University of Glasgow | School of Mathematics & Statistics

Asymptotic behaviour of solutions of integral equations with singular kernels via a new Gronwall inequality.

Jeff Webb

International Meeting on Differential Equations and Applications
January 12, 2022

Outline

- Problems of interest
- Fractional integral and derivatives
- A ‘fractional’ Gronwall inequality
- Asymptotic behaviour of solutions

Outline

- Problems of interest
- Fractional integral and derivatives
- A ‘fractional’ Gronwall inequality
- Asymptotic behaviour of solutions

This presentation is based on the paper
J. R. L. Webb, A fractional Gronwall inequality and the asymptotic behaviour of global solutions of Caputo fractional problems, *Elec. J. Diff. Eq. 2021*.

Problems of interest

We will investigate the asymptotic behaviour of global solutions (exist on $[0,T]$ for all $T>0$) of problems involving fractional integrals, integral operators with singular kernels

$$u(t) = u_0 + I^\alpha f(t,u(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))\, ds$$

where $0<\alpha<1$, which corresponds to solutions of a fractional differential equation

$$D_*^\alpha u(t) = f(t,u(t)), \quad \text{for a.e. } t>0, \quad u(0)=u_0.$$

$D_*^\alpha u$ is the Caputo derivative, defined precisely later.

Problems of interest

We also similarly study

$$u(t) = u_0 + I^\alpha f(t, u(t), D_*^\gamma u(t))$$
$$= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), D_*^\gamma u(s))\, ds.$$

which corresponds to solution of

$$D_*^\alpha u(t) = f(t, u(t), D_*^\gamma u(t)), \text{ for a.e. } t > 0, \quad u(0) = u_0,$$

when $0 < \gamma < \alpha < 1$ and f is continuous.

Problems of interest

We also similarly study

$$u(t) = u_0 + I^{\alpha} f(t, u(t), D_*^{\gamma} u(t))$$
$$= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), D_*^{\gamma} u(s))\, ds.$$

which corresponds to solution of

$$D_*^{\alpha} u(t) = f(t, u(t), D_*^{\gamma} u(t)), \text{ for a.e. } t > 0, \quad u(0) = u_0,$$

when $0 < \gamma < \alpha < 1$ and f is continuous.

I have done some other problems in the paper, including some of order between 1 and 2, but I do not discuss them in this presentation.

The fractional integral

We consider real valued functions defined on an arbitrary interval $[0, T]$, all functions are supposed measurable and all integrals are Lebesgue integrals.

L^1 is the space of integrable functions (defined on $[0, T]$ unless specified otherwise), $C[0, T]$ is the space of continuous functions, $AC[0, T]$ is the space of Absolutely Continuous functions which is the appropriate space for the fundamental theorem of calculus for Lebesgue integrals: $f \in AC$ if and only if f' exists almost everywhere (a.e.) and belongs to L^1 and

$$f(t) = f(0) + \int_0^t f'(s)\, ds.$$

The fractional integral

For $f \in L^1[0,T]$, as are well-known,
$If(t) := \int_0^t f(s)\,ds$ and $I(If)(t) = I^2 f(t) = \int_0^t (t-s)\,f(s)ds$,
$I^3 f(t) = \int_0^t \frac{(t-s)^2}{2} f(s)ds$ so the definition of the Riemann-Liouville (R-L) fractional integral for a real number $\alpha > 0$

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

is the natural generalization. The Gamma function is, for $\alpha > 0$, given by

$$\Gamma(\alpha) := \int_0^\infty s^{\alpha-1} \exp(-s)\,ds,$$

it extends the factorial function: $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

The fractional integral

For $I^\alpha f$ to be defined we want $f \in L^1[0,T]$ (at least), then the formula makes sense for a.e. t.

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds.$$

$I^\alpha f$ is the convolution $h * f$ of the L^1 functions h, f where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so, by well known results on convolutions, $I^\alpha f$ is defined as an L^1 function, in particular $I^\alpha f(t)$ is finite for a.e. t.
When $\alpha \geq 1$ this is a regular integral operator, but for $0 < \alpha < 1$ there is a singularity along the line $t = s$.

Fractional derivatives

The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a function u is informally defined by

$$D^\alpha u(t) = D\big(I^{1-\alpha}u(t)\big), \quad \text{where } Du = u'.$$

To be a valid and useful definition it is required that $I^{1-\alpha}u \in AC$.
Then $D^\alpha u(t)$ is defined for a.e. t and is an L^1 function.
For higher orders, if $m-1 < \beta < m$ the R-L derivative is defined for a.e. t by $D^\beta u(t) = D^m(I^{m-\beta}u)(t)$ provided that $I^{m-\beta}u \in AC^{m-1}$ (order $m-1$ derivative in AC).
Note that $I^{m-\beta}u$ is always the case of a singular kernel.

