Component-wise localization of solutions for nonlinear systems: a fixed point index approach

Jorge Rodríguez López

CITMAga & Departamento de Estatística, Análise matemática e Optimización

Universidade de Santiago de Compostela, Spain

Funding: Projects PID2020-113275GB-I00 / ED431C 2023/12

International Meetings on Differential Equations and Their Applications (IMDETA)

March 13, 2024

Consider the second order system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, & \\ x(0) = x(1) = 0 = y(0) = y(1), & \end{cases}$$

where $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions.

Consider the second order system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, \\ x(0) = x(1) = 0 = y(0) = y(1), \end{cases}$$

where $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions.

Main goal

Existence and localization of coexistence positive solutions (x, y) for the system, that is, with both x and y non-trivial.

Consider the second order system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, \\ x(0) = x(1) = 0 = y(0) = y(1), \end{cases}$$

where $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions.

We look for fixed points of $T = (T_1, T_2) : \mathcal{C}(I) \times \mathcal{C}(I) \to \mathcal{C}(I) \times \mathcal{C}(I)$ given by

$$T_i(x,y)(t) = \int_0^1 G(t,s) f_i(x(s), y(s))\, ds, \qquad i = 1, 2,$$

where G is the Green's function of the Dirichlet problem, *i.e.*,

$$G(t,s) = \begin{cases} (1-t)\, s, & \text{if } 0 \le s \le t \le 1, \\ t\,(1-s), & \text{if } 0 \le t < s \le 1. \end{cases}$$

Consider the second order system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, \qquad (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, \\ x(0) = x(1) = 0 = y(0) = y(1), \end{cases}$$

where $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions.

First attempt

Krasnosel'skiĭ compression–expansion fixed point theorem in cones.

We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$.

We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$. Moreover, $x \prec y$ if $y - x \in K \setminus \{0\}$.

We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$. Moreover, $x \prec y$ if $y - x \in K \setminus \{0\}$.

- $K = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a cone in $\mathbb{R}^2$.

We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$. Moreover, $x \prec y$ if $y - x \in K \setminus \{0\}$.

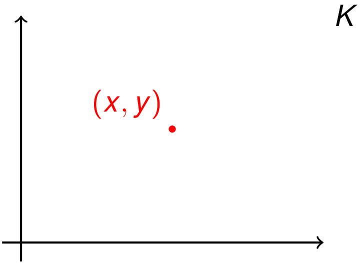
- $K = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a cone in $\mathbb{R}^2$.
- $P = \{x \in \mathcal{C}([0,1]) : x(t) \geq 0 \; \forall \, t \in [0,1]\}$ is a cone in $\mathcal{C}([0,1])$.

We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$. Moreover, $x \prec y$ if $y - x \in K \setminus \{0\}$.

- $K = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a cone in $\mathbb{R}^2$.
- $P = \{x \in \mathcal{C}([0, 1]) : x(t) \geq 0 \; \forall \, t \in [0, 1]\}$ is a cone in $\mathcal{C}([0, 1])$.

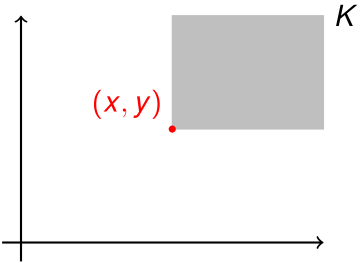


We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$. Moreover, $x \prec y$ if $y - x \in K \setminus \{0\}$.

- $K = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a cone in $\mathbb{R}^2$.
- $P = \{x \in \mathcal{C}([0,1]) : x(t) \geq 0 \ \forall\, t \in [0,1]\}$ is a cone in $\mathcal{C}([0,1])$.

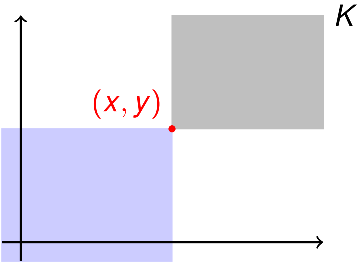


We will say that a closed and convex subset K of a normed space X is a cone if

- $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$;
- $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines a partial order relation in X: $x \preceq y$ if $y - x \in K$. Moreover, $x \prec y$ if $y - x \in K \setminus \{0\}$.

- $K = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a cone in $\mathbb{R}^2$.
- $P = \{x \in \mathcal{C}([0, 1]) : x(t) \geq 0 \; \forall \, t \in [0, 1]\}$ is a cone in $\mathcal{C}([0, 1])$.



Theorem (Krasnosel'skiĭ)

Let K be a cone of a normed space X and $T : K \to K$ a completely continuous map.
Assume that there exist $r, R > 0$, $r \neq R$, such that

$$Tx \not\preceq x \text{ for all } x \in K \text{ with } \|x\| = r,$$
$$Tx \not\succeq x \text{ for all } x \in K \text{ with } \|x\| = R.$$

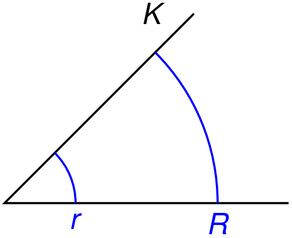
Then T has a fixed point $\overline{x} \in K$ s.t. $\min\{r, R\} < \|\overline{x}\| < \max\{r, R\}$.

Theorem (Krasnosel'skiĭ)

Let K be a cone of a normed space X and $T : K \to K$ a completely continuous map.
Assume that there exist $r, R > 0$, $r \neq R$, such that

$$
\begin{aligned}
&Tx \not\preceq x \text{ for all } x \in K \text{ with } \|x\| = r,\\
&Tx \not\succeq x \text{ for all } x \in K \text{ with } \|x\| = R.
\end{aligned}
$$

Then T has a fixed point $\overline{x} \in K$ s.t. $\min\{r, R\} < \|\overline{x}\| < \max\{r, R\}$.



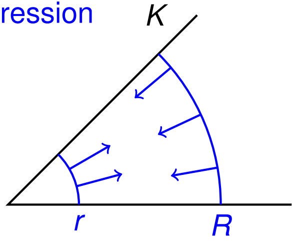
Theorem (Krasnosel'skiĭ)

Let K be a cone of a normed space X and $T : K \to K$ a completely continuous map.
Assume that there exist $r, R > 0$, $r \neq R$, such that

$$Tx \not\leq x \text{ for all } x \in K \text{ with } \|x\| = r,$$
$$Tx \not\geq x \text{ for all } x \in K \text{ with } \|x\| = R.$$

Then T has a fixed point $\overline{x} \in K$ s.t. $\min\{r, R\} < \|\overline{x}\| < \max\{r, R\}$.

Compression



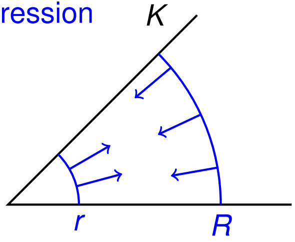
Theorem (Krasnosel'skiĭ)

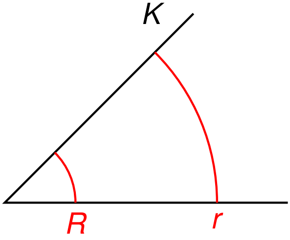
Let K be a cone of a normed space X and $T : K \to K$ a completely continuous map.
Assume that there exist $r, R > 0$, $r \neq R$, such that

$$Tx \not\preceq x \text{ for all } x \in K \text{ with } \|x\| = r,$$
$$Tx \not\succeq x \text{ for all } x \in K \text{ with } \|x\| = R.$$

Then T has a fixed point $\overline{x} \in K$ s.t. $\min\{r, R\} < \|\overline{x}\| < \max\{r, R\}$.

