Concentration phenomenon in aggregation-diffusion equations

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Presentation based on joint work with
G. Karch (Wrocław) and A. Lanar (Lyon).

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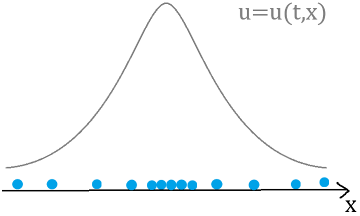

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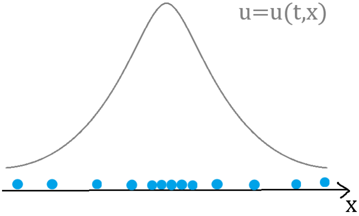

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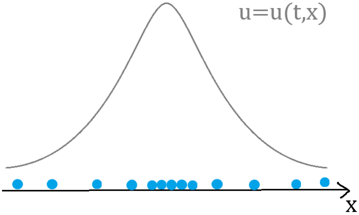

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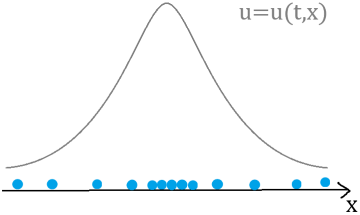

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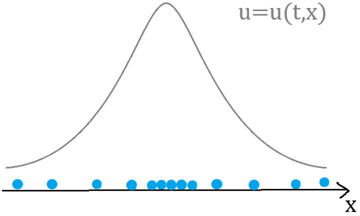

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*In fact, it is not always aggregation.

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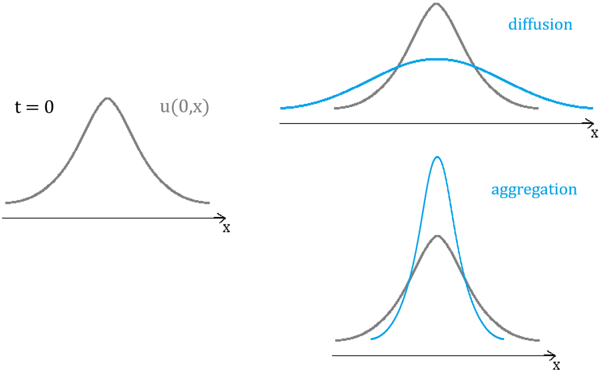

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- $k = 0$, $W_k(x) = \log(x)$ (critical case) - chemotaxis Keller-Segel model in $\mathbb{R}^2$

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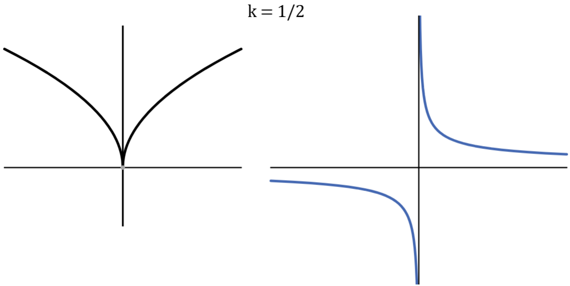

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- Local-in-time solutions

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$$\|\nabla W_k * v\|_q \leq C_{k,p} \|v\|_p, \quad \frac{1}{p} + \frac{1-k}{d} = 1 + \frac{1}{q}$$

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- Mass conservation

$$\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x$$

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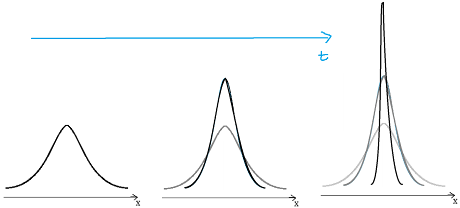

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- Can we describe the same effect for $k \in (0, 1)$ when $\varepsilon \to 0$?