Fractional derivatives

The Caputo fractional derivative is frequently defined with the derivative and fractional integral taken in the reverse order to that of the R-L derivative, that is for order $0 < \alpha < 1$, it is informally defined by

$$D_C^\alpha u(t) = (I^{1-\alpha} Du)(t), \quad \text{where } Du = u'.$$

It is required that $u \in AC$, then $D_C^\alpha u$ is defined for a.e. t.

Fractional derivatives

This definition has a disadvantage. We want solutions of the initial value problem (IVP)

$D_C^\alpha u(t) = f(t),\ u(0) = u_0$ (with f continuous)

to be given by

$u \in C[0, T]$ and $u(t) = u_0 + I^\alpha f(t)$.

Fractional derivatives

This definition has a disadvantage. We want solutions of the initial value problem (IVP)

$$D_C^\alpha u(t) = f(t),\ u(0) = u_0 \text{ (with } f \text{ continuous)}$$

to be given by

$$u \in C[0,T] \text{ and } u(t) = u_0 + I^\alpha f(t).$$

If $u \in AC$ satisfies $D_C^\alpha u(t) = f(t),\ u(0) = u_0$ then $u \in C[0,T]$ does satisfy $u(t) = u_0 + I^\alpha f(t)$.

Unfortunately I^α does not map all of $C[0,T]$ into $AC[0,T]$ (Weierstrass type function, as shown by Hardy and Littlewood 1928).

So $u \in C[0,T]$ and $u(t) = u_0 + I^\alpha f(t)$ does not imply $D_C^\alpha u$ exists without extra (often unwanted) conditions on f.

Alternative Caputo definition

The way round is to define, for $0 < \alpha < 1$ and $1 < \beta < 2$, as in Diethelm's well known text, the Caputo fractional operator (we only use this definition and call it the Caputo derivative)

$$D_*^\alpha u(t) = D^\alpha(u - u_0)(t); \quad D_*^\beta u(t) = D^\beta(u - u_0 - tu'(0))(t),$$

under the appropriate conditions for the R-L derivatives to exist.

Alternative Caputo definition

NOTE: Fractional integrals commute $I^{\alpha} I^{\beta} f = I^{\alpha+\beta} f$ (semigroup property) but fractional derivatives do not in general.

$$D_*^{\alpha}(D_*^{\beta} u) \neq D_*^{\beta}(D_*^{\alpha} u) \neq D_*^{\alpha+\beta} u,$$

equality can occur but it usually does not.

Alternative Caputo definition

NOTE: Fractional integrals commute $I^{\alpha} I^{\beta} f = I^{\alpha+\beta} f$ (semigroup property) but fractional derivatives do not in general.

$$D_*^{\alpha}(D_*^{\beta} u) \neq D_*^{\beta}(D_*^{\alpha} u) \neq D_*^{\alpha+\beta} u,$$

equality can occur but it usually does not.

For $0 < \alpha < 1$, when $u \in AC$ the two definitions coincide, $D_C^{\alpha} u(t) = D_*^{\alpha} u(t)$ for a.e. t. In fact,

$$\begin{aligned} D_*^{\alpha} u(t) = DI^{1-\alpha}(u - u_0) &= DI^{1-\alpha}(Iu') \\ &= DI\, I^{1-\alpha} u' = I^{1-\alpha} u' = D_C^{\alpha} u. \end{aligned}$$

Caputo advantages

The main advantages of the Caputo derivative $D_*^\alpha u$ over the R-L derivative are that $D_*^\alpha(c) = 0$ when c is a constant function (any $\alpha > 0$), whereas the R-L derivative (for $0 < \alpha < 1$) of a constant has a singularity at zero; and initial value problems for the Caputo derivative are well posed when initial values are prescribed on the function and its ordinary derivatives, fractional integrals and derivatives should be prescribed in the R-L case.

Caputo advantages

The main advantages of the Caputo derivative $D_*^\alpha u$ over the R-L derivative are that $D_*^\alpha(c) = 0$ when c is a constant function (any $\alpha > 0$), whereas the R-L derivative (for $0 < \alpha < 1$) of a constant has a singularity at zero; and initial value problems for the Caputo derivative are well posed when initial values are prescribed on the function and its ordinary derivatives, fractional integrals and derivatives should be prescribed in the R-L case.

Fractional derivatives are nonlocal operators, $D_*^\alpha u(t)$ depends on all values $u(s), s \in [0, t]$. Hence they are used in models where there are memory effects.

Some properties of fractional integrals.

Fractional integrals were studied in depth long ago, for example G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. I., *Math. Z.* 27 (1928), 565–606. There became renewed interest with the study of fractional differential equations, which has grown enormously in the last twenty years.

Some properties of fractional integrals.

For $0 < \alpha < 1$ (the singular kernel case) we list some properties of the fractional integrals (on a finite interval $[0, T]$).

- I^α is a bounded linear operator from $L^p[0, T]$ into L^p for all $1 \leq p \leq \infty$.
- For $p > 1/\alpha$, the fractional integral operator I^α is bounded from L^p into a Hölder space, that is for $f \in L^p$, $I^\alpha f$ is Hölder continuous with exponent $\alpha - 1/p$, thus $I^\alpha f$ is continuous.