Compression



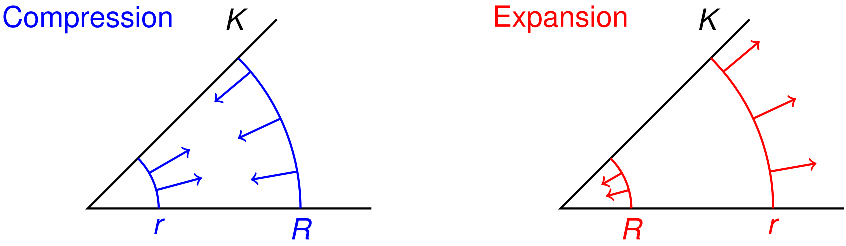


Theorem (Krasnosel'skiĭ)

Let K be a cone of a normed space X and $T : K \to K$ a completely continuous map.
Assume that there exist $r, R > 0$, $r \neq R$, such that

$$Tx \nleq x \text{ for all } x \in K \text{ with } \|x\| = r,$$
$$Tx \ngeq x \text{ for all } x \in K \text{ with } \|x\| = R.$$

Then T has a fixed point $\overline{x} \in K$ s.t. $\min\{r, R\} < \|\overline{x}\| < \max\{r, R\}$.

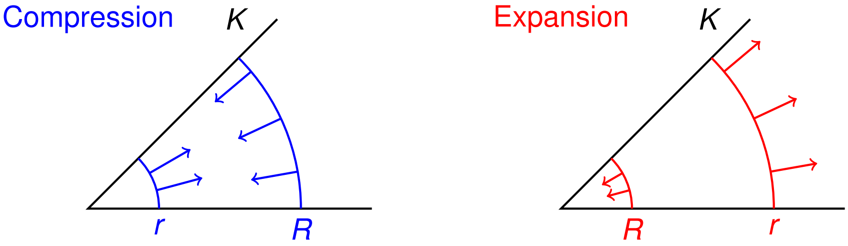


Theorem (homotopy version of Krasnosel'skiĭ)

Let K be a cone of a normed space X and $T : K \to K$ a completely continuous map.
Assume that there exist $r, R > 0$, $r \neq R$, and $h \in K \setminus \{0\}$ such that

$$T(u) + \mu h \neq u \quad \text{if } \|u\| = r \text{ and } \mu \geq 0,$$
$$T(u) \neq \lambda u \quad \text{if } \|u\| = R \text{ and } \lambda \geq 1.$$

Then T has a fixed point $\overline{x} \in K$ s.t. $\min\{r, R\} < \|\overline{x}\| < \max\{r, R\}$.



Coming back to the system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, & \\ x(0) = x(1) = 0 = y(0) = y(1), & \end{cases}$$

we can apply Krasnosel'skiĭ type theorem by using

- Normed space: $\mathcal{C}(I) \times \mathcal{C}(I)$, $\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$.

Coming back to the system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, & \\ x(0) = x(1) = 0 = y(0) = y(1), & \end{cases}$$

we can apply Krasnosel'skiĭ type theorem by using

- Normed space: $\mathcal{C}(I) \times \mathcal{C}(I)$, $\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$.
- Cone: $K \times K$, with $K = \left\{ x \in P : \min_{t \in [1/4, 3/4]} x(t) \geq \frac{1}{4} \|x\|_\infty \right\}$.

Coming back to the system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, \qquad (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, \\ x(0) = x(1) = 0 = y(0) = y(1), \end{cases}$$

we can apply Krasnosel'skiĭ type theorem by using

- Normed space: $\mathcal{C}(I) \times \mathcal{C}(I)$, $\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$.
- Cone: $K \times K$, with $K = \left\{ x \in P : \min_{t \in [1/4, 3/4]} x(t) \geq \frac{1}{4} \|x\|_\infty \right\}$.
- Operator: $T = (T_1, T_2)$, with

$$T_i(x,y)(t) = \int_0^1 G(t,s) f_i(x(s), y(s))\, ds, \quad t \in I, \quad i = 1, 2.$$

Krasnosel'skiĭ fixed point theorem can be applied and, under suitable assumptions, it guarantees the existence of a solution $(x, y) \in K \times K$ such that

$$\alpha \leq \|(x, y)\| \leq \beta,$$

for some $\alpha, \beta > 0$.

Krasnosel'skiĭ fixed point theorem can be applied and, under suitable assumptions, it guarantees the existence of a solution $(x, y) \in K \times K$ such that

$$\alpha \leq \|(x, y)\| \leq \beta,$$

for some $\alpha, \beta > 0$.

That means,

$$\alpha \leq \max\{\|x\|_\infty, \|y\|_\infty\} \leq \beta.$$

Krasnosel'skiĭ fixed point theorem can be applied and, under suitable assumptions, it guarantees the existence of a solution $(x, y) \in K \times K$ such that

$$\alpha \leq \|(x, y)\| \leq \beta,$$

for some $\alpha, \beta > 0$.

That means,

$$\alpha \leq \max\{\|x\|_\infty, \|y\|_\infty\} \leq \beta.$$

Hence, it could be $\|x\|_\infty = 0$ or $\|y\|_\infty = 0$.

Krasnosel'skiĭ fixed point theorem can be applied and, under suitable assumptions, it guarantees the existence of a solution $(x, y) \in K \times K$ such that

$$\alpha \leq \|(x, y)\| \leq \beta,$$

for some $\alpha, \beta > 0$.

That means,

$$\alpha \leq \max\{\|x\|_\infty, \|y\|_\infty\} \leq \beta.$$

Hence, it could be $\|x\|_\infty = 0$ or $\|y\|_\infty = 0$.

How to avoid this undesirable possibility?

Krasnosel'skiĭ fixed point theorem can be applied and, under suitable assumptions, it guarantees the existence of a solution $(x, y) \in K \times K$ such that

$$\alpha \leq \|(x, y)\| \leq \beta,$$

for some $\alpha, \beta > 0$.

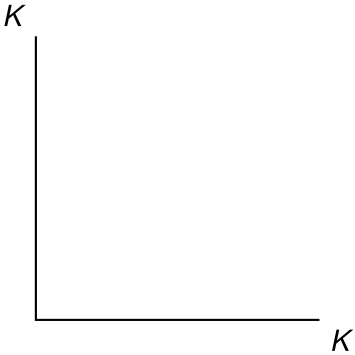
That means,

$$\alpha \leq \max\{\|x\|_\infty, \|y\|_\infty\} \leq \beta.$$

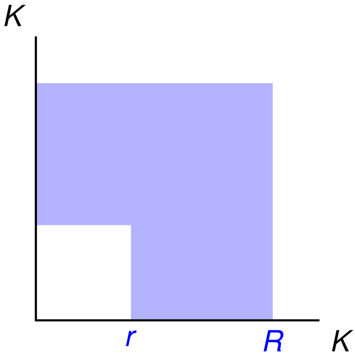
Hence, it could be $\|x\|_\infty = 0$ or $\|y\|_\infty = 0$.

