\frac{n}{s}}(B(R/2)) \cap C^{s_0 - s}(B(R/2)) \quad + \text{Estimate}$$

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Recall: Morrey embedding $W^{t,\frac{n}{s}} \subset C^{0,t-s}$

But $W^{t,\frac{n}{s}} \not\subset C^{0,s_0-s}$

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On the smallness condition: The condition

$$[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(R))} = \int_{B(R)}\int_{B(R)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} < \varepsilon^{\frac{n}{s}}$$

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UNIFORM REGULARITY

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Recall: Morrey embedding $W^{t,\frac{n}{s}} \subset C^{0,t-s}$

But $W^{t,\frac{n}{s}} \not\subset C^{0,s_0-s}$

Roughly speaking the proof follows classical result of Morrey for minimizing harmonic maps

Uniform regularity

On the smallness condition: The condition

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The regularity of *non-minimizing* $W^{s,\frac{n}{s}}$-harmonic maps is a major open problem (even in the local case of $W^{1,n}$-harmonic maps)

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Roughly speaking the proof follows classical result of Morrey for minimizing harmonic maps

Corollary: Limits of minimizers

Corollary

Let

$u_t \colon \Sigma \to \mathcal{N}$ - minimizing $W^{t,\frac{n}{s}}$-harmonic maps in a fixed homotopy class

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and u_s is a $W^{s,\frac{n}{s}}$-minimizing harmonic map in $\Sigma \setminus A$ in its homotopy class

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We need to study what happens in the points when the convergence fails!

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We need to study what happens in the points when the convergence fails!

But first let's check what does this mean

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It is quick to prove such a theorem for round target manifolds (using regularity theory)

Removability of singularities — continued

We need to construct a comparison map

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Problem?

Removability of singularities — continued

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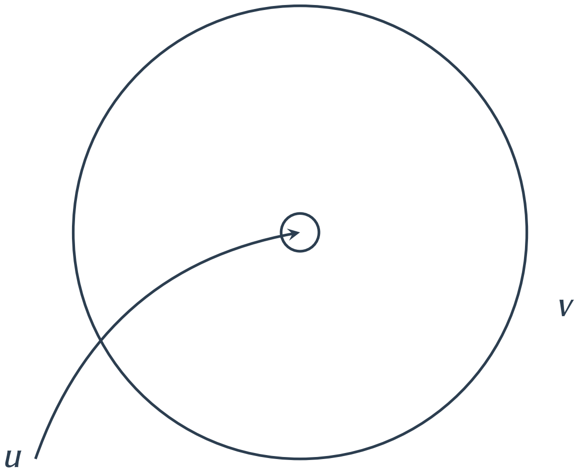

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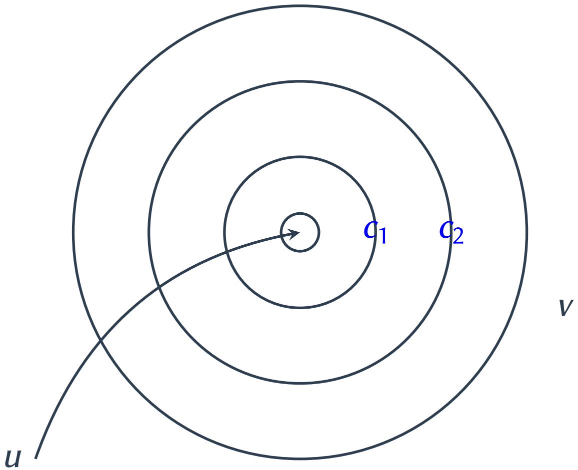

$c_1 \in \mathcal{N}$, $c_2 \in \mathcal{N}$

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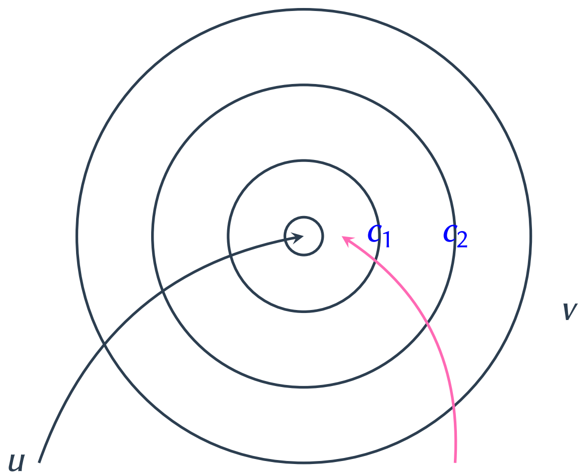

