Wrong *a priori* ideas in the study of boundary value problems and periodic solutions : some personal experiences

Jean Mawhin

Université catholique de Louvain

I. Nonresonance conditions for Dirichlet problems : topological vs variational methods

Nonresonance at the first eigenvalue

- $I = [0,\pi],\ f \in C(I \times \mathbb{R}, \mathbb{R}),\ F(x,u) := \int_0^u f(x,s)\,ds$
- $-u'' + f(x,u) = 0,\ u(0) = 0 = u(\pi)$ **(DP)**
  - Euler-Lagrange eqn for $\varphi(u) := \int_I [\frac{u'(x)^2}{2} - F(x,u(x))]\,dx$
- **thm** (LICHTENSTEIN, JRAM, 1915) : *(DP) has a solution if* $\overline{\lim}_{|u|\to\infty} F(x,u) < \infty$ *uniformly in* $x \in I$ **(L)**
- **thm** (HAMMERSTEIN, AcMa, 1930) : *(DP) has a solution if* $\overline{\lim}_{|u|\to\infty} \frac{2F(x,u)}{u^2} < 1$ *uniformly in* $x \in I$ **(H)**
  - **proof** : Ritz method – minimization – limit process
- (H) extends $\overline{\lim}_{|u|\to\infty} \frac{f(x,u)}{u} < 1$ uniformly in $x \in I$
- (H) sharp : $-u'' = u + \sin x,\ u(0) = 0 = u(\pi)$ not solvable

A surprising improvement

- $g \in C(\mathbb{R}),\ G(u) := \int_0^u g(s)\,ds,\ h \in C(I)$
- **thm** (FERNANDES-OMARI-ZANOLIN, DIE, 1989) :
  $-u'' = g(u) - h(x),\ u(0) = 0 = u(\pi)$ **(SDP)** *is solvable if*
  $\underline{\lim}_{u\to-\infty} \frac{2G(u)}{u^2} < 1$ *and* $\underline{\lim}_{u\to+\infty} \frac{2G(u)}{u^2} < 1$ **(FOZ)**
  - **improves (H)** when $\underline{\lim}_{u\to\mp\infty} \frac{G(u)}{u^2} < \overline{\lim}_{u\to\mp\infty} \frac{G(u)}{u^2}$
    i.e. when $G(u)/u^2$ **oscillates** at infinity :
    $G(u) = \frac{u^2}{2}\left[a + \sin(\log(u^2+1))\right]\ (0 < a < 1)$
    $\underline{\lim}_{u\to\mp\infty} \frac{2G(u)}{u^2} = a < 1,$
    $\overline{\lim}_{|u|\to\infty} \frac{g(u)-h(x)}{u} = a + \sqrt{2} > a + 1 = \overline{\lim}_{|u|\to\infty} \frac{2G(u)-hu}{u^2}$
  - **proof** uses LERAY-SCHAUDER degree + time maps
    (very technical !)

Wrong *a priori* ideas

- The Fernandes-Omari-Zanolin result (FOZ)
  - used Leray-Schauder degree to obtain better results than the variational approach for variational probblems
  - killed two of my (many) **wrong** *a priori* ideas :
    - for variational problems, the variational approach gives better results than the Leray-Schauder degree
    - the Leray-Schauder degree cannot prove existence with assumptions on $F(x, u) = G(u) - h(x)u$ only
- in 1988, at a conference in Paris on “Variational Problems”, I
  - lectured on the FOZ result
  - did not receive any hint for a possible variational proof

A way to a variational proof for FOZ

- FONDA-GOSSEZ-ZANOLIN (DIE, 1991) : proof of FOZ thm using **lower and upper solutions**
  - $\alpha \in C^2(I)$ (resp. $\beta \in C^2(I)$) **lower** (resp. **upper) solution** of (DP) if $-\alpha''(x) \leq f(x, \alpha(x)),\ \alpha(0) \leq 0,\ \alpha(\pi) \leq 0$ (resp. $-\beta''(x) \geq f(x, \beta(x)),\ \beta(0) \geq 0,\ \beta(\pi) \geq 0)$)
  - if $\alpha \leq \beta$ let $\gamma(x, u) := \max\{\alpha(x), \min\{u, \beta(x)\}\}$