How to avoid this undesirable possibility?

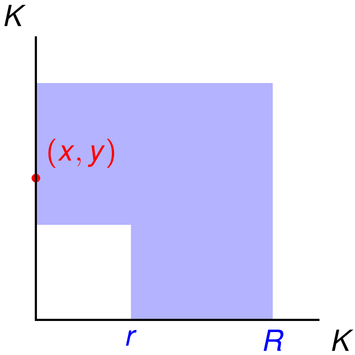
R. Precup,
A vector version of Krasnosel'skiĭ's fixed point theorem in cones and positive periodic solutions of nonlinear systems,
J. Fixed Point Theory Appl., **2** (2007) 141–151.



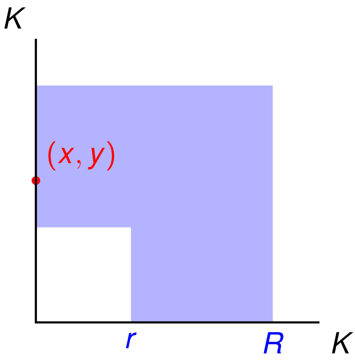
Krasnosel'skiĭ F. P. T.



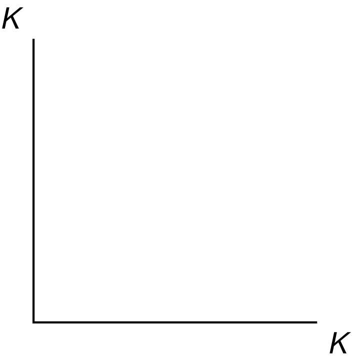
Krasnosel'skiĭ F. P. T.



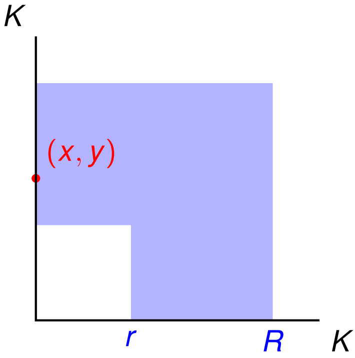
Krasnosel'skiĭ F. P. T.



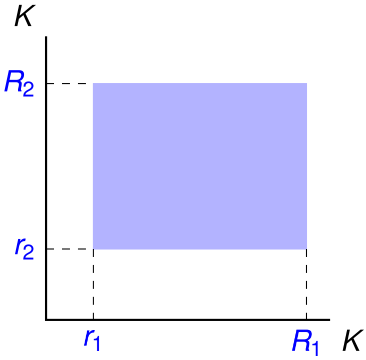
Krasnosel'skiĭ F. P. T.



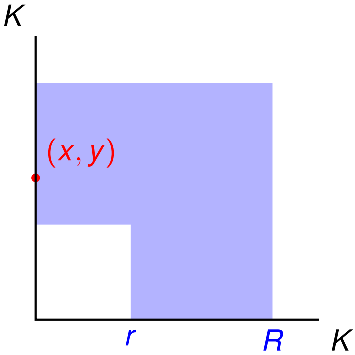
Vector version



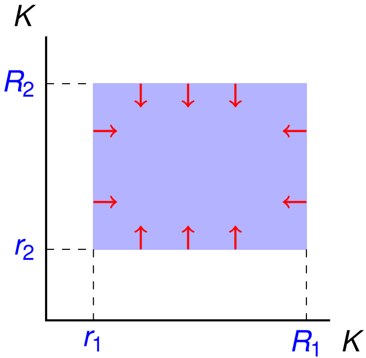
Krasnosel'skiĭ F. P. T.



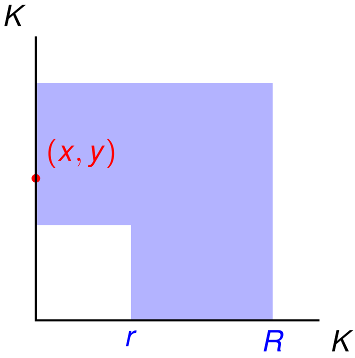
Vector version



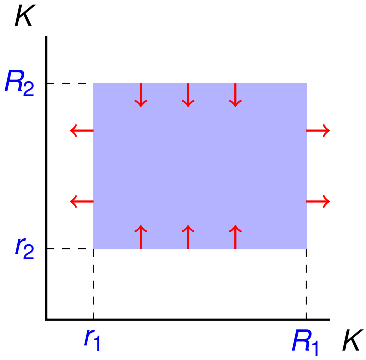
Krasnosel'skiĭ F. P. T.



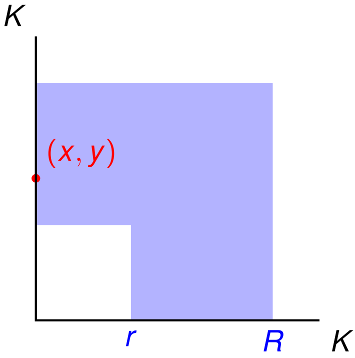
Vector version



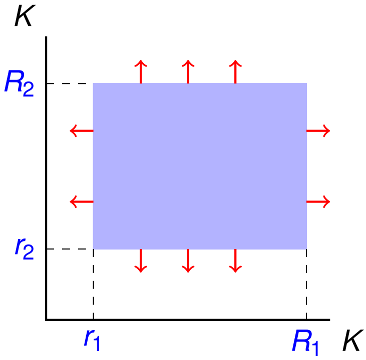
Krasnosel'skiĭ F. P. T.



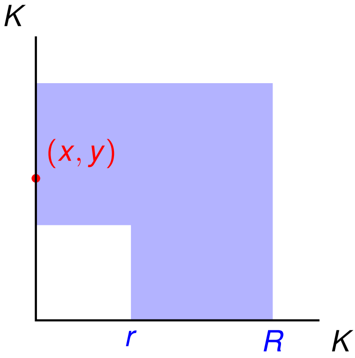
Vector version



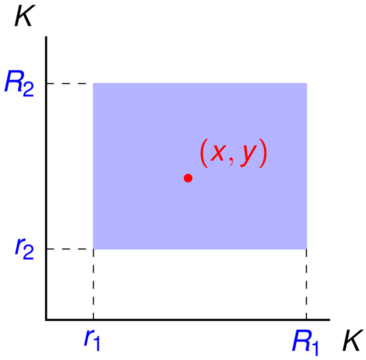
Krasnosel'skiĭ F. P. T.



Vector version



Krasnosel'skiĭ F. P. T.



Vector version

Let K_1, K_2 two cones of a normed space X and $K := K_1 \times K_2$.