Opening technique by Brezis–Li connects $u \in W^{s,\frac{n}{s}}$ to a constant

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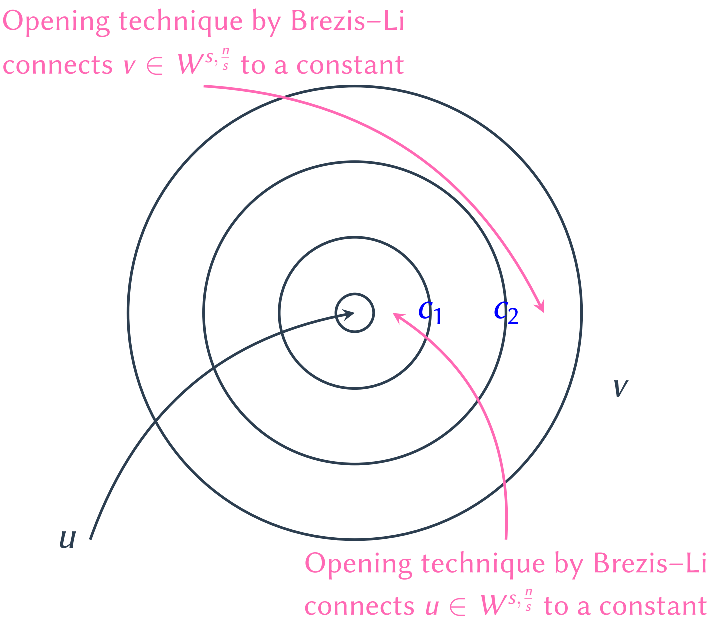

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Removability of singularities — continued

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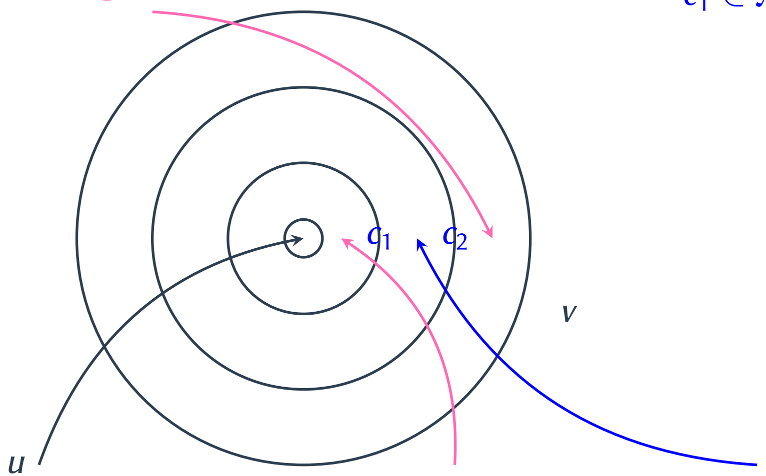
Opening technique by Brezis–Li connects $v \in W^{s,\frac{n}{s}}$ to a constant

$c_1 \in \mathcal{N}$, $c_2 \in \mathcal{N}$



Connecting points in critical Sobolev space is cheap (energywise)

Opening technique by Brezis–Li connects $u \in W^{s,\frac{n}{s}}$ to a constant

Removability of singularities — continued

We need to construct a comparison map

mixed terms

Problem? Yes

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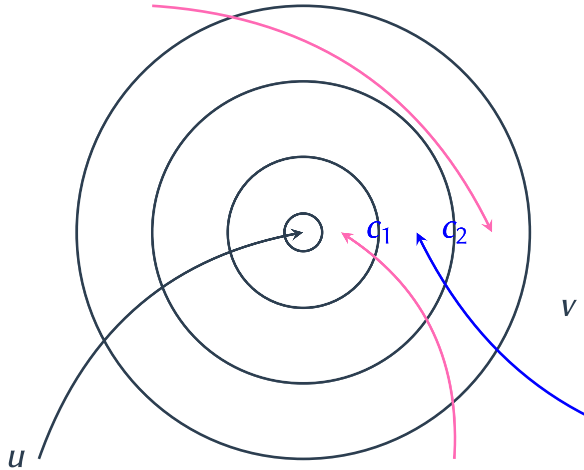