Some properties of fractional integrals.

For $0 < \alpha < 1$ (the singular kernel case) we list some properties of the fractional integrals (on a finite interval $[0, T]$).

- I^α is a bounded linear operator from $L^p[0, T]$ into L^p for all $1 \leq p \leq \infty$.
- For $p > 1/\alpha$, the fractional integral operator I^α is bounded from L^p into a Hölder space, that is for $f \in L^p$, $I^\alpha f$ is Hölder continuous with exponent $\alpha - 1/p$, thus $I^\alpha f$ is continuous.
- I^α maps $C[0, T]$ into $C[0, T]$ and $AC[0, T]$ into $AC[0, T]$.
- I^α does not map $C^1[0, T]$ into $C^1[0, T]$.
- I^α does not map $C[0, T]$ into $AC[0, T]$ (it maps $C[0, T]$ into the Hölder space $C^{0,\alpha}$).

Classical Gronwall inequality

THEOREM (Simple version of the classical Gronwall inequality) *Suppose that* $u \in C_+[0,T]$ *satisfies*

$$u(t) \leq a + \int_0^t \phi(s)u(s)\,ds$$

for $t \in [0,T]$, *where* $a > 0$ *and* $\phi \in L^1_+[0,T]$*. Then*

$$u(t) \leq a \exp\left(\int_0^t \phi(s)\,ds\right)$$

for $t \in [0,T]$.

If also $\phi \in L^1[0,\infty)$ *then* u *is uniformly bounded for all* $t > 0$*.*

Subscript $+$ means functions are non-negative.

Classical Gronwall inequality

THEOREM (Simple version of the classical Gronwall inequality) *Suppose that* $u \in C_+[0,T]$ *satisfies*

$$u(t) \le a + \int_0^t \phi(s)u(s)\,ds$$

for $t \in [0,T]$, *where* $a > 0$ *and* $\phi \in L^1_+[0,T]$. *Then*

$$u(t) \le a\exp\Big(\int_0^t \phi(s)\,ds\Big)$$

for $t \in [0,T]$.

If also $\phi \in L^1[0,\infty)$ *then* u *is uniformly bounded for all* $t > 0$.

Subscript $+$ means functions are non-negative.

The proof of this simple case is easy. Let $w(t) := a + \int_0^t \phi(s)u(s)\,ds$, then $w(t) \ge w(0) = a > 0$, $w'(t) = \phi(t)u(t) \le \phi(t)w(t)$. Thus $w'(t)/w(t) \le \phi(t)$ which can be integrated to give the result.

Singular kernel Gronwall inequality

For Gronwall type inequalities with singular kernels the pioneering work was done by Dan Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics, No. 840. Springer-Verlag, Berlin-New York, 1981.

He used an iteration method and gave results as infinite series. A version of one result is as follows.

THEOREM. *Let* a, b *be positive and let* $0 < \alpha < 1$*. If* $u \in L^\infty_+[0,T]$ *satisfies the inequality*

$$u(t) \leq a + b \int_0^t (t-s)^{\alpha-1} u(s)ds, \; t \in [0,T],$$

then $u(t) \leq aE_\alpha(b\Gamma(\alpha)t^\alpha)$ *where* E_α *is the Mittag-Leffler function.*

Exponential function version

The Mittag-Leffler function is an entire function of $z \in \mathbb{C}$ defined by a power series $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$. The special case $E_1(z)$ is the exponential function.

Exponential function version

The Mittag-Leffler function is an entire function of $z \in \mathbb{C}$ defined by a power series $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$. The special case $E_1(z)$ is the exponential function.

In *Nonlinear Evolution Equations–Global Behavior of Solutions*, Lecture Notes in Mathematics, No. 841, Springer–Verlag, Berlin–New York, 1981

Haraux proved, for the special case $\alpha = 1/2$, using a method he attributed to Pazy of reducing to the classical Gronwall inequality, an estimate involving the exponential function instead of the Mittag-Leffler function.

Previous exponential function version

I used the ideas of that proof to extend Haraux's result to all $\alpha \in (0,1)$, also with u replaced by $t^{-\gamma}u$, with $0 < \gamma < \alpha$, in the paper
JRLW, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.* 471 (2019), 692–711.

To make progress on the asymptotic behaviour I realized that a more general result was required, and could be done by the same method.

Previous exponential function version

I used the ideas of that proof to extend Haraux's result to all $\alpha \in (0, 1)$, also with u replaced by $t^{-\gamma}u$, with $0 < \gamma < \alpha$, in the paper
JRLW, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.* 471 (2019), 692–711.

To make progress on the asymptotic behaviour I realized that a more general result was required, and could be done by the same method.
Medved (1997) used Hölder's inequality and other techniques to prove a version involving exponential functions of a different type, not simple enough to write here. Also Zhu (2020) has used Medved's ideas to get some other versions.