- **ULS thm** : *if (DP) has a LS α and an US β with $\alpha \leq \beta$, (DP) has a solution u_0 with $\alpha \leq u_0 \leq \beta$*
  - $-u'' + u = \gamma(t, u) + f(t, \gamma(t, u)),\ u(0) = 0 = u(\pi)$ **(MDP)**
  - *if u solves (MDP), then $\alpha \leq u \leq \beta$ and u solves (DP)*
  - *(MDP) has a solution* (by SCHAUDER's FPT for example)

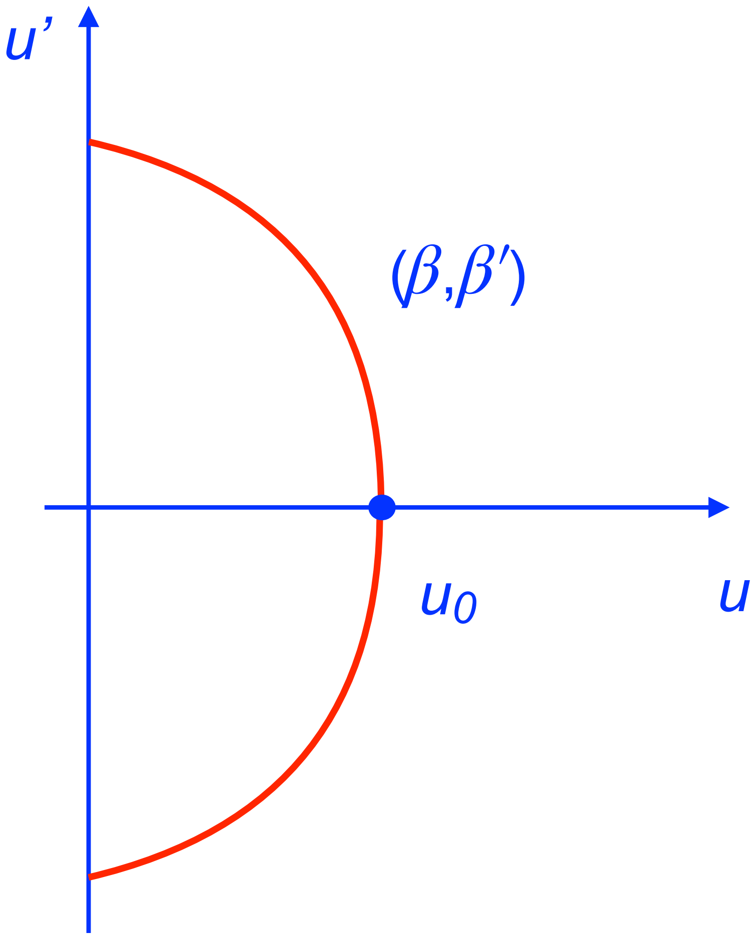
Variational version of LUS thm

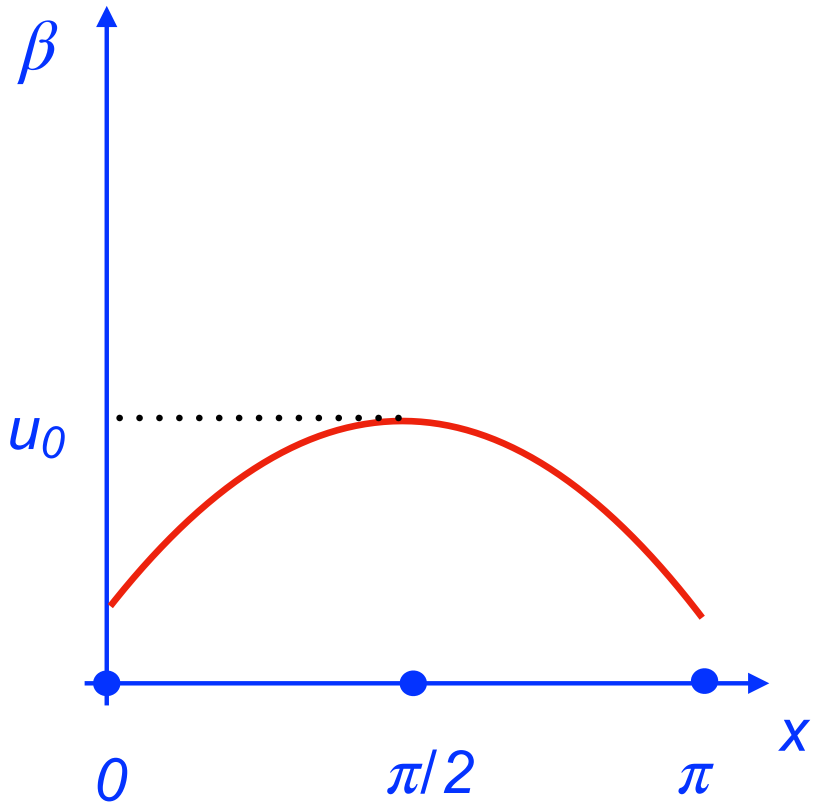
- $C = \{v \in H_0^1(I) : \alpha \leq v \leq \beta\}$
- **VarULS thm** : *if (DP) has a LS* α *and an US* β *with* $\alpha \leq \beta$, *it has a solution* u_0 *with* $\varphi(u_0) = \min_{v \in C} \varphi(v)$
  - $\widetilde{F}(x,u) := \int_0^u [\gamma(x,s) + f(x,\gamma(x,s))]\, ds$
    $\widetilde{\varphi}(u) := \int_I [u'^2/2 + u^2/2 - \widetilde{F}(x,u)]\, dx$
  - for $\alpha(x) \leq u \leq \beta(x)$, $\widetilde{F}(x,u) = \frac{u^2}{2} + F(x,u) + a(x)$,
    $a(x) := \int_0^{\alpha(x)} [\gamma(x,s) - s + f(x,\gamma(x,s)) - f(x,s)]\, ds$
  - $\widetilde{\varphi} \in C^1(H_0^1(I))$ wlsc, coercive reaches its minimum at say u_0
  - u_0 solves (MDP), hence (DP) and minimizes $\widetilde{\varphi}$
  - hence u_0 minimizes $\varphi = \widetilde{\varphi} + \int_I a$ on C

FoGZ proof of FOZ result

- show that (SDP) has LS and US with $\alpha < 0 < \beta$ (say for β)
- if g unbounded below on $\mathbb{R}_+$ take $\beta \in \mathbb{R}_+ : g(\beta) < -|h|_\infty$