Let K_1, K_2 two cones of a normed space X and $K := K_1 \times K_2$.
For $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, with $0 < r_i < R_i$ $(i = 1, 2)$, we denote

$$\overline{K}_{r,R} := \{u = (u_1, u_2) \in K \,:\, r_i \leq \|u_i\| \leq R_i \quad \text{for } i = 1, 2\}.$$

Let K_1, K_2 two cones of a normed space X and $K := K_1 \times K_2$.
For $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, with $0 < r_i < R_i$ ($i = 1, 2$), we denote

$$\overline{K}_{r,R} := \{u = (u_1, u_2) \in K \; : \; r_i \leq \|u_i\| \leq R_i \quad \text{for } i = 1, 2\}.$$

Theorem (Krasnosel'skiĭ-Precup)

Assume that $T = (T_1, T_2) : \overline{K}_{r,R} \to K$ is a compact map and for each $i \in \{1, 2\}$ there exists $h_i \in K_i \setminus \{0\}$ such that one of the following conditions is satisfied in $\overline{K}_{r,R}$:

(a) $T_i(u) + \mu\, h_i \neq u_i$ *if* $\|u_i\| = r_i$ *and* $\mu \geq 0$, *and* $T_i(u) \neq \lambda\, u_i$ *if* $\|u_i\| = R_i$ *and* $\lambda \geq 1$;

(b) $T_i(u) \neq \lambda\, u_i$ *if* $\|u_i\| = r_i$ *and* $\lambda \geq 1$, *and* $T_i(u) + \mu\, h_i \neq u_i$ *if* $\|u_i\| = R_i$ *and* $\mu \geq 0$.

Then T has at least a fixed point $u = (u_1, u_2) \in K$ with $r_i < \|u_i\| < R_i$ ($i = 1, 2$).

Remark

For each fixed $u_2 \in (\overline{K}_2)_{r_2,R_2}$, the operator

$$T_1(\cdot, u_2) : (\overline{K}_1)_{r_1,R_1} \to K_1$$

is under the assumptions of Krasnosel'skiĭ theorem.

Remark

For each fixed $u_2 \in (\overline{K}_2)_{r_2,R_2}$, the operator

$$T_1(\cdot, u_2) : (\overline{K}_1)_{r_1,R_1} \to K_1$$

is under the assumptions of Krasnosel'skiĭ theorem. Hence, there exists $v_{u_2} \in (\overline{K}_1)_{r_1,R_1}$ such that $v_{u_2} = T_1(v_{u_2}, u_2)$.

Remark

For each fixed $u_2 \in (\overline{K}_2)_{r_2,R_2}$, the operator

$$T_1(\cdot, u_2) : (\overline{K}_1)_{r_1,R_1} \to K_1$$

is under the assumptions of Krasnosel'skiĭ theorem. Hence, there exists $v_{u_2} \in (\overline{K}_1)_{r_1,R_1}$ such that $v_{u_2} = T_1(v_{u_2}, u_2)$.

Analogously, for each fixed $u_1 \in (\overline{K}_1)_{r_1,R_1}$ the operator

$$T_2(u_1, \cdot) : (\overline{K}_2)_{r_2,R_2} \to K_2$$

has a fixed point in $(\overline{K}_2)_{r_2,R_2}$, that is, there is $w_{u_1} \in (\overline{K}_2)_{r_2,R_2}$ such that $w_{u_1} = T_2(u_1, w_{u_1})$.

Remark

For each fixed $u_2 \in (\overline{K}_2)_{r_2,R_2}$, the operator

$$T_1(\cdot, u_2) : (\overline{K}_1)_{r_1,R_1} \to K_1$$

is under the assumptions of Krasnosel'skiĭ theorem. Hence, there exists $v_{u_2} \in (\overline{K}_1)_{r_1,R_1}$ such that $v_{u_2} = T_1(v_{u_2}, u_2)$.

Analogously, for each fixed $u_1 \in (\overline{K}_1)_{r_1,R_1}$ the operator

$$T_2(u_1, \cdot) : (\overline{K}_2)_{r_2,R_2} \to K_2$$

has a fixed point in $(\overline{K}_2)_{r_2,R_2}$, that is, there is $w_{u_1} \in (\overline{K}_2)_{r_2,R_2}$ such that $w_{u_1} = T_2(u_1, w_{u_1})$.

Precup's result ensures that there is $(v, w) \in (\overline{K}_1)_{r_1,R_1} \times (\overline{K}_2)_{r_2,R_2}$ such that

$$(v, w) = (T_1(v, w), T_2(v, w)).$$

Remark

The operator T may exhibit a different behavior (compression or expansion) in each component. More exactly, the following options are possible:

- (*i*) both operators T_1 and T_2 are compressive;
- (*ii*) both operators T_1 and T_2 are expansive;
- (*iii*) one of the operators T_1 or T_2 is compressive, while the other one is expansive.

Remark

The operator T may exhibit a different behavior (compression or expansion) in each component. More exactly, the following options are possible:

(*i*) both operators T_1 and T_2 are compressive;

(*ii*) both operators T_1 and T_2 are expansive;

(*iii*) one of the operators T_1 or T_2 is compressive, while the other one is expansive.

Remark

The proof due to Precup is based on

- Schauder fixed point theorem;
- a trick to transform expansion conditions into compression ones.

Fixed point index

Let P be a cone of a normed linear space, $U \subset P$ be a bounded relatively open set and $T : \overline{U} \to P$ be a compact map such that T has no fixed points on the boundary of U (denoted by ∂U).

Fixed point index

Let P be a cone of a normed linear space, $U \subset P$ be a bounded relatively open set and $T : \overline{U} \to P$ be a compact map such that T has no fixed points on the boundary of U (denoted by ∂U).

The fixed point index of T in P over U, $i_P(T, U)$, is well-defined.

Fixed point index

Let P be a cone of a normed linear space, $U \subset P$ be a bounded relatively open set and $T : \overline{U} \to P$ be a compact map such that T has no fixed points on the boundary of U (denoted by ∂U).

The fixed point index of T in P over U, $i_P(T, U)$, is well-defined.

Roughly speaking, it can be seen as an algebraic count of the fixed points of T in U.

Fixed point index

Let P be a cone of a normed linear space, $U \subset P$ be a bounded relatively open set and $T : \overline{U} \to P$ be a compact map such that T has no fixed points on the boundary of U (denoted by ∂U).

The fixed point index of T in P over U, $i_P(T, U)$, is well-defined.

Roughly speaking, it can be seen as an algebraic count of the fixed points of T in U.

H. Amann,
Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces,
SIAM Rev., **18** 4 (1976), 620–709.

Proposition

The fixed point index $i_P(T, U)$ has the following properties:

1. *(Additivity) Let U be the disjoint union of two open sets U_1 and U_2. If $0 \notin (I - T)(\overline{U} \setminus (U_1 \cup U_2))$, then*

$$i_P(T, U) = i_P(T, U_1) + i_P(T, U_2).$$

Proposition

The fixed point index $i_P(T, U)$ has the following properties:

1. *(Additivity) Let U be the disjoint union of two open sets U_1 and U_2. If $0 \notin (I - T)(\overline{U} \setminus (U_1 \cup U_2))$, then*

$$i_P(T, U) = i_P(T, U_1) + i_P(T, U_2).$$

2. *(Existence) If $i_P(T, U) \neq 0$, then there exists $u \in U$ such that $u = Tu$.*

Proposition

The fixed point index $i_P(T, U)$ has the following properties:

