Two-parameters formulas for general solution to planar weakly delayed linear discrete systems with multiple delays, equivalent non-delayed systems, and conditional stability

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Abstract

Weakly delayed planar linear discrete systems with multiple delays

$$x(k+1) = Dx(k) + \sum_{l=1}^{n} H^l x(k - m_l), \quad k = 0, 1, \dots$$

are considered where $0 < m_1 < m_2 < \cdots < m_n$ are fixed integers, D, H^1, ..., H^n are nonzero 2×2 real constant matrices and $x\colon \{-m_n, -m_n + 1, \dots\} \to \mathbb{R}^2$.
Formulas for general solutions are found and simplified, equivalent non-delayed planar linear discrete systems are constructed and conditional stability is analyzed.

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Introduction

The notation used throughout the paper is the following: Θ is the zero 2×2 matrix, E is the unit 2×2 matrix, θ is the zero 2×1 vector and, for integers s, q , $s \leq q$, $\mathbb{Z}_s^q = \{s, s+1, \dots, q\}$. The set $\mathbb{Z}_s^\infty$ is defined in much the same way.

In the paper, weakly delayed (WD) discrete planar systems with multiple delays

$$x(k+1) = Dx(k) + \sum\nolimits_{l=1}^{n} H^l x(k - m_l) \tag{1}$$

are considered, where $m_1, m_2, \dots, m_n$ are fixed integer delays, $0 < m_1 < m_2 < \cdots < m_n$ (in the sequel, we will write m for short rather than m_n if this does not cause ambiguity), $k \in \mathbb{Z}_0^\infty$, $D, H^1, \ldots, H^n$ are constant 2×2 matrices, $D = \{d_{ij}\}_{i,j=1}^2$, $\det D \neq 0$, $H^l = \{h_{ij}^l\}_{i,j=1}^2 \neq \Theta$, $l = 1, 2, \dots, n$, $n \geq 1$, and $x \colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^2$.

The function x in (1) is called a solution if the equation holds for every $k \in \mathbb{Z}_0^\infty$. A unique solution of (1) can be determined by an initial problem

$$x(k) = \xi(k), \quad k = -m, -m+1, \ldots, 0 \tag{2}$$

with given function $\xi \colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^2$.

WD systems

The system (1) is a WD system if, for every $\mu \in \mathbb{C} \setminus \{0\}$,

$$\mathcal{D}(\mu) \equiv \det\left(D - \mu E\right), \tag{3}$$

where

$$\mathcal{D}(\mu) := \det\left(D + \sum\nolimits_{l=1}^{n} \mu^{-m_l} H^l - \mu E\right).$$

If we look for a solution of (1) in the form $x(k) = \chi\mu^k$, $k \in \mathbb{Z}^{\infty}_{-m}$, where $\mu = \mathrm{const}$, $\mu \neq 0$ and χ is a nonzero constant vector, then the characteristic equations $\mathcal{D}(\mu) = 0$ to system (1) and

$$x(k+1) = Dx(k)$$

are equivalent. Since the value $\mu = 0$ is excluded, throughout the paper we assume that no eigenvalue of D equals zero. This is guaranteed by the assumption $\det D \neq 0$. WD systems are invariant to nonsingular transformations.
From formula (3), the following criterion can be derived.

Theorem

System (1) *is a WD system if and only if*

$$h_{11}^{l} + h_{22}^{l} = 0, \quad \begin{vmatrix} h_{11}^{l} & h_{12}^{l} \\ h_{21}^{v} & h_{22}^{v} \end{vmatrix} = 0, \quad \begin{vmatrix} d_{11} & d_{12} \\ h_{21}^{l} & h_{22}^{l} \end{vmatrix} + \begin{vmatrix} h_{11}^{l} & h_{12}^{l} \\ d_{21} & d_{22} \end{vmatrix} = 0 \tag{4}$$

where $l, v = 1, 2, \ldots, n$.