- if not any positive solution on I of $-u'' = g(u) + M$ **(AE)** with $M > |h|_\infty$, $g(u) + M \geq 1$ on $\mathbb{R}_+$ is a positive US of (SDP)
- write (AE) $-u'' = V'(u)$ with $V(u) = G(u) + Mu$
- (FOZ) $\Rightarrow$ $\exists\, \varepsilon > 0,\ \exists\, (u_n) \to +\infty : \frac{(1-\varepsilon)u_n^2}{2} - V(u_n) \to +\infty$
- $\exists\, u_0 > 0 : \frac{(1-\varepsilon)u^2}{2} - V(u) \leq \frac{(1-\varepsilon)u_0^2}{2} - V(u_0),\ \forall\, u \in [0, u_0]$
- **time-map** $T = \int_0^{u_0} \frac{du}{\sqrt{2[V(u_0)-V(u)]}} \geq \frac{\pi}{2\sqrt{1-\varepsilon}}$
- the positive solution $\beta(x)$ of (AE) with $\beta(\pi/2) = u_0$, $\beta'(\pi/2) = 0$ vanishes at $\frac{\pi}{2} - T$ and $\frac{\pi}{2} + T$ with $T > \pi/2$
- $\beta(x)$ is a positive US for (SDP)

β in pictures





Remarks and open problems

- FOZ thm : **both** a topological and a variational proof
- condition (H) can be extended to **systems**
  $-u'' = \nabla_u F(x,u),\ u(0) = 0 = u(\pi)$ in the form
  $\limsup_{|u|\to\infty} 2F(x,u)/|u|^2 < 1$ *uniformly in* $x \in I$
  - **open** : extension of condition (FOZ) to systems)
- FOZ thm and its proofs are easily extended to $h \in L^\infty(I)$
  - **open** : case where $h \in L^p(I)\ (1 \le p < \infty)$
- **extensions of FOZ thm** to other ODEs or radial solutions of PDEs
  - **open** : sharp conditions for radial solutions
- corresponding results for the **Neumann or periodic problems** : replace 1 by 0 in (FOZ) – similar but more easy proofs
- **open** : **non variational proof** for Hammerstein's condition