1. *(Additivity) Let U be the disjoint union of two open sets U_1 and U_2. If $0 \notin (I - T)(\overline{U} \setminus (U_1 \cup U_2))$, then*

$$i_P(T, U) = i_P(T, U_1) + i_P(T, U_2).$$

2. *(Existence) If $i_P(T, U) \neq 0$, then there exists $u \in U$ such that $u = Tu$.*
3. *(Homotopy invariance) If $H : \overline{U} \times [0, 1] \to P$ is a compact homotopy and $0 \notin (I - H)(\partial U \times [0, 1])$, then*

$$i_P(H(\cdot, 0), U) = i_P(H(\cdot, 1), U).$$

Proposition

The fixed point index $i_P(T, U)$ has the following properties:

1. *(Additivity) Let U be the disjoint union of two open sets U_1 and U_2. If $0 \notin (I - T)(\overline{U} \setminus (U_1 \cup U_2))$, then*
$$i_P(T, U) = i_P(T, U_1) + i_P(T, U_2).$$
2. *(Existence) If $i_P(T, U) \neq 0$, then there exists $u \in U$ such that $u = Tu$.*
3. *(Homotopy invariance) If $H : \overline{U} \times [0, 1] \to P$ is a compact homotopy and $0 \notin (I - H)(\partial U \times [0, 1])$, then*
$$i_P(H(\cdot, 0), U) = i_P(H(\cdot, 1), U).$$
4. *(Normalization) If T is a constant map with $T(u) = u_0$ for every $u \in \overline{U}$, then*
$$i_P(T, U) = \begin{cases} 1, & \text{if } u_0 \in U, \\ 0, & \text{if } u_0 \notin \overline{U}. \end{cases}$$

Proposition

Let U be a bounded relatively open subset of a cone P such that $0 \in U$ and $T : \overline{U} \to P$ be a compact map.

1. *If $T(u) \neq \lambda\, u$ for all $u \in \partial\, U$ and all $\lambda \geq 1$, then $i_P(T, U) = 1$.*

Proposition

Let U be a bounded relatively open subset of a cone P such that $0 \in U$ and $T : \overline{U} \to P$ be a compact map.

1. *If $T(u) \neq \lambda\, u$ for all $u \in \partial\, U$ and all $\lambda \geq 1$, then $i_P(T, U) = 1$.*
2. *If there exists $h \in P \setminus \{0\}$ such that $T(u) + \lambda\, h \neq u$ for every $\lambda \geq 0$ and all $u \in \partial\, U$, then $i_P(T, U) = 0$.*

Proposition

Let U be a bounded relatively open subset of a cone P such that $0 \in U$ and $T : \overline{U} \to P$ be a compact map.

1. *If $T(u) \neq \lambda u$ for all $u \in \partial U$ and all $\lambda \geq 1$, then $i_P(T, U) = 1$.*
2. *If there exists $h \in P \setminus \{0\}$ such that $T(u) + \lambda h \neq u$ for every $\lambda \geq 0$ and all $u \in \partial U$, then $i_P(T, U) = 0$.*

Remark

The homotopy version of Krasnosel'skiĭ fixed point theorem can be easily proven by means of fixed point index theory.

In the case of systems we need the following auxiliary result.

Lemma

Let U and V be bounded relatively open subsets of K_1 and K_2, respectively, such that $0 \in U$.

Assume that $T : \overline{U \times V} \to K$, $T = (T_1, T_2)$, is a compact map and there exists $h \in K_2 \setminus \{0\}$ such that

$$T_1(u,v) \neq \lambda\, u \quad \text{for } u \in \partial_{K_1} U,\ v \in \overline{V} \text{ and } \lambda \geq 1;$$
$$T_2(u,v) + \mu\, h \neq v \quad \text{for } u \in \overline{U},\ v \in \partial_{K_2} V \text{ and } \mu \geq 0.$$

Then $i_K(T, U \times V) = 0$.

In the case of systems we need the following auxiliary result.

Lemma

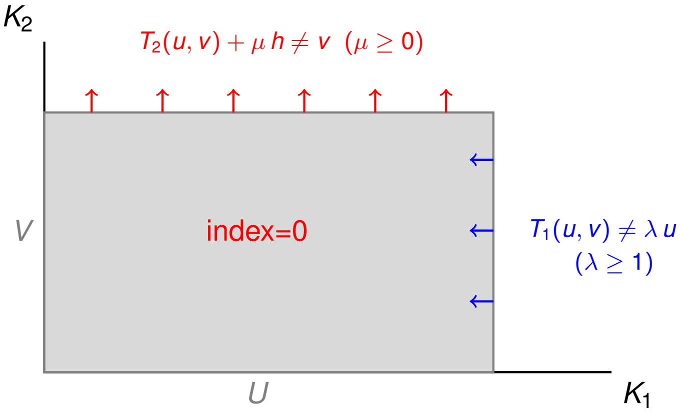
Let U and V be bounded relatively open subsets of K_1 and K_2, respectively, such that $0 \in U$.

Assume that $T : \overline{U \times V} \to K$, $T = (T_1, T_2)$, is a compact map and there exists $h \in K_2 \setminus \{0\}$ such that

$$T_1(u,v) \neq \lambda\, u \quad \text{for } u \in \partial_{K_1} U,\ v \in \overline{V} \text{ and } \lambda \geq 1;$$
$$T_2(u,v) + \mu\, h \neq v \quad \text{for } u \in \overline{U},\ v \in \partial_{K_2} V \text{ and } \mu \geq 0.$$

Then $i_K(T, U \times V) = 0$.

R. Precup and J. R-L,
Multiplicity Results for Operator Systems via Fixed Point Index,
Results Math., **74**:25 (2019), 1–14.


K_2
$T_2(u,v) + \mu\, h \neq v \ \ (\mu \geq 0)$
index=0
V
$T_1(u,v) \neq \lambda\, u$
$(\lambda \geq 1)$
U
K_1

For $r, R \in \mathbb{R}^2_+$, $0 < r_i < R_i$ ($i = 1, 2$), fixed, our aim is to compute the fixed point index of a compact operator $T = (T_1, T_2) : \overline{K}_{r,R} \to K$ in the relatively open set

$$K_{r,R} := \{u = (u_1, u_2) \in K \, : \, r_i < \|u_i\| < R_i \quad \text{for } i = 1, 2\}$$

under the conditions of Krasnosel'skiĭ-Precup fixed point theorem.

For $r, R \in \mathbb{R}^2_+$, $0 < r_i < R_i$ $(i = 1, 2)$, fixed, our aim is to compute the fixed point index of a compact operator $T = (T_1, T_2) : \overline{K}_{r,R} \to K$ in the relatively open set

$$K_{r,R} := \{u = (u_1, u_2) \in K \ : \ r_i < \|u_i\| < R_i \quad \text{for } i = 1, 2\}$$

under the conditions of Krasnosel'skiĭ-Precup fixed point theorem.

Case 1: $T_1,\ T_2$ are compressive.