Glueing along a buffer zone, $\eta \in (0,1)$

$$E_{s,p}(u,\Sigma) \leq \left(1+\frac{C}{(1-\eta)^{sp+1}}\right) E_{s,p}(u,\Sigma\cap B(r)) + \left(1+\frac{C\eta^n}{1-\eta}\right) E_{s,p}(u,\Sigma\setminus\overline{B(\eta r)})$$

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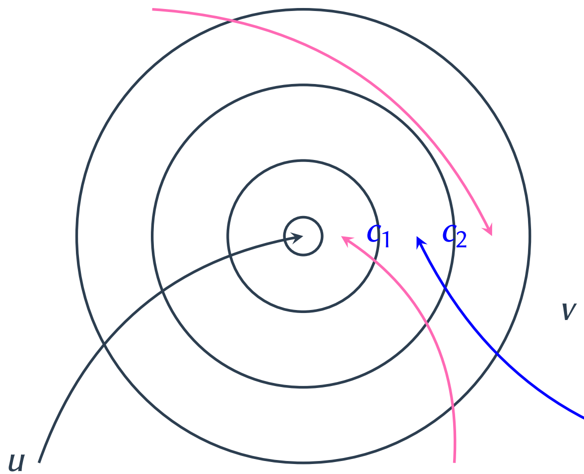
C independent of Σ, r, η



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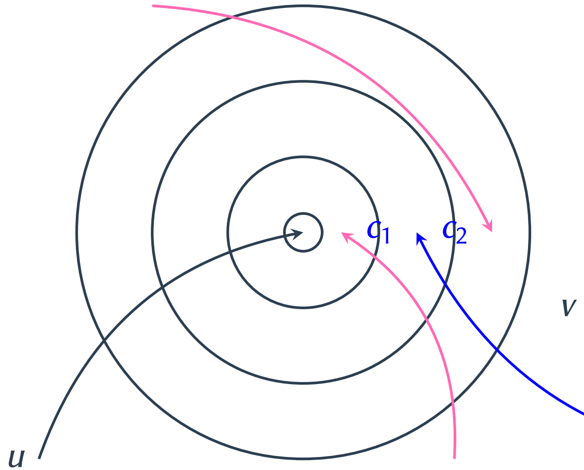

Sets $\Sigma\cap B(r)$, $\Sigma\setminus\overline{B(\eta r)}$ overlap

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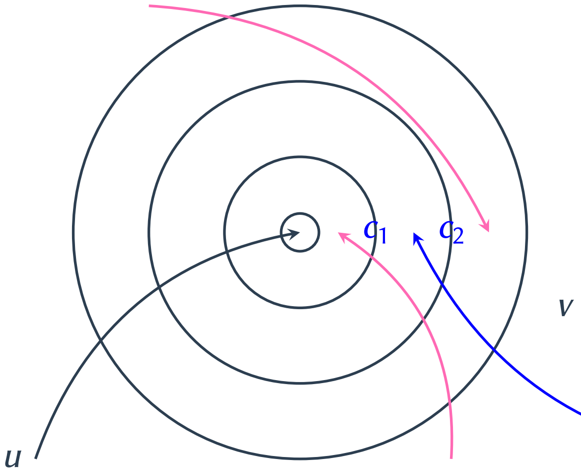

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Construction similar to the one by Monteil–Van Schaftingen

Balanced energy estimates for $\Sigma = \mathbb{S}^n$

Theorem

- $0 < s < s_0 < 1$
- $\rho \in (0, \sqrt{\frac{4}{5}})$
- $t \in (s, s_0]$
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As a consequence we get that the energy of our approximate map cannot concentrate only in a single point and vanish everywhere else

Balanced energy estimates — idea of proof

Consider the simple case $\Sigma = \mathbb{R}^n$

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$$
\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v_\lambda(x)-v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \lambda^{n\left(\frac{t-s}{s}\right)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y
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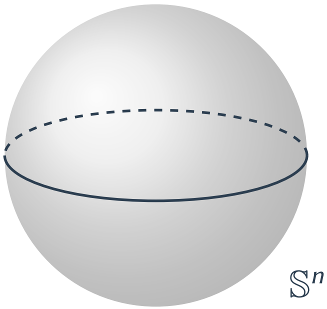