Concentration around the origin

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then there exist numbers $T_*, C_*, \varepsilon_*, \nu>0$, independent of ε, such that for all $\varepsilon\in(0,\varepsilon_*)$ we have

$$\int_0^{T_*}\int_{B_{(\nu\varepsilon)^{1/k}}}u_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t\geq C_*.$$

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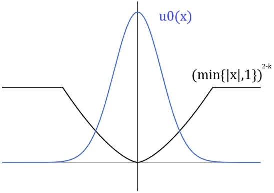

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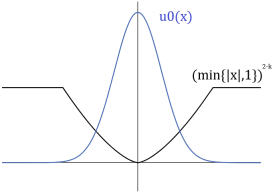

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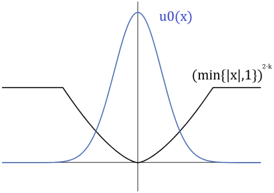

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- Initial condition has to be sufficiently concentrated around the origin.

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$$\sup_{t\in[0,T_*]} \|u_\varepsilon(t)\|_\infty \to +\infty \quad \text{when} \quad \varepsilon \to 0.$$

A priori estimates for L^p-norms

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- $d \geq 1$, $k \in (0,1)$ and $p \geq p_k$ $(2 \text{ or } 1/k)$

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- $d \geq 1$, $k \in (0,1)$ and $p \geq p_k$ (2 or $1/k$)
- u_ε - non-negative solution with $\varepsilon > 0$ and $u_0 \in L^1 \cap L^p(\mathbb{R}^d)$

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$$\|u_\varepsilon(t)\|_p \leq \max\left\{\|u_0\|_p, C(k,p)M^{1+\frac{1}{\eta}}\varepsilon^{-\frac{1}{\eta}}\right\},$$

where $\eta = kp/(d(p-1))$

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where

$$\frac{1}{s} + \frac{1}{q} + \frac{1-k}{d} = \frac{3}{2}.$$

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$$\|\nabla u^{p/2}\|_2 \geq CM^{(\beta-1)p/d}\|u\|_p^{(d/2+1-\beta)p/d}.$$

A priori estimates - proof (3)

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p \leq \frac{4}{p^2}\|\nabla u^{p/2}\|_2 \times \Big(-\varepsilon C_1 M^{(\beta-1)p/d}\|u\|_p^{(d/2+1-\beta)p/d} + C_2 M^{(1-\alpha)(1-\beta)p/2+1-\gamma}\|u\|_p^{(\alpha+\beta-\alpha\beta)p/2+\gamma} \Big)$$

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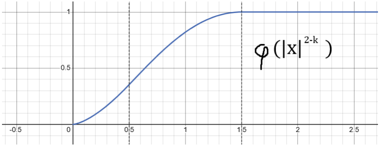

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$$I'(t) = \int_{\mathbb{R}^d} \varepsilon \Delta u \varphi(|x|^{2-k})\,\mathrm{d}x + \int_{\mathbb{R}^d} \nabla \cdot (u \nabla W_k * u) \varphi(|x|^{2-k})\,\mathrm{d}x$$

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$$\int_{\mathbb{R}^d} \varphi(|x|^{2-k})\Delta u(t,x)\,\mathrm{d}x = -(2-k)\int_{\mathbb{R}^d} \varphi'\left(|x|^{2-k}\right)\frac{x}{|x|^k}\cdot\nabla u(t,x)\,\mathrm{d}x$$

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$$= -\sigma_d (2-k) \int_0^{+\infty} \varphi'\left(r^{2-k}\right) r^{d-k} u_r(t,r) \,\mathrm{d}r$$

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$$\begin{aligned}
= \int_0^{+\infty} & \Big(\sigma_d(2-k)^2\varphi''\left(r^{2-k}\right) r^{d+1-2k} \\
& +\sigma_d(2-k)(d-k)\varphi'\left(r^{2-k}\right) r^{d-k-1}\Big)\, u(t,r)\,\mathrm{d}r \\
& - \sigma_d(2-k)\left[\varphi'\left(r^{2-k}\right) r^{d-k} u(t,r)\right]_{r=0}^{r=+\infty}
\end{aligned}$$

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$$= -\sigma_d(2-k)\int_0^{+\infty} \varphi'\left(r^{2-k}\right) r^{d-k} u_r(t,r)\,\mathrm{d}r$$

$$= \int_0^{+\infty} \left(\sigma_d(2-k)^2\varphi''\left(r^{2-k}\right) r^{d+1-2k} + \sigma_d(2-k)(d-k)\varphi'\left(r^{2-k}\right) r^{d-k-1}\right) u(t,r)\,\mathrm{d}r - \sigma_d(2-k)\left[\varphi'\left(r^{2-k}\right) r^{d-k} u(t,r)\right]_{r=0}^{r=+\infty}$$

$$\leq \sigma_d(2-k)(d-k)\int_0^{3/2} u(t,r) r^{d-k-1}\,\mathrm{d}r = \mathcal{D}(u)(t)$$