Theorem

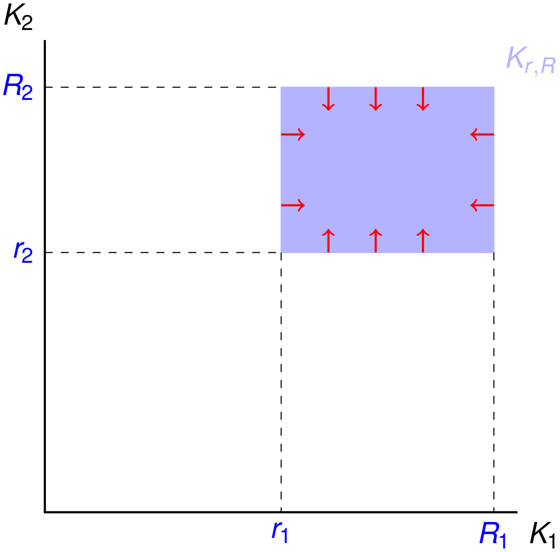
Assume that $T = (T_1, T_2) : \overline{K}_{r,R} \to K$ is a compact map and for each $i \in \{1, 2\}$ there exists $h_i \in K_i \setminus \{0\}$ such that the following conditions are satisfied in $\overline{K}_{r,R}$:

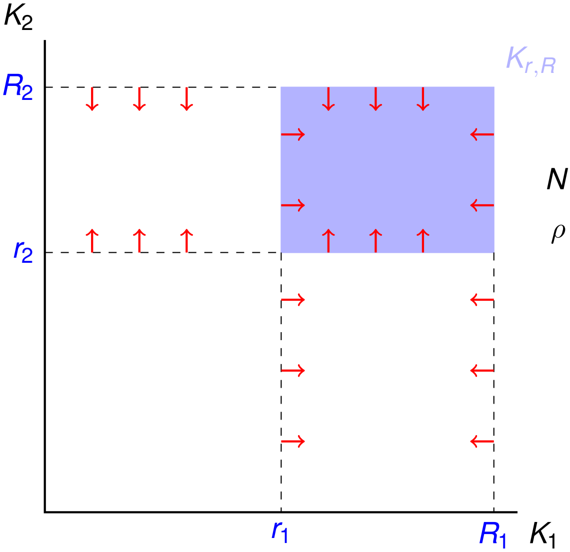
(i) $T_i(u) + \mu\, h_i \neq u_i$ *if* $\|u_i\| = r_i$ *and* $\mu \geq 0$*;*

(ii) $T_i(u) \neq \lambda\, u_i$ *if* $\|u_i\| = R_i$ *and* $\lambda \geq 1$*.*

Then

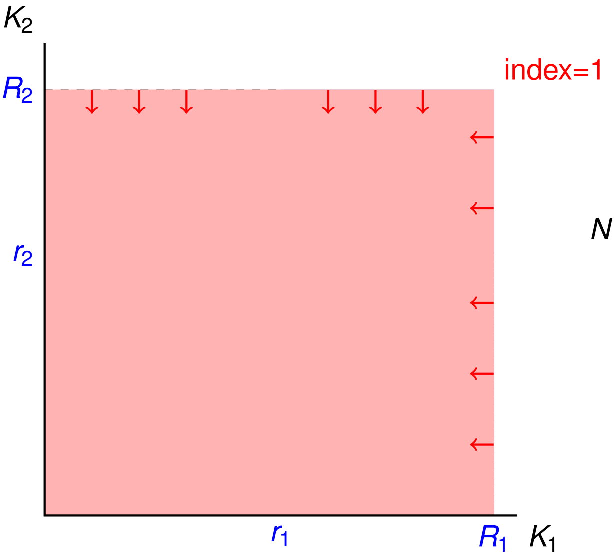
$$i_K(T, K_{r,R}) = 1.$$


K_2
$K_{r,R}$
R_2
r_2
r_1
R_1
K_1

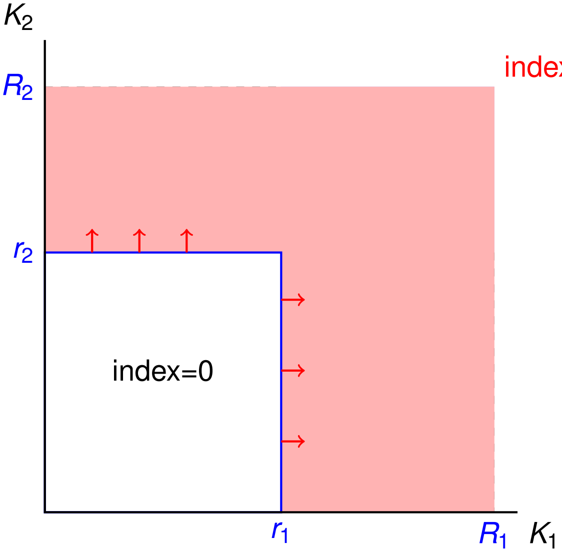


$N = T \circ \rho,$

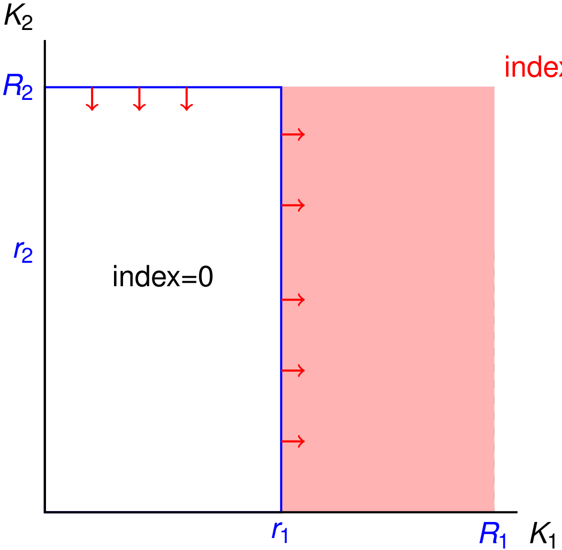
$\rho : \overline{K}_R \to \overline{K}_{r,R}$ retraction



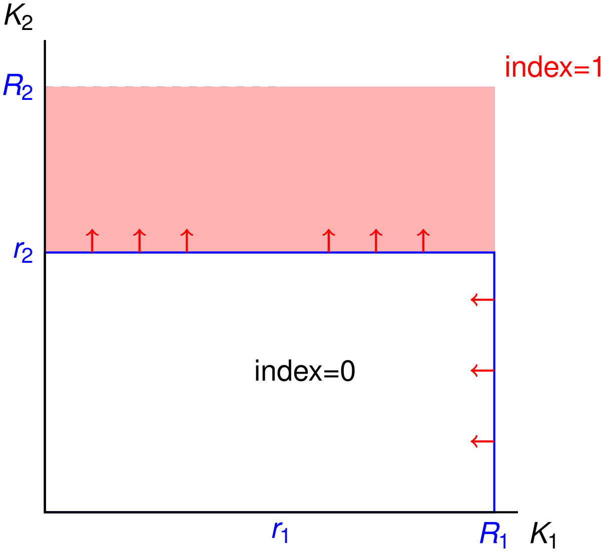
$N(u) \neq \lambda\, u, \;\; \lambda \geq 1$