II. The coincidence degree for periodic solutions of some autonomous differential equations

Periodic solutions of Duffing equations

- DING TONGREN, IANNACCI, ZANOLIN (JMAA, 1991)
- $x'' + g(x) = p(t, x, x')$ **(DEp)**
  - p T-periodic in t and bounded on $\mathbb{R}^3$, $g \in C(\mathbb{R})$
- **T-periodic solution (TPS)** of (DEp) : $x(t+T) = x(t), \forall\, t \in \mathbb{R}$
  - $C_T^k := \{x \in C^k(\mathbb{R}) : x \text{ T-periodic}\}$
- **method** : continuation thm of coincidence degree for the homotopy $x'' + g(x) = \lambda p(t, x, x')\ \ (\lambda \in [0,1])$
- **problem** : compute the coincidence degree $d_L[L + \gamma_0, D]$ for
  - $L : C_T^2 \to C_T^0,\ Lx = x''$
  - $\gamma_p : C_T^1 \to C_T^0,\ \gamma_p(x) = g(x) - p(\cdot, x, x')$
  - open bounded $D \subset C_T^1$: $Lx + \gamma_0(x) \neq 0$ on ∂D

The frailty of non constant closed orbits

- $x'' + g(x) = 0$ **(DE0)**
- **observation** : it is very easy to kill the **non constant** TPS of (DE0) :
  - for $\varepsilon \neq 0$: $x'' + \varepsilon x' + g(x) = 0$ **(DDE0)**
  - x TPS of (DDE0) $\Rightarrow \int_0^T [x''x' + \varepsilon x'^2 + g(x)x']\,dt = 0$
    $\Rightarrow \int_0^T x'^2\,dt = 0 \Rightarrow x(t) = c \Rightarrow g(c) = 0$ (equilibrium)
- $d_L[L + \gamma_0, D]$ remains the same for small perturbations of γ_0
  $\Rightarrow$ for $\delta : C_T^1 \to C_T^0,\ x \mapsto x',$ and $|\varepsilon| \ll 0,$
  $d_L[L + \varepsilon\delta + \gamma_0, D] = d_L[L + \gamma_0, D]$
- as $L + \varepsilon\delta + \gamma_0$ has only **constant** zeros, one can hope to find a formula for $d_L[L + \varepsilon\delta + \gamma_0, D]$

The computation of $d_L[L+\gamma_0, D]$

- $\forall\, \varepsilon \neq 0,\ \forall\, T > 0,\ \forall\, \lambda \in (0,1)$, *the TPS of*
  $x'' + \lambda(1-\lambda)\varepsilon x' + \frac{1-\lambda}{T}\int_0^T g(x(s))\,ds + \lambda g(x) = 0$ **(HDE0)**
  are the zeros of g (proved like in previous slide)
- **lem** (Ding-Iannacci-Zanolin, 1991) : *if* $0 \notin (L+\gamma_0)(\partial D)$, *then* $0 \notin g(\partial D \cap \mathbb{R})$ *and* $d_L[L+\gamma_0, D] = d_B[g, D\cap\mathbb{R}, 0]$
  - homotopy $L + \Gamma(\cdot,\lambda)$ with $\Gamma : C_T^1 \times [0,1] \to C_T^0$,
    $x \mapsto \lambda(1-\lambda)\varepsilon\delta(x) + \frac{(1-\lambda)}{T}\int_0^T g(x(s))\,ds + \lambda\gamma_0(x)$
  - $\lambda \in (0,1),\ Lx + \Gamma(x,\lambda) = 0 \ \Rightarrow\ x(t) = c,\ g(c) = 0$
  - $\lambda = 0 : Lx + \frac{1}{T}\int_0^T g(x(s))\,ds = 0 \ \Rightarrow\ x(t) = c,\ g(c) = 0$
  - in both cases $Lx + \Gamma(x,\lambda) \neq 0$ on ∂D
  - homotopy invariance and reduction thm of coincidence degree :
    $$d_L[L+\gamma_0, D] = d_L[L+\Gamma(\cdot,1), D] = d_L[L+\Gamma(\cdot,0), D] = d_B[g, D\cap\mathbb{R}, 0]$$

A continuation theorem for (DEp)