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$$-\sigma_d(2-k)\left[\varphi'\left(r^{2-k}\right) r^{d-k} u(t,r)\right]_{r=0}^{r=+\infty}$$

$$\leq \sigma_d(2-k)(d-k)\int_0^{3/2} u(t,r) r^{d-k-1}\,\mathrm{d}r = \mathcal{D}(u)(t)$$

or else

$$\mathcal{D}(u)(t) = (2-k)(d-k)\int_{B_{3/2}} \frac{u(t,x)}{|x|^k}\,\mathrm{d}x,$$

Differential inequality (5)

$$I'(t) \leq \varepsilon \mathcal{D}(u)(t) + \int_{\mathbb{R}^d} \nabla \cdot (u \nabla W_k * u) \varphi(|x|^{2-k}) \, \mathrm{d}x$$

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$$= -\frac{2-k}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(t,x) u(t,y) \Phi(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

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$$\Phi(x,y) + \Phi(x,-y) \geq 2^k, \quad (x,y) \in B_{1/2} \times B_{1/2}$$

Differential inequality (7)

$$J(t) = -\frac{2-k}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(t,x) u(t,y) \Phi(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

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$$= -\frac{2-k}{4} \int_{B_{1/2}} \int_{B_{1/2}} u(t,x)u(t,y)\left(\Phi(x,y) + \Phi(x,-y)\right)\, \mathrm{d}x\, \mathrm{d}y,$$

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$$J(t) = -\frac{2-k}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)u(t,y)\Phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\begin{aligned}
J_1(t) &= -\frac{2-k}{2}\int_{B_{1/2}}\int_{B_{1/2}} u(t,x)u(t,y)\Phi(x,y)\,\mathrm{d}x\,\mathrm{d}y,\\
&= -\frac{2-k}{4}\int_{B_{1/2}}\int_{B_{1/2}} u(t,x)u(t,y)\left(\Phi(x,y)+\Phi(x,-y)\right)\,\mathrm{d}x\,\mathrm{d}y,\\
&\leq -\frac{2-k}{2^{2-k}}\int_{B_{1/2}}\int_{B_{1/2}} u(t,x)u(t,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

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$$J_1(t) \leq -\frac{2-k}{2^{2-k}} \int_{B_{1/2}} \int_{B_{1/2}} u(t,x)u(t,y)\,\mathrm{d}x\,\mathrm{d}y.$$

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$$\begin{aligned} I'(t) &\le \varepsilon \mathcal{D}(u)(t) + J_1(t) + J(t) - J_1(t) \\ &\le \varepsilon \mathcal{D}(u)(t) - C_1 M^2 + C_2 M I(t) + |J(t) - J_1(t)| \end{aligned}$$

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$$\int_0^T \mathcal{D}(u_\varepsilon)(t) e^{-\omega_k Mt} \, \mathrm{d}t \geq \frac{L}{\varepsilon}$$

Main result (1)

$$\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} \frac{u(t,x)}{|x|^k} \, \mathrm{d}x \, \mathrm{d}t$$

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$$\|u_\varepsilon(t)\|_p \leq \max\left\{ \|u_0\|_p, C(k,p) M^{1+\frac{1}{\eta}} \varepsilon^{-\frac{1}{\eta}} \right\}$$

Main result (2)

$$\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} \frac{u(t,x)}{|x|^k} \,\mathrm{d}x\,\mathrm{d}t \leq C T^{\alpha} \nu^{\beta} \varepsilon^{-1} \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t \right)^{\gamma}$$

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$$\int_0^{T_*} \int_{B_{(\nu\varepsilon)^{1/k}}} u_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \ge C_*.$$

Additional comments

- moment is localized, so local behaviour of the kernel is sufficient i.e. $|x|^k$ only around the origin

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- possible generalization for full range of parameter $k \in (0, 2)$

Thank you

Thank you for your attention.