Coefficient criteria for WD systems

Assume that, after a suitable transformation,

$$x(k) = \mathcal{T} y(k) \tag{5}$$

with 2×2 regular matrix $\mathcal{T}$, WD system (1) is transformed to a new WD system

$$y(k+1) = \mathcal{J}(D) y(k) + \sum_{l=1}^{n} H^{*l} y(k - m_l), \; k \in \mathbb{Z}_0^{\infty}, \tag{6}$$

where $\mathcal{J}(D) = \mathcal{T}^{-1} D \mathcal{T}$ is the Jordan form of D and $H^{*l} = \mathcal{T}^{-1} H^l \mathcal{T}$, $H^{*l} = \{h_{ij}^{*l}\}_{i,j=1}^{2}$, $l = 1, \dots, n$.
Initial data to (6) can be derived from (2) and (5) by the formula

$$y(k) = \eta(k) := \mathcal{T}^{-1} \xi(k), \quad k = -m, -m+1, \dots, 0. \tag{7}$$

In the following, we will use the notation

$$\omega_i^l(s) := h_{i1}^{*l} \eta_1(s) + h_{i2}^{*l} \eta_2(s), \tag{8}$$

where $i = 1, 2$, $s = -m, \dots, 0$, $l = 1, \dots, n$.

Coefficient criteria for detecting a WD system for every possible Jordan form can be deduced.
If the characteristic equation $\mathcal{D}(\mu) = 0$ has two real distinct roots μ_1, μ_2, then

$$\mathcal{J}(D) = \mathcal{J}_1 := \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},$$

in the case of one double real root μ, we have either

$$\mathcal{J}(D) = \mathcal{J}_2 := \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

if μ has geometrical multliplicity 2 or

$$\mathcal{J}(D) = \mathcal{J}_3 := \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$$

if μ has geometrical multliplicity 1, and, finally, if the roots are complex conjugate, i.e. $\mu_{1,2} = \alpha \pm i\beta$ with $\beta \neq 0$, then

$$\mathcal{J}(D) = \mathcal{J}_4 := \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

If $\mathcal{J}(D) = \mathcal{J}_1$, the necessary and sufficient conditions (4) for system (6) are reduced to

$$h_{11}^{*l} = h_{22}^{*l} = 0, \quad h_{12}^{*l} h_{21}^{*v} = 0, \tag{9}$$

where $l, v = 1, 2, \ldots, n$. Condition $H^l \neq \Theta$, $l = 1, 2, \ldots, n$ and (9) imply the following. If $h_{12}^{*l} = 0$ for a value l, then $h_{21}^{*v} \neq 0$ for all $v = 1, 2, \ldots, n$.

If $\mathcal{J}(D) = \mathcal{J}_2$, the necessary and sufficient conditions (4) for system (6) are reduced to

$$h_{11}^{*l} + h_{22}^{*l} = 0, \quad h_{11}^{*l} h_{22}^{*v} - h_{12}^{*l} h_{21}^{*v} = 0, \tag{10}$$

where $l, v = 1, 2, \ldots, n$.

If $\mathcal{J}(D) = \mathcal{J}_3$, the necessary and sufficient conditions (4) for system (6) are reduced to

$$h_{11}^{*l} = h_{22}^{*l} = h_{21}^{*l} = 0, \;\; l = 1, 2, \ldots, n. \tag{11}$$

Then, $h_{12}^{*l} \neq 0, \;\; l = 1, 2, \ldots, n$ since, in the opposite case, $H^l = \Theta$, $l = 1, 2, \ldots, n$, and this is a contradiction with our assumptions.

Let $\mathcal{J}(D) = \mathcal{J}_4$. Then, there exists no WD system (1). The necessary and sufficient conditions (4) imply $h_{11}^{*l} = h_{22}^{*l} = h_{21}^{*l} = h_{12}^{*l} = 0$,
$l = 1, 2, \ldots, n$.
This is a contradiction with the assumption $H^l \neq \Theta$, $l = 1, 2, \ldots, n$.