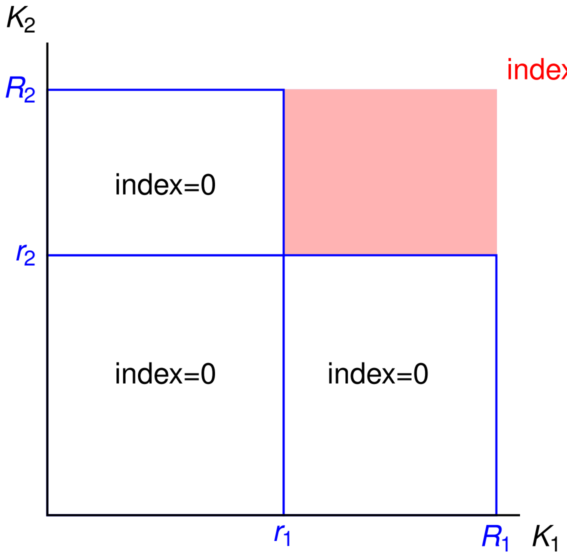
$$N(u) + \mu\, h \neq u, \;\; \mu \geq 0$$



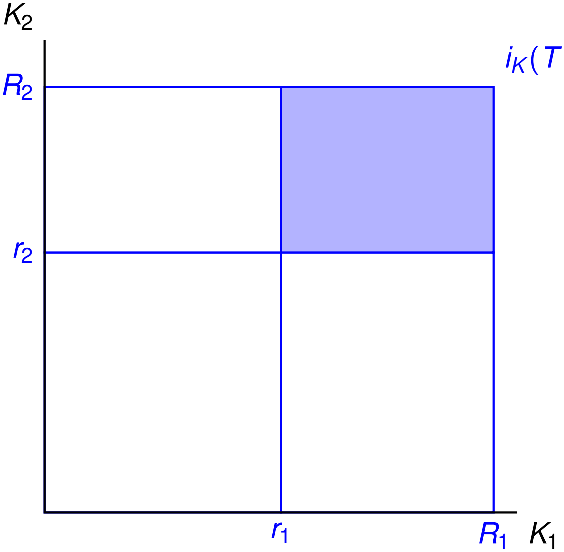
LEMMA



LEMMA



Additivity property



$i_K(T, K_{r,R}) = 1$

$N = T$ in $K_{r,R}$

Case 2: T_1 is compressive and T_2 is expansive.

Theorem

Assume that $T = (T_1, T_2) : \overline{K}_{r,R} \to K$ *is a compact map and for each* $i \in \{1, 2\}$ *there exists* $h_i \in K_i \setminus \{0\}$ *such that the following conditions are satisfied in* $\overline{K}_{r,R}$*:*

(*i*) $T_1(u) + \mu\, h_1 \neq u_1$ *if* $\|u_1\| = r_1$ *and* $\mu \geq 0$*, and* $T_1(u) \neq \lambda\, u_1$ *if* $\|u_1\| = R_1$ *and* $\lambda \geq 1$*;*

(*ii*) $T_2(u) + \mu\, h_2 \neq u_2$ *if* $\|u_2\| = R_2$ *and* $\mu \geq 0$*, and* $T_2(u) \neq \lambda\, u_2$ *if* $\|u_2\| = r_2$ *and* $\lambda \geq 1$*.*

Then

$$i_K(T, K_{r,R}) = -1.$$

Case 3: Both $T_1,\ T_2$ are expansive.

Theorem

Assume that $T = (T_1, T_2) : \overline{K}_{r,R} \to K$ *is a compact map and for each* $i \in \{1, 2\}$ *there exists* $h_i \in K_i \setminus \{0\}$ *such that the following conditions are satisfied in* $\overline{K}_{r,R}$*:*

(*i*) $T_i(u) + \mu\, h_i \neq u_i$ *if* $\|u_i\| = R_i$ *and* $\mu \geq 0$*;*

(*ii*) $T_i(u) \neq \lambda\, u_i$ *if* $\|u_i\| = r_i$ *and* $\lambda \geq 1$*.*

Then

$$i_K(T, K_{r,R}) = 1.$$

Several generalizations are possible:

- Different domains for the operator T:

Several generalizations are possible:

- Different domains for the operator T:

  T defined in $\left(\overline{U}_1 \setminus V_1\right) \times \left(\overline{U}_2 \setminus V_2\right)$, where

  - $0 \in V_i \subset \overline{V}_i \subset U_i$,
  - U_i and V_i bounded and (relatively) open in K_i,
  - $\overline{U}_i \setminus V_i$ is a retract of $\overline{U}_i$.

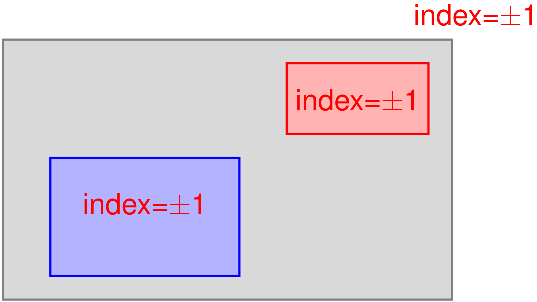
Several generalizations are possible:

- Different domains for the operator T:

  T defined in $(\overline{U}_1 \setminus V_1) \times (\overline{U}_2 \setminus V_2)$, where

  - $0 \in V_i \subset \overline{V}_i \subset U_i$,
  - U_i and V_i bounded and (relatively) open in K_i,
  - $\overline{U}_i \setminus V_i$ is a retract of $\overline{U}_i$.

- Multiple fixed points of T.



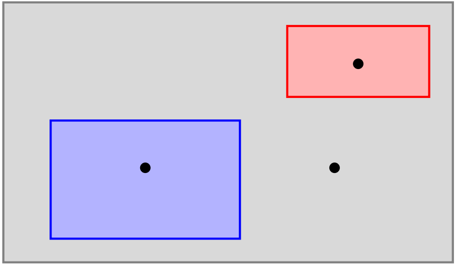
Several generalizations are possible:

- Different domains for the operator T:

  T defined in $(\overline{U}_1 \setminus V_1) \times (\overline{U}_2 \setminus V_2)$, where

  - $0 \in V_i \subset \overline{V}_i \subset U_i$,
  - U_i and V_i bounded and (relatively) open in K_i,
  - $\overline{U}_i \setminus V_i$ is a retract of $\overline{U}_i$.

- Multiple fixed points of T.



In the case $X = \mathbb{R}$ and $K_1 = K_2 = \mathbb{R}_+$, we obtain an equivalent version of Poincaré-Miranda theorem:

Theorem

Assume that $g = (g_1, g_2) : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^2$ is a continuous function and for each $i \in \{1, 2\}$ one of the following conditions is satisfied:

(a) $g_i(x_1, x_2) > 0$ *if* $x_i = a_i$, *and* $g_i(x_1, x_2) < 0$ *if* $x_i = b_i$;

(b) $g_i(x_1, x_2) < 0$ *if* $x_i = a_i$, *and* $g_i(x_1, x_2) > 0$ *if* $x_i = b_i$.

Then there exists $(\bar{x}_1, \bar{x}_2)$ *such that* $g(\bar{x}_1, \bar{x}_2) = 0$.

In the case $X = \mathbb{R}$ and $K_1 = K_2 = \mathbb{R}_+$, we obtain an equivalent version of Poincaré-Miranda theorem:

Theorem

Assume that $g = (g_1, g_2) : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^2$ *is a continuous function and for each* $i \in \{1, 2\}$ *one of the following conditions is satisfied:*

(a) $g_i(x_1, x_2) > 0$ *if* $x_i = a_i$, *and* $g_i(x_1, x_2) < 0$ *if* $x_i = b_i$;

(b) $g_i(x_1, x_2) < 0$ *if* $x_i = a_i$, *and* $g_i(x_1, x_2) > 0$ *if* $x_i = b_i$.