New singular kernel Gronwall inequality

THEOREM. *Let* $a > 0$, $0 < \beta < 1$ *and let* ϕ *be non-increasing*, $\phi \in L^1_+[0, T]$ *for all* $T > 0$. *Suppose that* $u \in C_+[0, \infty)$ *satisfies the inequality*

$$u(t) \leq a + \int_0^t (t-s)^{-\beta} \phi(s) u(s)\, ds, \text{ for } t > 0,$$

If there exists $r \in (0, 1)$ *and* $t_r > 0$ *such that* $(I^{1-\beta}\phi)(t) \leq \dfrac{r}{\Gamma(1-\beta)}$ *for* $0 \leq t \leq t_r$, *then*

$$u(t) \leq \frac{a}{1-r} \exp\Big(\frac{1}{t_r^{\beta}(1-r)} \int_0^t \phi(s)\, ds \Big), \text{ for every } t > 0.$$

New singular kernel Gronwall inequality

Moreover, if, in addition, $\phi \in L^1_+[0,\infty)$ *then*

$$u(t) \leq \frac{a}{1-r}\exp\Big(\frac{1}{t_r^\beta(1-r)}\int_0^\infty \phi(s)\,ds\Big),$$

so that $u(t)$ *is uniformly bounded.*
Often $r_1 = 1/2$ is an allowed value (as in the next slide) giving the simpler expression

$$u(t) \leq 2a\exp\Big(2\,t_{r_1}^{-\beta}\int_0^t \phi(s)ds\Big).$$

New singular kernel Gronwall inequality

The result allows ϕ to have a singularity at 0. A suitable t_r exists for any $r \in (0,1)$ (e.g. $r = 1/2$) if $I^{1-\beta}\phi(t) \to 0$ as $t \to 0+$. Example, if for some $0 < \eta < 1-\beta$, $\phi(t) = t^{-\eta}v(t)$, v bounded for all small t, then

$$\begin{aligned} I^{1-\beta}\phi(t) &= \tfrac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} s^{-\eta} v(s)\, ds \\ &= \tfrac{1}{\Gamma(1-\beta)} t^{1-\beta-\eta} \int_0^1 (1-\sigma)^{-\beta} \sigma^{-\eta} v(t\sigma)\, d\sigma, \end{aligned}$$

which proves the limit is 0, since v is bounded and $\int_0^1 (1-\sigma)^{-\beta}\sigma^{-\eta}\, d\sigma = B(1-\beta, 1-\eta)$. Beta function.

New singular kernel Gronwall inequality

The result allows ϕ to have a singularity at 0. A suitable t_r exists for any $r \in (0,1)$ (e.g. $r = 1/2$) if $I^{1-\beta}\phi(t) \to 0$ as $t \to 0+$. Example, if for some $0 < \eta < 1-\beta$, $\phi(t) = t^{-\eta}v(t)$, v bounded for all small t, then

$$I^{1-\beta}\phi(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} s^{-\eta} v(s)\, ds$$
$$= \frac{1}{\Gamma(1-\beta)} t^{1-\beta-\eta} \int_0^1 (1-\sigma)^{-\beta} \sigma^{-\eta} v(t\sigma)\, d\sigma,$$

which proves the limit is 0, since v is bounded and $\int_0^1 (1-\sigma)^{-\beta}\sigma^{-\eta}\, d\sigma = B(1-\beta, 1-\eta)$. Beta function.

$B(p,q) := \int_0^1 (1-s)^{p-1} s^{q-1}\, ds$ well defined Lebesgue integral for $p > 0, q > 0$ and $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Proof of the Gronwall type inequality

We will use the notation $u^*(t) := \max_{s\in[0,t]} u(s)$. Let $t > 0$ and let $\tau \in (0,t]$ be arbitrary (fixed). For $\tau \leq t_r$ we have

$$
\begin{aligned}
u(\tau) \leq a + \int_0^\tau (\tau - s)^{-\beta}\phi(s)u(s)\,ds \\
&\leq a + u^*(t)\int_0^\tau (\tau - s)^{-\beta}\phi(s)\,ds \\
&= a + \Gamma(1-\beta)\big(I^{1-\beta}\phi\big)(\tau)u^*(t) \leq a + ru^*(t).
\end{aligned}
$$

Proof of the Gronwall type inequality

Now we consider the case when $\tau > t_r$. We have

$$u(\tau) \leq a + \int_0^{\tau - t_r} (\tau - s)^{-\beta} \phi(s) u(s)\, ds + \int_{\tau - t_r}^{\tau} (\tau - s)^{-\beta} \phi(s) u(s)\, ds.$$

In the first integral we use the fact that $\tau - s \geq t_r$ so that $(\tau - s)^{-\beta} \leq t_r^{-\beta}$, while in the second integral we use $s \geq s - (\tau - t_r) \geq 0$ and the fact that ϕ is non-increasing. This gives

$$\begin{aligned} u(\tau) \leq & a + \int_0^{\tau - t_r} t_r^{-\beta} \phi(s) u^*(s)\, ds \\ & + u^*(t) \int_{\tau - t_r}^{\tau} (\tau - s)^{-\beta} \phi(s - (\tau - t_r))\, ds. \end{aligned}$$