General Solutions of WD Systems

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_1$. Then, the problem (1), (2) has a solution

$$x(k) = \mathcal{T} y(k), \;\; y(k) = (y_1(k), y_2(k))^T, \;\; k \in \mathbb{Z}_{-m}^{\infty}$$

and, for $k \in \mathbb{Z}_0^{\infty}$,

$$y_1(k) = \mu_1^k \eta_1(0) + \frac{\eta_2(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{12}^{*l} \binom{k - m_l}{0} \left[\mu_1^{k-m_l} - \mu_2^{k-m_l} \right] + \sum_{l=1}^{n} h_{12}^{*l} \sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s) \binom{k - s}{0} \mu_1^{k-s}, \quad (12)$$

$$y_2(k) = \mu_2^k \eta_2(0) + \frac{\eta_1(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{21}^{*l} \binom{k - m_l}{0} \left[\mu_1^{k-m_l} - \mu_2^{k-m_l} \right] + \sum_{l=1}^{n} h_{21}^{*l} \sum_{s=1}^{m_l} \eta_1(-m_l - 1 + s) \binom{k - s}{0} \mu_2^{k-s}. \quad (13)$$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. Then, the problem (1), (2) *has a solution*

$$x(k) = \mathcal{T} y(k), \;\; y(k) = (y_1(k), y_2(k))^T, \;\; k \in \mathbb{Z}_{-m}^{\infty}$$

and, for $k \in \mathbb{Z}_0^{\infty}$,

$$y_1(k) = \mu^k \eta_1(0) + \sum_{l=1}^{n} \left[\sum_{s=1}^{m_l} \omega_1^l(-m_l - 1 + s) \binom{k-s}{0} \mu^{k-s} \right] + \sum_{l=1}^{n} (k - m_l) \binom{k-m_l}{0} \omega_1^l(0) \mu^{k-m_l-1}, \quad (14)$$

$$y_2(k) = \mu^k \eta_2(0) + \sum_{l=1}^{n} \left[\sum_{s=1}^{m_l} \omega_2^l(-m_l - 1 + s) \binom{k-s}{0} \mu^{k-s} \right] + \sum_{l=1}^{n} (k - m_l) \binom{k-m_l}{0} \omega_2^l(0) \mu^{k-m_l-1}. \quad (15)$$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_3$. Then, the problem (1), (2) has a solution

$$x(k) = \mathcal{T} y(k), \;\; y(k) = (y_1(k), y_2(k))^T, \;\; k \in \mathbb{Z}_{-m}^{\infty}$$

and, for $k \in \mathbb{Z}_0^{\infty}$,

$$\begin{aligned} y_1(k) = {} & \mu^k \eta_1(0) + k\mu^{k-1}\eta_2(0) \\ & + \sum\nolimits_{l=1}^{n} \left[\sum\nolimits_{s=1}^{m_l} \omega_1^l(-m_l - 1 + s) \binom{k-s}{0} \mu^{k-s} \right] \\ & + \sum\nolimits_{l=1}^{n} (k - m_l) \binom{k-m_l}{0} \omega_1^l(0) \mu^{k-m_l-1} \end{aligned} \quad (16)$$

and

$$y_2(k) = \mu^k \eta_2(0). \quad (17)$$

Simple Formulas for General Solutions

The case $\mathcal{J}(D) = \mathcal{J}_1$

$$C_1 := \eta_1(0) + \frac{\eta_2(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{12}^{*l} \mu_1^{-m_l} + \sum_{l=1}^{n} h_{12}^{*l} \sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s) \mu_1^{-s}, \tag{18}$$

$$C_2 := \eta_2(0) - \frac{\eta_1(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{21}^{*l} \mu_2^{-m_l} + \sum_{l=1}^{n} h_{21}^{*l} \sum_{s=1}^{m_l} \eta_1(-m_l - 1 + s) \mu_2^{-s}. \tag{19}$$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_1$. Then, the problem (1), (2) *has a solution* $x(k) = \mathcal{T} y(k)$, $y(k) = (y_1(k), y_2(k))^T$, $k \in \mathbb{Z}_{-m}^{\infty}$ *and, for* $k \in \mathbb{Z}_{m}^{\infty}$,

$$y_1(k) = C_1\mu_1^k - C_2\frac{\mu_2^k}{\mu_1 - \mu_2}\sum_{l=1}^{n} h_{12}^{*l}\mu_2^{-m_l}, \tag{20}$$

$$y_2(k) = C_1\frac{\mu_1^k}{\mu_1 - \mu_2}\sum_{l=1}^{n} h_{21}^{*l}\mu_1^{-m_l} + C_2\mu_2^k, \tag{21}$$

where C_1, C_2 *are arbitrary parameters connected with arbitrary initial values* (2) *through formulas* (7), (18), (19).