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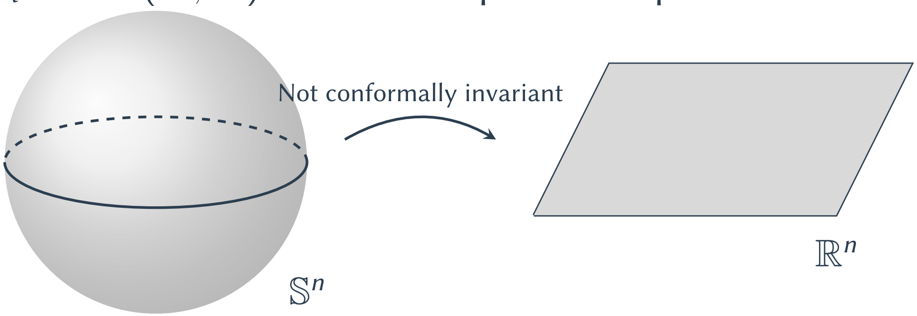

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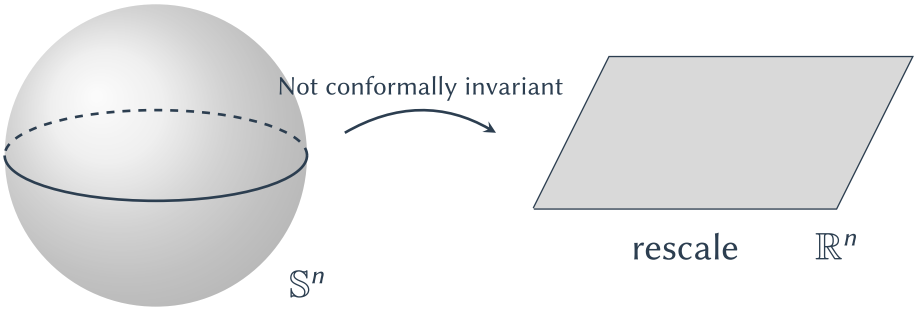
Which is only possible if $v = const$

For $u_t \in W^{t,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$ consider comparison map constructed as



Balanced energy estimates — idea of proof

Consider the simple case $\Sigma = \mathbb{R}^n$

Let $v \in W^{t,\frac{n}{s}}(\mathbb{R}^n, \mathcal{N})$ be minimizing in its homotopy class

Since $t > s$ the energy $E_{t,\frac{n}{s}}$ is not scaling invariant

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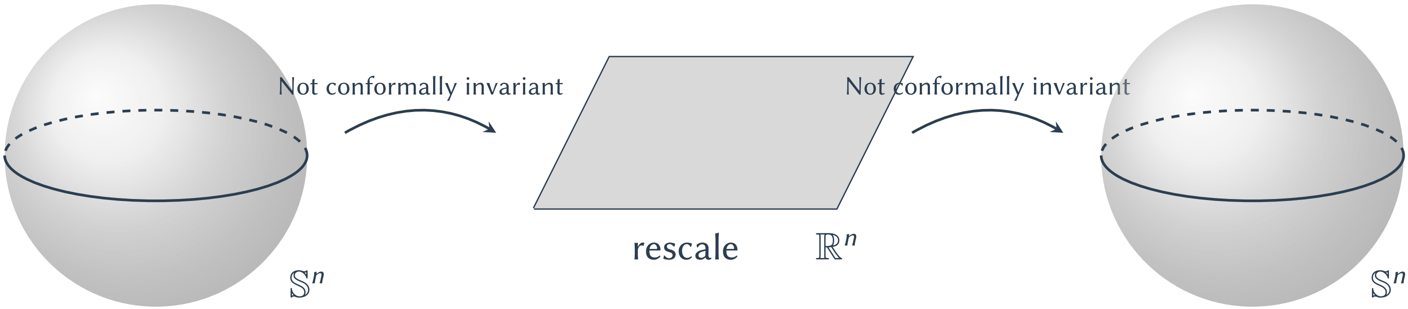

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Fix a homotopy class X in $C^0(\Sigma, \mathcal{N})$

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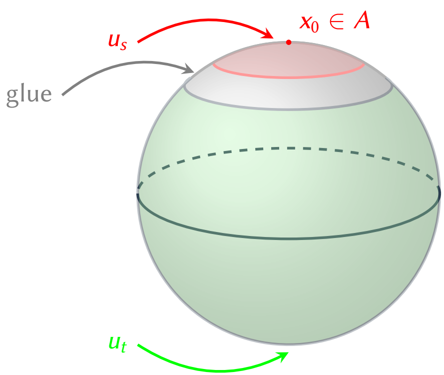