Proof of the Gronwall type inequality

In the second integral we now let $\sigma = s - (\tau - t_r)$ and it becomes

$$\int_0^{t_r} (t_r - \sigma)^{-\beta} \phi(\sigma) d\sigma = \Gamma(1-\beta)(I^{1-\beta}\phi)(t_r) \leq r.$$

Proof of the Gronwall type inequality

In the second integral we now let $\sigma = s - (\tau - t_r)$ and it becomes

$$\int_0^{t_r} (t_r - \sigma)^{-\beta} \phi(\sigma) d\sigma = \Gamma(1-\beta)(I^{1-\beta}\phi)(t_r) \le r.$$

Thus we get

$$\begin{aligned} u(\tau) &\le a + b \int_0^{\tau - t_r} t_r^{-\beta} \phi(s) u^*(s)\, ds + r u^*(t) \\ &\le a + b \int_0^{\tau} t_r^{-\beta} \phi(s) u^*(s)\, ds + r u^*(t). \end{aligned}$$

Proof continuing

With the first part we see that the last equation holds for all $\tau \in (0, t]$. Taking the $\sup_{\tau \in (0,t]}$, we obtain

$$u^*(t) \leq a + \int_0^t t_r^{-\beta} \phi(s) u^*(s)\, ds + r u^*(t).$$

Hence we have

$$u^*(t) \leq \frac{a}{1-r} + \frac{1}{1-r} \int_0^t t_r^{-\beta} \phi(s) u^*(s)\, ds, \quad \text{for all } t > 0.$$

Proof continuing

This is now the classical Gronwall inequality, and we can immediately deduce that

$$u(t) \leq u^*(t) \leq \frac{a}{1-r} \exp\left(\frac{1}{1-r}\, t_r^{-\beta} \int_0^t \phi(s) ds\right), \quad \text{for all } t > 0.$$

Proof continuing

This is now the classical Gronwall inequality, and we can immediately deduce that

$$u(t) \le u^*(t) \le \frac{a}{1-r} \exp\Big(\frac{1}{1-r} \, t_r^{-\beta} \int_0^t \phi(s) ds \Big), \quad \text{for all } t > 0.$$

The uniform boundedness if $\phi \in L^1[0, \infty)$ is now obvious.

Boundedness property

The following fact will be important in the following discussions. The non-increasing property of the function is important for our proof and prevents ϕ having any spikes.

THEOREM. *Let* $0 < \alpha < 1$ *and suppose that* ϕ *is non-increasing,* $\phi \in L^1_+(0, \infty)$ *and there exist* $t_1 > 0$ *and a constant* $M > 0$ *such that* $(I^\alpha \phi)(t) \leq M$ *for* $0 \leq t \leq t_1$ *(no blow-up at* 0*). Then* $I^\alpha \phi(t)$ *is uniformly bounded for all* $t > 0$*.*

Boundedness property

The following fact will be important in the following discussions. The non-increasing property of the function is important for our proof and prevents ϕ having any spikes.

THEOREM. *Let $0 < \alpha < 1$ and suppose that ϕ is non-increasing, $\phi \in L^1_+(0,\infty)$ and there exist $t_1 > 0$ and a constant $M > 0$ such that $(I^\alpha \phi)(t) \leq M$ for $0 \leq t \leq t_1$ (no blow-up at 0). Then $I^\alpha \phi(t)$ is uniformly bounded for all $t > 0$.*

The proof uses the same method of splitting the integral as in the proof of the Gronwall type inequality.

Boundedness property

The following fact will be important in the following discussions. The non-increasing property of the function is important for our proof and prevents ϕ having any spikes.

THEOREM. *Let* $0 < \alpha < 1$ *and suppose that* ϕ *is non-increasing,* $\phi \in L^1_+(0, \infty)$ *and there exist* $t_1 > 0$ *and a constant* $M > 0$ *such that* $(I^\alpha \phi)(t) \leq M$ *for* $0 \leq t \leq t_1$ *(no blow-up at* 0*). Then* $I^\alpha \phi(t)$ *is uniformly bounded for all* $t > 0$*.*

The hypotheses do not imply that $I^\alpha \phi \in L^1[0, \infty)$.

Example

EXAMPLE. Let $f(t) = \frac{t^{-\gamma}}{1+t}$ for $0 < \gamma < 1$. Then $f \in L^1[0,\infty)$ and for $0 < \gamma < \alpha < 1$ we have $I^\alpha f(t) \to 0$ as $t \to 0+$. $I^\alpha f$ is uniformly bounded but $I^\alpha f \notin L^1[0,\infty)$.
In fact,

$$
\begin{aligned}
I^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{s^{-\gamma}}{1+s}\, ds \\
&\geq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{t^{-\gamma}}{1+t}\, ds \\
&= \frac{t^{\alpha-\gamma}}{\Gamma(\alpha+1)(1+t)},
\end{aligned}
$$

which shows that $I^\alpha f \notin L^1[0,\infty)$ since $\alpha - \gamma > 0$.