Then there exists $(\bar{x}_1, \bar{x}_2)$ *such that* $g(\bar{x}_1, \bar{x}_2) = 0$.

J. R-L,
A fixed point index approach to Krasnosel'skiĭ-Precup fixed point theorem in cones and applications,
Nonlinear Anal., **226** No. 113138 (2023), 1–19.

Consider the second order system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, \\ x(0) = x(1) = 0 = y(0) = y(1), \end{cases}$$

where $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions.

Consider the second order system

$$\begin{cases} x''(t) + f_1(x(t), y(t)) = 0, & (t \in I = [0,1]) \\ y''(t) + f_2(x(t), y(t)) = 0, \\ x(0) = x(1) = 0 = y(0) = y(1), \end{cases}$$

where $f_1, f_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions.

We look for fixed points of $T = (T_1, T_2) : K \times K \to K \times K$ given by

$$T_i(x,y)(t) = \int_0^1 G(t,s) f_i(x(s), y(s))\, ds, \qquad i = 1, 2,$$

where

$$K = \left\{ x \in P : \min_{t \in [1/4, 3/4]} x(t) \geq \frac{1}{4} \|x\|_\infty \right\}.$$

Now, let us fix some notations:

$$m := \min_{t\in[1/4,3/4]} \int_{1/4}^{3/4} G(t,s)\, ds, \quad M := \max_{t\in[0,1]} \int_{0}^{1} G(t,s)\, ds.$$

Now, let us fix some notations:

$$m := \min_{t\in[1/4,3/4]} \int_{1/4}^{3/4} G(t,s)\,ds, \quad M := \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds.$$

In addition, for $\alpha_i, \beta_i > 0$, $\alpha_i \neq \beta_i$, $i = 1, 2$, denote

$$\begin{aligned}
f_1^{\alpha,\beta} &:= \min\{f_1(u_1,u_2) \,:\, c_1\,\beta_1 \le u_1 \le \beta_1,\ c_2\,r_2 \le u_2 \le R_2\},\\
f_2^{\alpha,\beta} &:= \min\{f_2(u_1,u_2) \,:\, c_1\,r_1 \le u_1 \le R_1,\ c_2\,\beta_2 \le u_2 \le \beta_2\},\\
F_1^{\alpha,\beta} &:= \max\{f_1(u_1,u_2) \,:\, 0 \le u_1 \le \alpha_1,\ 0 \le u_2 \le R_2\},\\
F_2^{\alpha,\beta} &:= \max\{f_2(u_1,u_2) \,:\, 0 \le u_1 \le R_1,\ 0 \le u_2 \le \alpha_2\},
\end{aligned}$$

where $r_i := \min\{\alpha_i, \beta_i\}$ and $R_i := \max\{\alpha_i, \beta_i\}$.

Theorem

Suppose that there exist positive numbers $\alpha_i, \beta_i > 0$ *with* $\alpha_i \neq \beta_i$*,* $i = 1, 2$*, such that*

$$f_i^{\alpha,\beta} > \beta_i/m, \qquad F_i^{\alpha,\beta} < \alpha_i/M \quad (i = 1, 2).$$

Then the system has at least one positive solution $(u_1, u_2) \in K \times K$ *such that* $r_i < \|u_i\|_\infty < R_i$ $(i = 1, 2)$*.*

Theorem

Suppose that there exist positive numbers $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that

$$f_i^{\alpha,\beta} > \beta_i/m, \qquad F_i^{\alpha,\beta} < \alpha_i/M \quad (i = 1, 2).$$

Then the system has at least one positive solution $(u_1, u_2) \in K \times K$ such that $r_i < \|u_i\|_\infty < R_i$ $(i = 1, 2)$.

Idea of proof

- $f_i^{\alpha,\beta} > \beta_i/m \Longrightarrow T_i(u_1, u_2) + \mu \mathbf{1} \neq u_i$ if $\|u_i\|_\infty = \beta_i$ and $\mu \geq 0$.

Theorem

Suppose that there exist positive numbers $\alpha_i, \beta_i > 0$ *with* $\alpha_i \neq \beta_i$, $i = 1, 2$*, such that*

$$f_i^{\alpha,\beta} > \beta_i/m, \qquad F_i^{\alpha,\beta} < \alpha_i/M \quad (i = 1, 2).$$

Then the system has at least one positive solution $(u_1, u_2) \in K \times K$ *such that* $r_i < \|u_i\|_\infty < R_i$ $(i = 1, 2)$.

Idea of proof

- $f_i^{\alpha,\beta} > \beta_i/m \Longrightarrow T_i(u_1, u_2) + \mu \mathbf{1} \neq u_i$ if $\|u_i\|_\infty = \beta_i$ and $\mu \geq 0$.
- $F_i^{\alpha,\beta} < \alpha_i/M \Longrightarrow T_i(u_1, u_2) \neq \lambda\, u_i$ if $\|u_i\|_\infty = \alpha_i$ and $\lambda \geq 1$.

Example

Consider the system

$$\begin{aligned} &-x'' = h(x)(1+\sin^2(y)),\\ &-y'' = y^2(1+\sin^2(x)),\\ &x(0) = x(1) = 0 = y(0) = y(1), \end{aligned}$$

with

$$h(x) = \begin{cases} \sqrt[3]{x}, & \text{if } x \in [0,1],\\ x^3, & \text{if } x \in (1,10),\\ \sqrt[3]{x-10} + 1000, & \text{if } x \in [10,+\infty). \end{cases}$$

Example

Consider the system

$$
\begin{aligned}
-x'' &= h(x)(1+\sin^2(y)),\\
-y'' &= y^2(1+\sin^2(x)),\\
x(0) &= x(1) = 0 = y(0) = y(1),
\end{aligned}
$$

with

$$
h(x) = \begin{cases} \sqrt[3]{x}, & \text{if } x \in [0,1],\\ x^3, & \text{if } x \in (1,10),\\ \sqrt[3]{x-10} + 1000, & \text{if } x \in [10,+\infty). \end{cases}
$$

Based on the multiplicity results the system has at least three positive solutions (u_1,u_2), (v_1,v_2) and (w_1,w_2) such that

$$
\begin{aligned}
1/512 < \|u_1\|_\infty < 1/4, &\quad 2 < \|u_2\|_\infty < 512,\\
64 < \|v_1\|_\infty < 522, &\quad 2 < \|v_2\|_\infty < 512,\\
1/4 \le \|w_1\|_\infty \le 64, &\quad 2 < \|w_2\|_\infty < 512.
\end{aligned}
$$

Thank you for your attention!

Component-wise localization of solutions for nonlinear systems: a fixed point index approach

Jorge Rodríguez López

CITMAga & Departamento de Estatística, Análise matemática e Optimización

Universidade de Santiago de Compostela, Spain

Funding: Projects PID2020-113275GB-I00 / ED431C 2023/12

International Meetings on Differential Equations and Their Applications (IMDETA)

March 13, 2024