The case $\mathcal{J}(D) = \mathcal{J}_2$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. Assume $h_{11}^{*1} \neq 0$. Then, the problem (1), (2) has a solution $x(k) = \mathcal{T} y(k)$, $y(k) = (y_1(k), y_2(k))^T$, $k \in \mathbb{Z}_{-m}^{\infty}$ and, for $k \in \mathbb{Z}_m^{\infty}$,

$$y_1(k) = C_1\mu^k + C_2\left(\sum_{l=1}^{n} h_{11}^{*l}\mu^{-m_l-1}\right)k\mu^k, \qquad (22)$$

$$\begin{aligned} y_2(k) &= (h_{11}^{*1}/h_{12}^{*1})(-C_1 + C_2)\mu^k \\ &\quad - (h_{11}^{*1}/h_{12}^{*1})C_2\left(\sum_{l=1}^{n} h_{11}^{*l}\mu^{-m_l-1}\right)k\mu^k, \end{aligned} \qquad (23)$$

where C_1, C_2 are arbitrary parameters.

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. Assume $h_{11}^{*1} = 0$. Then, the problem (1), (2) has a solution $x(k) = \mathcal{T} y(k)$, $y(k) = (y_1(k), y_2(k))^T$, $k \in \mathbb{Z}_{-m}^{\infty}$ and, for $k \in \mathbb{Z}_m^{\infty}$,

$$y_1(k) = C_1\mu^k + C_2 k\mu^k \sum_{l=1}^{n} h_{12}^{*l}\mu^{-m_l-1},$$

$$y_2(k) = C_2\mu^k + C_1 k\mu^k \sum_{l=1}^{n} h_{21}^{*l}\mu^{-m_l-1},$$

where C_1, C_2 are arbitrary parameters.

The case $\mathcal{J}(D) = \mathcal{J}_3$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_3$. Then, the problem (1), (2) has a solution $x(k) = \mathcal{T} y(k)$, $y(k) = (y_1(k), y_2(k))^T$, $k \in \mathbb{Z}_{-m}^{\infty}$ and, for $k \in \mathbb{Z}_{m}^{\infty}$,

$$y_1(k) = C_1 \mu^k + C_2 k \mu^{k-1} \left[1 + \sum_{l=1}^{n} h_{12}^{*l} \mu^{-m_l} \right],$$

$$y_2(k) = C_2 \mu^k,$$

where C_1, C_2 are arbitrary parameters.

Merging of solutions

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_1$. Let constants $C_1 := C_1^*$ and $C_2 := C_2^*$ be fixed. If the initial values (7) satisfy

$$\eta_1(0) + \frac{\eta_2(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{12}^{*l} \mu_1^{-m_l} + \sum_{l=1}^{n} h_{12}^{*l} \sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s)\mu_1^{-s} = C_1^*,$$

$$\eta_2(0) - \frac{\eta_1(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{21}^{*l} \mu_2^{-m_l} + \sum_{l=1}^{n} h_{21}^{*l} \sum_{s=1}^{m_l} \eta_1(-m_l - 1 + s)\mu_2^{-s} = C_2^*,$$

then these initial values define, for $k \in \mathbb{Z}_m^\infty$, the same solutions of (1).

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. *Assume* $h_{11}^{*1} \neq 0$. *Let constants*

$$C_1 := C_1^*, \quad C_2 := C_2^*$$

be fixed. If the initial values (7) *satisfy*

$$\eta_1(0) + \sum_{l=1}^{n} \left[-m_l \omega_1^l(0) \mu^{-m_l-1} + \sum_{s=1}^{m_l} \omega_1^l(-m_l - 1 + s) \mu^{-s} \right] = C_1^*,$$

$$\omega_1^l(0)/h_{11}^* = C_2^*,$$

then these initial values define, for $k \in \mathbb{Z}_m^\infty$, *the same solutions of* (1).