Asymptotic behaviour, order $\alpha < 1$

The asymptotic behaviour of global solutions of a problem such as

$$u(t) = u_0 + I^\alpha f(t, u(t)),$$

where $0 < \alpha < 1$, clearly depends on the behaviour of $I^\alpha f(t, u(t))$ as $t \to \infty$, which depends on the behaviour of the term $f(t, u)$. Our standard type of hypothesis is that f has linear growth in the dependent variable, that is $|f(t, u)| \leq \phi(t)(1 + |u|)$ for suitable ϕ.

Asymptotic behaviour, order $\alpha < 1$

We first study a simple case of the asymptotic behaviour of solutions (in $C[0,T]$ for all $T > 0$) of the integral equation

$$u(t) = u_0 + I^\alpha(f(t, u(t))) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds,$$

with $0 < \alpha < 1$ and f satisfying $|f(t,u)| \leq \phi(t)(1 + |u|)$. This corresponds to solutions of

$$D_*^\alpha u(t) = f(t, u(t)), \text{ for a.e. } t > 0, \quad u(0) = u_0.$$

Asymptotic behaviour, order $\alpha < 1$

Our result for this problem is the following.

THEOREM. *Let* $0 < \alpha < 1$, $\phi \in L^1_+[0, \infty)$, *and let* ϕ *be non-increasing and suppose that* $I^\alpha \phi(t) \to 0$ *as* $t \to 0$. *Let* f *satisfy* $|f(t, u)| \le \phi(t)(1 + |u|)$ *for all* $t \in [0, \infty)$ *and all* $u \in \mathbb{R}$. *If* u *is a global solution of the integral equation*

$$u(t) = u_0 + I^\alpha f(t, u(t))$$

then $|u|$ *is uniformly bounded on* $[0, \infty)$.

Asymptotic behaviour, order $\alpha < 1$

Proof. We note that $I^\alpha \phi(t) \le C_0$ for every $t > 0$ by the boundedness property. Then we have

$$\begin{aligned} |u(t)| &\le |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s)(1 + |u(s)|)\, ds \\ &\le |u_0| + C_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) |u(s)|)\, ds. \end{aligned}$$

By the fractional Gronwall inequality, for t_1 chosen so that $\Gamma(\alpha)(I^\alpha \phi)(t) < 1/2$ for $t \le t_1$, we obtain
$u(t) \le 2(|u_0| + C_0) \exp\left(2t_1^{\alpha-1} \int_0^t \phi(s)\, ds\right),$ for every $t > 0$,
hence $|u(t)| \le 2(|u_0| + C_0) \exp\big(2t_1^{\alpha-1} \int_0^\infty \phi(s)\, ds\big)$, that is $|u|$ is uniformly bounded.

f depends on fractional derivatives

We now turn to the much trickier case, when $0 < \gamma < \alpha < 1$, of global solutions of the integral equation

$$\begin{aligned} u(t) &= u_0 + I^\alpha f(t, u(t), D_*^\gamma u(t)) \\ &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), D_*^\gamma u(s))\, ds. \end{aligned}$$

This corresponds to solutions of the fractional derivative equation

$$D_*^\alpha u(t) = f(t, u(t), D_*^\gamma u(t)), \text{ for a.e. } t > 0, \quad u(0) = u_0,$$

f depends on fractional derivatives

These can only be equivalent problems in an appropriate space, in this case

$$X := \{u \in C[0,T], D_*^\gamma u \in C[0,T]\}$$

endowed with the norm $\|u\|_X := \|u\|_\infty + \|D_*^\gamma u\|_\infty$. Thus $u \in X$ is equivalent to $u \in C[0,T]$ and $I^{1-\gamma}(u - u_0) \in C^1[0,T]$.

We will not discuss existence but assume a global solution exists, that is we assume that for every $T > 0$ a solution exists in the Banach space X.

Bounded asymptotic behaviour

THEOREM. *Let* $0 < \gamma < \alpha < 1$. *Let* f *satisfy* $|f(t,u,p)| \leq \phi(t)(1 + |u| + |p|)$ *for all* $t \in [0,\infty)$ *and all* $u, p \in \mathbb{R}$, *where* $(I^{\alpha-\gamma}\phi)(t) \to 0$ *as* $t \to 0+$, *and* $\phi(s)$ *and* $s^\gamma \phi(s)$ *are non-increasing* $L^1[0,\infty)$ *functions. If* u *is a global solution of*

$$u(t) = u_0 + I^\alpha f(t, u(t), D_*^\gamma u(t)),$$

then $|u|$ *and* $|D_*^\gamma u|$ *are uniformly bounded on* $[0,\infty)$.