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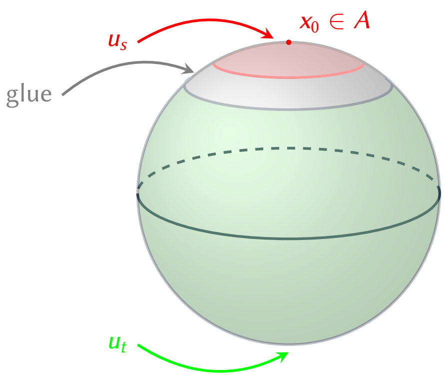

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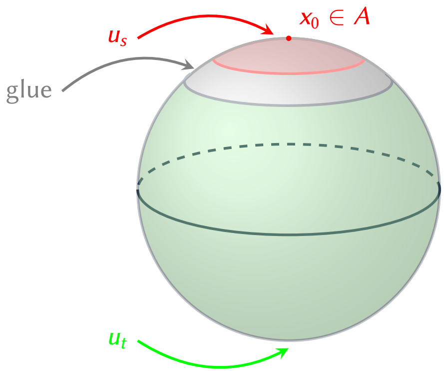

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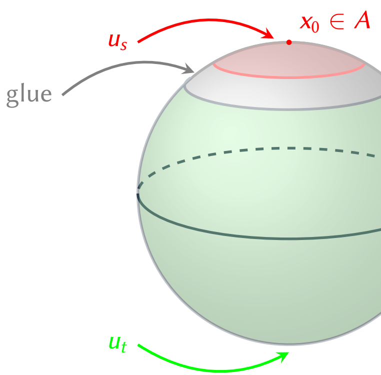

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From regularity theory we have strong convergence in x_0

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where $\quad \#\Gamma := \inf_{u \in \Gamma \cap W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})} E_{s,\frac{n}{s}}(u, \mathbb{S}^n) = \lim_{t \to s^+} \inf_{u \in \Gamma \cap W^{t,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})} E_{t,\frac{n}{s}}(u, \mathbb{S}^n)$

Corollary

There exists a number $k \in \mathbb{Z}$, $k \neq 0$ such that

$$\inf \left\{ E_{s,\frac{n}{s}}(u, \mathbb{S}^n) : \; u \in C^0 \cap W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^n), \; \deg u = k \right\}$$

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Yes, if $\frac{n}{s} = 2$ (Mironescu)

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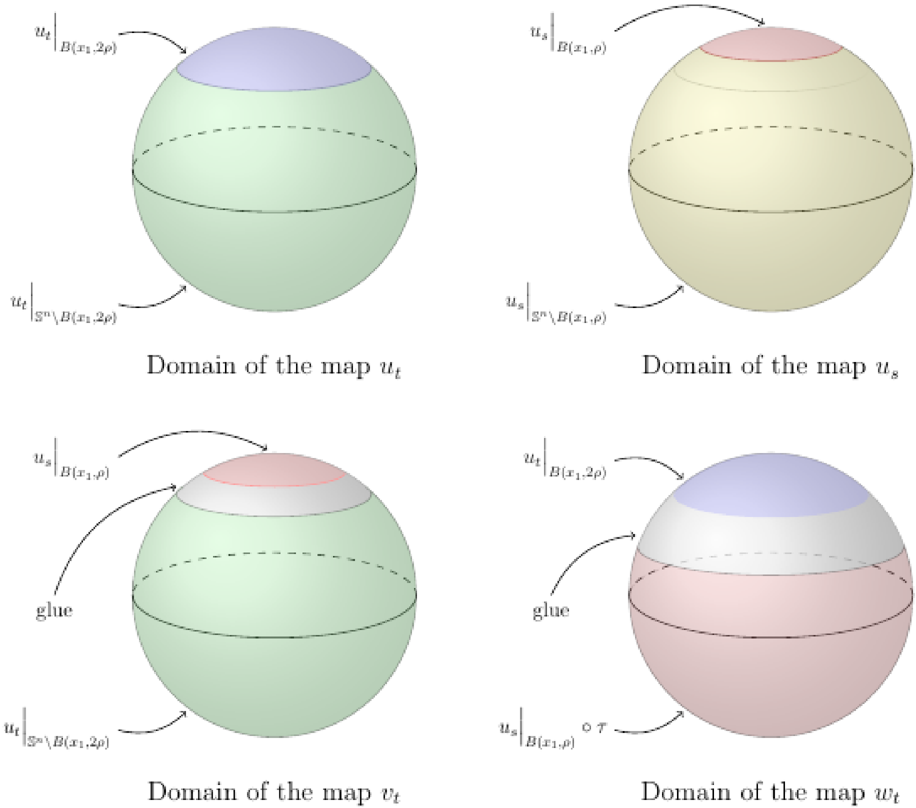