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. *Assume* $h_{11}^{*1} = 0$. *Let constants*

$$C_1 := C_1^*, \quad C_2 := C_2^*$$

be fixed. If the initial values (7) *satisfy*

$$\eta_1(0) + \sum_{l=1}^{n} h_{12}^{*l} \left[-m_l \eta_2(0) \mu^{-m_l-1} + \sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s) \mu^{-s} \right] = C_1^*,$$

$$\eta_2(0) + \sum_{l=1}^{n} h_{21}^{*l} \left[-m_l \eta_1(0) \mu^{-m_l-1} + \sum_{s=1}^{m_l} \eta_1(-m_l - 1 + s) \mu^{-s} \right] = C_2^*,$$

then these initial values define, for $k \in \mathbb{Z}_m^{\infty}$, *the same solutions of* (1).

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_3$. *Let constants*

$$C_1 := C_1^*, \quad C_2 := C_2^*$$

be fixed. If initial values (7) *satisfy*

$$\begin{aligned} &\eta_1(0) \\ &\ + \sum_{l=1}^{n} h_{12}^{*l} \left[\sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s)\mu^{-s} - \eta_2(0) \sum_{l=1}^{n} m_l \mu^{-m_l - 1} \right] \\ &\quad = C_1^*, \\ &\eta_2(0) = C_2^*, \end{aligned}$$

then these initial values define, for $k \in \mathbb{Z}_m^{\infty}$, *the same solutions of* (1).

Non-delayed Systems Equivalent to WD Systems

Consider a non-delayed planar linear discrete system

$$w(k+1) = Gw(k), \quad k \in \mathbb{Z}_m^\infty \tag{24}$$

where $G = \{g_{ij}\}_{i,j=1,2}^2$.

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_1$. Then, for $k \in \mathbb{Z}_m^\infty$, the general solution of the problem (1) is given by the formula $x(k) = \mathcal{T} w(k)$, where $w(k) = (w_1(k), w_2(k))^T$ is the general solution of non-delayed system (24) and G is concretized as follows,

$$G := \begin{pmatrix} \mu_1 & \sum_{l=1}^n h_{12}^{*l} \mu_2^{-m_l} \\ \sum_{l=1}^n h_{21}^{*l} \mu_1^{-m_l} & \mu_2 \end{pmatrix}. \tag{25}$$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. Assume $h_{11}^{*1} \neq 0$. Then, for $k \in \mathbb{Z}_m^{\infty}$, the general solution of the problem (1) *is given by the formula $x(k) = \mathcal{T} w(k)$, where $w(k) = (w_1(k), w_2(k))^T$ is the general solution of non-delayed system* (24) *and G is concretized as follows,*

$$G := \begin{pmatrix} \mu + \sum_{l=1}^{n} h_{11}^{*l} \mu^{-m_l} & h_{12}^{*1} (h_{11}^{*1})^{-1} \sum_{l=1}^{n} h_{11}^{*l} \mu^{-m_l} \\ h_{11}^{*1} (h_{12}^{*1})^{-1} \sum_{l=1}^{n} h_{22}^{*l} \mu^{-m_l} & \mu + \sum_{l=1}^{n} h_{22}^{*l} \mu^{-m_l} \end{pmatrix}.$$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. Assume $h_{11}^{*1} = 0$. Then, for $k \in \mathbb{Z}_m^{\infty}$, the general solution of the problem (1) *is given by formula $x(k) = \mathcal{T} w(k)$, where $w(k) = (w_1(k), w_2(k))^T$ is the general solution of non-delayed system* (24) *and G is concretized as follows,*

$$G := \begin{pmatrix} \mu & \sum_{l=1}^{n} h_{12}^{*l} \mu^{-m_l} \\ \sum_{l=1}^{n} h_{21}^{*l} \mu^{-m_l} & \mu \end{pmatrix}.$$

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_3$. *Then, for* $k \in \mathbb{Z}_m^\infty$, *the general solution of the problem* (1) *is given by the formula* $x(k) = \mathcal{T} w(k)$, *where* $w(k) = (w_1(k), w_2(k))^T$ *is the general solution of non-delayed system* (24) *and* G *is concretized as follows,*

$$G := \begin{pmatrix} \mu & 1 + \sum_{l=1}^{n} h_{12}^{*l} \mu^{-m_l} \\ 0 & \mu \end{pmatrix}.$$