Useful fact

We need the following result.
Let $f \in L^1[0,T]$ and $\alpha \geq \beta > 0$. Then

$$\frac{\Gamma(\alpha)(I^\alpha|f|)(t)}{t^\alpha} \leq \frac{\Gamma(\beta)(I^\beta|f|)(t)}{t^\beta} \text{ for a.e. } t \in (0,T).$$

Useful fact

We need the following result.
Let $f \in L^1[0,T]$ and $\alpha \geq \beta > 0$. Then

$$\frac{\Gamma(\alpha)(I^\alpha|f|)(t)}{t^\alpha} \leq \frac{\Gamma(\beta)(I^\beta|f|)(t)}{t^\beta} \text{ for a.e. } t \in (0,T).$$

We can suppose $f \geq 0$, then for a.e. t we have

$$\Gamma(\alpha)I^\alpha f(t) = \int_0^t (t-s)^{\alpha-1} f(s)\,ds = \int_0^t t^{\alpha-1}(1-s/t)^{\alpha-1} f(s)\,ds$$
$$\leq \int_0^t t^{\alpha-1}(1-s/t)^{\beta-1} f(s)\,ds = \int_0^t t^{\alpha-\beta}(t-s)^{\beta-1} f(s)\,ds$$
$$= t^{\alpha-\beta}\Gamma(\beta)I^\beta f(t).$$

Proof of Theorem.

If $u \in X$ and $u(t) = u_0 + I^\alpha\big(f(t, u(t), D_*^\gamma u(t))\big)$, then, using the useful fact above, we obtain

$$|u - u_0| = |I^\alpha(f)| \le I^\alpha|f| \le t^\gamma\Gamma(\alpha-\gamma)I^{\alpha-\gamma}|f|,$$

hence $|u| \le |u_0| + t^\gamma\Gamma(\alpha-\gamma)I^{\alpha-\gamma}|f|$.

Also we have, by definition, and the semigroup property that

$$D_*^\gamma u = D(I^{1-\gamma})(u-u_0) = D(I^{1-\gamma}I^\alpha)f = DII^{\alpha-\gamma}f = I^{\alpha-\gamma}f,$$

and hence $|D_*^\gamma u(t)| \le I^{\alpha-\gamma}|f|$ a.e.

Proof

We now have

$$I^{\alpha-\gamma}|f| = \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}|f(s,u(s),D_*^\gamma u(s))|\,ds$$

$$\leq \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)(1+|u(s)|+|D_*^\gamma u(s))|\,ds$$

$$\leq \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\times$$

$$\times\left(1+|u_0|+s^\gamma\Gamma(\alpha-\gamma)I^{\alpha-\gamma}|f|+I^{\alpha-\gamma}|f|\right)ds.$$

Proof

Write $v(t) := I^{\alpha-\gamma}|f|(t)$, then v satisfies the inequality

$$
\begin{aligned}
v(t) \leq\ & \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} \phi(s) \times \\
& \times \big(1 + |u_0| + s^\gamma \Gamma(\alpha-\gamma) v(s) + v(s)\big)\, ds, \\
& = (1+|u_0|)(I^{\alpha-\gamma}\phi)(t) \\
& + \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} \phi(s) \big(\Gamma(\alpha-\gamma) s^\gamma + 1)\big) v(s)\, ds
\end{aligned}
$$

Proof

Now we note that $I^{\alpha-\gamma}\phi$ is bounded on $[0,\infty)$ by the boundedness property, say $I^{\alpha-\gamma}\phi(t) \le M$ for all $t \ge 0$. Then

$$v(t) \le M(1+|u_0|) + \int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)v(s)\,ds.$$

Proof

Now we note that $I^{\alpha-\gamma}\phi$ is bounded on $[0,\infty)$ by the boundedness property, say $I^{\alpha-\gamma}\phi(t) \le M$ for all $t \ge 0$. Then

$$v(t) \le M(1+|u_0|) + \int_0^t (t-s)^{\alpha-\gamma-1}\phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)v(s)\,ds.$$

By the fractional Gronwall inequality we deduce that,

$$\begin{aligned} v(t) &\le 2M(1+|u_0|)\exp\Big(2\,t_1^{1+\gamma-\alpha}\int_0^t \phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)\,ds\Big), \\ &\le 2M(1+|u_0|)\exp\Big(2\,t_1^{1+\gamma-\alpha}\int_0^\infty \phi(s)\big(s^\gamma + \frac{1}{\Gamma(\alpha-\gamma)}\big)\,ds\Big), \end{aligned}$$

a constant, M_1 (say). Since $|D_*^\gamma u(t)| \le I^{\alpha-\gamma}|\phi f| = v(t)$ this has proved that $|D_*^\gamma u(t)| \le M_1$.

Proof

Next we have $u(t) = u_0 + I^\alpha f$ thus $|u(t)| \leq |u_0| + I^\alpha |f|$. This gives

$$\begin{aligned}
|u(t)| &\leq |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s), D_*^\gamma u(s))| \, ds \\
&\leq |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s)(1 + |u(s)| + M_1) \, ds \\
&= |u_0| + (1 + M_1) I^\alpha \phi(t) + \int_0^t (t-s)^{\alpha-1} \phi(s) |u(s)| \, ds.
\end{aligned}$$

Proof concluded

Since $I^\alpha \phi(t)$ is uniformly bounded, we have

$$|u(t)| \leq M_2 + \int_0^t (t-s)^{\alpha-1} \phi(s) |u(s)| \, ds.$$

By the fractional Gronwall inequality we deduce that $|u(t)| \leq M_3$ for all $t > 0$. This completes the proof.