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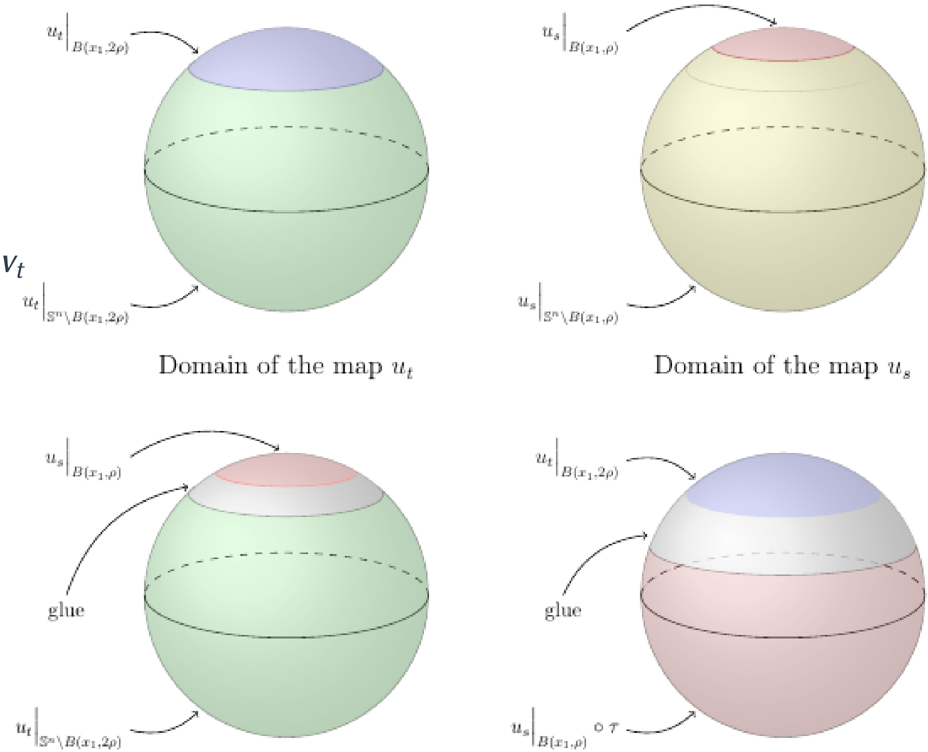
Γ_2 - homotopy class of w_t



Domain of the map u_t — Domain of the map u_s

Domain of the map v_t — Domain of the map w_t

Idea of the proof, $\pi_n(\mathcal{N}) \neq 0$

A - set of points

Fix a homotopy class Γ_0 in $C^0(\mathbb{S}^n, \mathcal{N})$ take u_t - sequence of minimizers of $E_{t,\frac{n}{s}}$ in Γ_0

From regularity: $\exists u_s \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$ such that $u_t \to u_s$ strongly in $W^{s_0,\frac{n}{s}}(\mathbb{S}^n \setminus A)$,

From removability: $u_s \in W^{s_0,\frac{n}{s}}(\mathbb{S}^n)$

Either A is empty (first case)