Conditional Stability

Definition

An unstable trivial solution of (1) *is said to be conditionally stable (CS) if, given* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that every initial function* $\xi(k)$*,* $k \in Z^0_{-m}$ *where* $\|\xi(r)\| < \delta$*,* $r = -m, \ldots, 0$ *defines a solution* x *to* (1) *satisfying* $\|x(k)\| < \varepsilon$ *for all* $k > 0$*, provided that the coordinates*

$$(\xi(-m), \xi(-m+1), \ldots, \xi(0)) \in \mathcal{M} \subset \mathbb{R}^{2(m+1)},$$

where $\mathcal{M}$ *is a set of the initial values and* $1 \leq \dim \mathcal{M} < 2(m+1)$*. If, moreover,* $\lim_{k \to \infty} \|x(k)\| = 0$*, we say that the trivial solution is conditionally asymptotically stable (CAS).*

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_1$.

(*i*) *If*

$$\eta_1(0) + \frac{\eta_2(0)}{(\mu_1 - \mu_2)} \sum_{l=1}^{n} h_{12}^{*l} \mu_1^{-m_l} + \sum_{l=1}^{n} h_{12}^{*l} \sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s)\mu_1^{-s} = 0, \quad (26)$$

then, the trivial solution is CS if $|\mu_2| \leq 1$ *and CAS if* $|\mu_2| < 1$.

(*ii*) *If*

$$\eta_2(0) - \frac{\eta_1(0)}{\mu_1 - \mu_2} \sum_{l=1}^{n} h_{21}^{*l} \mu_2^{-m_l} + \sum_{l=1}^{n} h_{21}^{*l} \sum_{s=1}^{m_l} \eta_1(-m_l - 1 + s)\mu_2^{-s} = 0, \quad (27)$$

then the trivial solution is CS if $|\mu_1| \leq 1$ *and CAS if* $|\mu_1| < 1$.

(*iii*) *If* (26) *and* (27) *both hold, then the trivial solution is CAS.*

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$*. Assume* $h_{11}^{*1} \neq 0$*.*

(i) *If*

$$\eta_1(0) + \sum_{l=1}^{n} \sum_{s=1}^{m_l} \omega_1^l(-m_l - 1 + s)\mu^{-s} = 0, \tag{28}$$

$$\omega_1^1(0) = 0, \tag{29}$$

then, the trivial solution is CAS.

(ii) *If* $|\mu| = 1$ *and* (29) *holds, then the trivial solution is CS.*

(iii) *If* $|\mu| = 1$ *and* $\sum_{l=1}^{n} h_{11}^{*l}\mu^{-m_l-1} = 0$*, then the trivial solution is stable.*

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$. *Assume* $h_{11}^{*1} = h_{12}^{*1} = 0$.

(*i*) *If*

$$\eta_1(0) = 0, \qquad (30)$$

$$\eta_2(0) + \sum_{l=1}^{n} h_{21}^{*l} \sum_{s=1}^{m_l} \eta_1(-m_l - 1 + s)\mu^{-s} = 0,$$

then, the trivial solution is CAS.

(*ii*) *If* $|\mu| = 1$ *and* (30) *holds, then the trivial solution is CS.*

(*iii*) *If* $|\mu| = 1$ *and* $\sum_{l=1}^{n} h_{21}^{*l}\mu^{-m_l} = 0$, *then the trivial solution is stable.*

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_2$*. Assume* $h_{11}^{*1} = h_{21}^{*1} = 0$*.*

(i) *If*

$$\eta_1(0) + \sum_{l=1}^{n} h_{12}^{*l} \sum_{s=1}^{m_l} \eta_2(-m_l - 1 + s)\mu^{-s} = 0, \tag{31}$$

$$\eta_2(0) = 0, \tag{32}$$

then, the trivial solution is CAS.