Proof concluded

Since $I^{\alpha}\phi(t)$ is uniformly bounded, we have

$$|u(t)| \leq M_2 + \int_0^t (t-s)^{\alpha-1}\phi(s)|u(s)|\, ds.$$

By the fractional Gronwall inequality we deduce that $|u(t)| \leq M_3$ for all $t > 0$. This completes the proof.

The apparently more general case when $|f(t,u,p)| \leq \phi(t)(a + b|u| + c|p|)$ is really the same since $a + b|u| + c|p| \leq \max\{a,b,c\}(1 + |u| + |p|)$.

This problem was recently studied by Kassim and Tatar (Electron. J. Diff. Eqs 2020) and they proved boundedness of $|u|$ and $|D_*^{\gamma}u|$ assuming f satisfies a special multiplicative type inequality which seems more restrictive than our sum inequality.

Other problems

We also investigated some higher order problems such as the integral equation

$$u(t) = u_0 + a(t)u_1 + I^{\alpha+\beta} f(t, u(t)), \; t > 0,$$

where a is continuous, u_0, u_1 are constants, $0 < \alpha, \beta \leq 1$ with $1 < \alpha + \beta < 2$.

For $a(t) = t$ this arises from seeking solutions of the IVP

$$D_*^{\alpha+\beta}(t) = f(t, u(t)), \; u(0) = u_0, u'(0) = u_1,$$

and for $a(t) = t^\beta$ with $0 < \beta < 1$ it arises from the similar problem

$$D_*^{\alpha}(D_*^{\beta} u)(t) = f(t, u(t)), \;\; u(0) = u_0, D_*^{\beta}(0) = u_1 \Gamma(\beta + 1).$$

Other problems

We assume
$\phi \in L^1[0,\infty), a\,\phi \in L^1[0,\infty), t^{\alpha+\beta-1}\phi(t) \in L^1[0,\infty)$,
Here the classical Gronwall inequality is sufficient, then ϕ does not have to be non-increasing. The result found is
$u(t) - \left(u_0 + a(t)u_1 + Lt^{\alpha+\beta-1}\right) \to 0$ as $t \to \infty$,
where $L = \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty f(s, u(s))\, ds$, proved to exist so is a constant.

Other problems

We also considered the problem

$u(t) = u_0 + I^{\beta} b_1 + I^{\alpha+\beta} f(t, u(t), D_*^{\gamma} u(t))$,

for $0 < \gamma \leq \beta \leq 1$, $0 < \alpha \leq \alpha + \beta - \gamma < 1$ and $\alpha + \beta > 1$; (no ordering between α and β). This corresponds to the sequential fractional differential problem

$$D_*^{\alpha}\big(D_*^{\beta} u(t)\big) = f(t, u(t), D_*^{\gamma} u(t)), a.e.$$

$$u(0) = u_0,\ D_*^{\beta} u(0) = b_1.$$

In this case, under some hypotheses, there is L such that $u(t) - \big(u_0 + \frac{b_1}{\Gamma(\beta+1)} t^{\beta} + L t^{\alpha+\beta-1}\big) \to 0$ as $t \to \infty$.

Since $\alpha < 1$ the term with L can be discarded. This used a combination of Classical and fractional Gronwall inequalities.

References

K. Diethelm, *The analysis of fractional differential equations*. Lecture Notes in Mathematics No. 2004. Springer-Verlag, Berlin, 2010.

A. Haraux, *Nonlinear Evolution Equations, Global Behavior of Solutions*, Lecture Notes in Mathematics, No. 841, Springer-Verlag, Berlin, New York, 1981

D. Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics, No. 840. Springer-Verlag, Berlin, New York, 1981

G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals. I., *Math. Z.* 27 (1928), 565–606.

M. Medveď, A new approach to an analysis of Henry type integral inequalities and their Bihari type versions, *J. Math. Anal. Appl.* 214, (1997), 349–366.

More references

T. Zhu, Fractional integral inequalities and global solutions of fractional differential equations, *Electron. J. Qual. Theory Differ. Equ.* 2020, Paper No. 5, 16 pp.

J. R. L. Webb, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.* 471 (2019), 692–711.

M. D. Kassim, N.-E. Tatar, Convergence of solutions of fractional differential equations to power-type functions, *Electron. J. Differential Equations*, Vol. 2020 (2020), No. 111, 14 pp.

J. R. L. Webb, A fractional Gronwall inequality and the asymptotic behaviour of global solutions of Caputo fractional problems, *Electron. J. Differential Equations*, Vol. 2021, No. 80, 22 pp.

Thank you for listening!