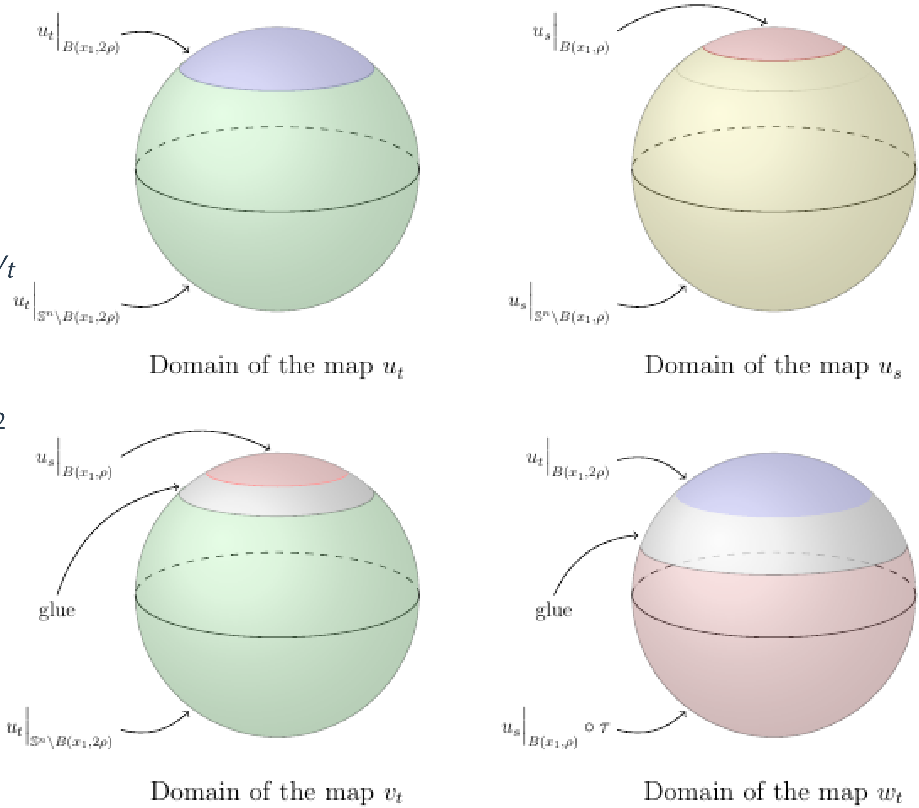
Or there is a blow-up point

Say, the North Pole

Then if Γ_1 - homotopy class of v_t

Γ_2 - homotopy class of w_t

$\pi_1(\mathcal{N})\gamma_0 \subset \pi_1(\mathcal{N})\gamma_1 + \pi_1(\mathcal{N})\gamma_2$



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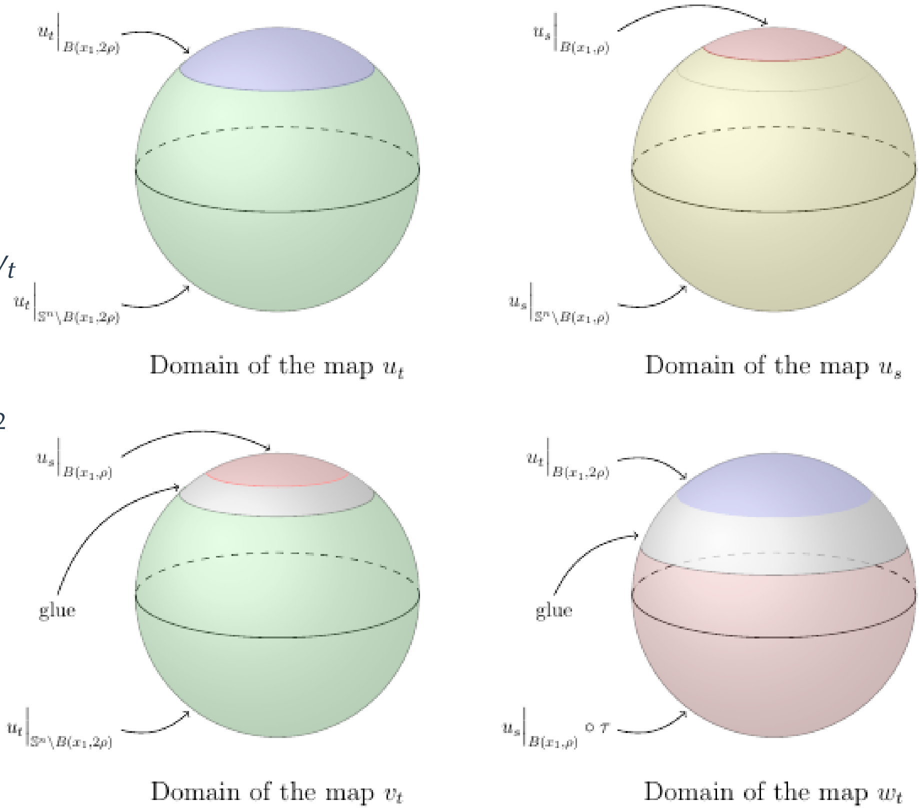
Careful application

of previous results

give the estimates



Thank you for your attention!