(ii) *If* $|\mu| = 1$ *and* (32) *holds, then the trivial solution is CS.*

(iii) *If* $|\mu| = 1$ *and* $\sum_{l=1}^{n} h_{12}^{*l}\mu^{-m_l} = 0$*, then the trivial solution is stable.*

Theorem

Let $\mathcal{J}(D) = \mathcal{J}_3$*.*

(i) *If* (31)*,* (32) *both hold, then the trivial solution is CAS.*

(ii) *If* $|\mu| = 1$ *and* (32) *holds, then, the trivial solution is CS.*

(iii) *If* $|\mu| = 1$ *and* $\sum_{l=1}^{n} h_{12}^{*l}\mu^{-m_l} = -1$*, then the trivial solution is stable.*

Example

Let a system (1), where $n = 3$, $m_1 = 1$, $m_2 = 2$ and $m = m_3 = 3$, be of the form

$$\begin{aligned} x_1(k+1) = \quad & 3x_1(k) + x_2(k) + x_1(k-1) + x_2(k-1) \\ & + 2x_1(k-2) + 2x_2(k-2) + 3x_1(k-3) \\ & + 3x_2(k-3), \end{aligned} \tag{33}$$

$$\begin{aligned} x_2(k+1) = & -2x_1(k) \qquad\qquad - x_1(k-1) - x_2(k-1) \\ & - 2x_1(k-2) - 2x_2(k-2) - 3x_1(k-3) - 3x_2(k-3). \end{aligned} \tag{34}$$

We have

$$D = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}, \quad H^1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad H^2 = 2H^1, \quad H^3 = 3H^1$$

and $\mu_1 = 2$, $\mu_2 = 1$.

Transforming (33), (34) by (5) where

$$\mathcal{T} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathcal{T}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

we derive a system of the type (6)

$$y_1(k+1) = 2y_1(k) + y_2(k-1) + 2y_2(k-2) + 3y_2(k-3), \qquad (35)$$

$$y_2(k+1) = \quad y_2(k) \qquad (36)$$

where

$$\mathcal{J}(D) = \mathcal{J}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad H^{*1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H^{*2} = 2H^{*1}, \quad H^{*3} = 3H^{*1}.$$

System (12), (13) reduces, for $k \geq m = 3$, to

$$y_1(k) = 2^k \left[\eta_1(0) + \frac{11}{8}\eta_2(0) + \frac{11}{8}\eta_2(-1) + \frac{7}{4}\eta_2(-2) + \frac{3}{2}\eta_2(-3)\right] - 6\eta_2(0), \tag{37}$$

$$y_2(k) = \eta_2(0). \tag{38}$$

We can write (37), (38) in the form

$$y_1(k) = C_1 2^k - 6C_2, \quad y_2(k) = C_2 \tag{39}$$

where C_1, C_2 are arbitrary parameters defined by (37), (38) as

$$C_1 = \eta_1(0) + \frac{11}{8}\eta_2(0) + \frac{11}{8}\eta_2(-1) + \frac{7}{4}\eta_2(-2) + \frac{3}{2}\eta_2(-3),$$
$$C_2 = \eta_2(0)$$

and $\eta_1(0)$, $\eta_2(k)$, $k = -3, -2, -1, 0$ are connected with initial data (2) through formula (7), i.e.,

$$\eta(k) = \begin{pmatrix} \eta_1(k) \\ \eta_2(k) \end{pmatrix} := \mathcal{T}^{-1}\xi(k)$$
$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1(k) \\ \xi_2(k) \end{pmatrix} = \begin{pmatrix} 2\xi_1(k) + \xi_2(k) \\ \xi_1(k) + \xi_2(k) \end{pmatrix}. \tag{40}$$

If $k \geq 3$ from (5) and (39), we derive

$$x_1(k) = C_1 2^k - 7C_2, \;\; x_2(k) = -C_1 2^k + 8C_2 \tag{41}$$

and arbitrary parameters C_1, C_2 can be expressed through the initial values (2) as

$$\begin{aligned} C_1 = {} & \frac{27}{8}\xi_1(0) + \frac{19}{8}\xi_2(0) + \frac{11}{8}\left(\xi_1(-1) + \xi_2(-1)\right) \\ & + \frac{7}{4}\left(\xi_1(-2) + \xi_2(-2)\right) \\ & + \frac{3}{2}\left(\xi_1(-3) + \xi_2(-3)\right), \end{aligned} \tag{42}$$

$$C_2 = \xi_1(0) + \xi_2(0). \tag{43}$$

Some of the initial values given by $C_1 = C_2 = 0$ are shown in Table 1 with the solutions $x(k) = (x_1(k), x_2(k))^T$ defined by them being visualized in Figure 1 by a sequence of points $(k, x_1(k), x_2(k))$, where $k \geq -3$, fitted with a line (represented by a suitable spline with the corresponding colour).

Table: 1

Initial values given by $C_1 = C_2 = 0$.

colouring	$\xi_1(-3)$	$\xi_2(-3)$	$\xi_1(-2)$	$\xi_2(-2)$	$\xi_1(-1)$	$\xi_2(-1)$	$\xi_1(0)$	$\xi_2(0)$
red	1	-3	0	-1	$14/11$	0	3	-3
blue	1	-3	1	0	0	-14/11	3	-3
green	1	-3	0	-2	$28/11$	0	3	-3
yellow	1	-3	0	2	0	-28/11	3	-3

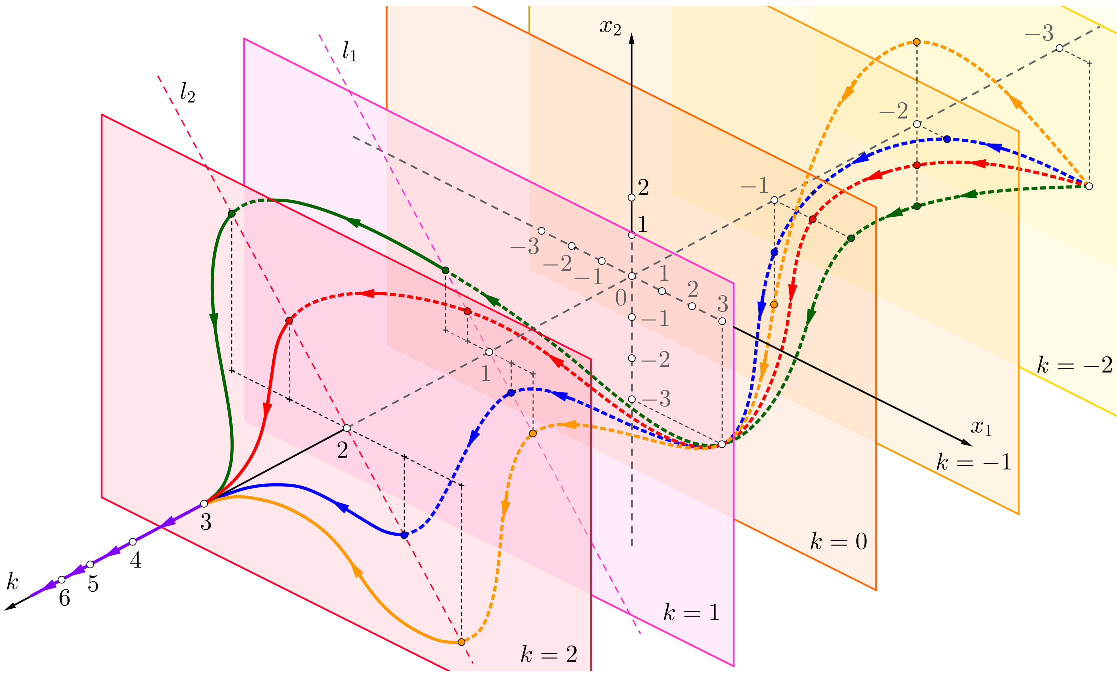

Figure: Solutions of the problem (33)−(34).

Moreover, if values of C_1, C_2 are arbitrary but fixed, then equations (42), (43) determine the sets of the initial values generating the same solutions for $k \in \mathbb{Z}_3^\infty$.
That is, after three steps, such solutions, although defined by different initial data, are merged into a single solution.
Finally, for $k \in \mathbb{Z}_3^\infty$, a non-delayed system (24) with the matrix

$$G = \begin{pmatrix} 2 & 6 \\ 0 & 1 \end{pmatrix}$$

has the same solutions as the system (35), (36) and, for $k \in \mathbb{Z}_3^\infty$, solutions $x(k) = \mathcal{T} w(k)$ coincide with solutions of system (33), (34).

Thank you for your attention!