- **thm** (DING-IANNACCI-ZANOLIN, JMAA, 1991) : *if* $\exists\, R \geq d > 0$ *:*
  (i) *sgn x* $\cdot\, g(x) > |p|_\infty$ *for* $|x| \geq d$
  (ii) $\forall\, \lambda \in [0,1]$, *any possible TPS* x *of*
  $x'' + g(x) = \lambda p(t,x,x')$ *satisfies* $\max_{\mathbb{R}} x(t) \neq R$
  then (DEp) has a TPS with $max_{\mathbb{R}} x < R$

  - sign condition on g $\Rightarrow$ *a priori* estimate $-S < x(t) < R$, $|x'|_\infty < N$ for the possible TPS of the homotopy
  - Ding-Iannacci-Zanolin lemma for
    $D = \{x \in C^1_T(\mathbb{R}) : -S < x(t) < R,\ |x'(t)| < N\}$
    $\Rightarrow\ d_L[L + \gamma_p, D] = d_L[L + \gamma_0, D] = d_B[g, D \cap \mathbb{R}, 0]$
  - sign condition on g $\Rightarrow$ $d_B[g, D \cap \mathbb{R}, 0] = 1$

A natural question

- can one extend the degree computation to sets of TPS of an **arbitrary autonomous system** $x' = f(x)$ **(AS)** with $f \in C(\mathbb{R}^n, \mathbb{R}^n)$, in the space C_T of T-periodic continuous $x : \mathbb{R} \to \mathbb{R}^n$?
  - the special trick of the **added friction** does not work anymore
  - $L : D(L) \subset C_T \to C_T, x \mapsto x', \ \phi : C_T \to C_T, x \mapsto f(x)$
- a (very) **partial answer** (MAWHIN, CBMS No 40, 1979) : *if* $f(x) = V'(x)$ *for some* $V \in C^1(\mathbb{R}^n, \mathbb{R})$ *and* $V'(x) \neq 0$ *for* $|x| = r,$ *then* $d_L[L - \phi, B(r)] = d_B[V', B(r), 0]$
  - homotopy $Lx - \frac{1-\lambda}{T} \int_0^T V'(x(s))\, ds - \lambda\phi(x) \ \ (\lambda \in [0,1])$
  - all its TPS are constant and zeros of V'
  - $d_L[L - \phi, B(r)] = d_L[L - (1/T) \int_0^T V'(x(s))\, ds, B(r)]$
    $= d_B[V', B(r), 0]$

A non trivial answer

- **lem** (CAPIETTO-MAWHIN-ZANOLIN, TAMS, 1992) : *if* $\Omega \subset C_T$ *is open bounded such that no TPS of (AS) lies in* $\partial\Omega$, *then* $d_L[L-\phi,\Omega] = d_B[f,\Omega\cap\mathbb{R}^n,0]$
  - general case reduced to the generic one where equilibria and closed orbits are isolated using **Kupka-Smale thm**
  - homotopy
    - first to a system $x' = \lambda^* f(x)$ (for some $\lambda^* \in (0,1)$) having only equilibria as TPS
    - then to the averaged system $x' = (\lambda^*/T)\int_0^T f(x(s))\,ds$
  - use of reduction thm to express its coincidence degree by the Brouwer degree of f

A general continuation theorem

- **thm** (CAPIETTO, MAWHIN, ZANOLIN, TAMS, 1992) : *let* $h(t, x, \lambda)$ *be T-periodic in* t, *continuous, with* $h(t, x, 0) = f(x)$ *and* $\Omega \in C_T$ *open bounded and such that*

  (i) $\forall \lambda \in [0, 1)$, $x' = h(t, x, \lambda)$ *has no TPS in* $\partial\Omega$

  (ii) $d_B[f, \Omega \cap \mathbb{R}^n, 0] \neq 0$

  then $x' = h(t, x, 1)$ *has at least one TPS in* $\overline{\Omega}$
- for $x' = f(x) + p(t)$ with $p \in C_T$, examples show that this thm may give better results than the homotopy $x' = \lambda[f(x) + p(t)]$
- extensions to TPS of **functional differential equations** and of differential equations in some **nonlinear spaces**
- many applications, including **perturbations problems**
- another proof (BARTSCH-MAWHIN, JDE, 1991) uses a **reduction theorem for LS-degree of** S^1**-invariant mappings in** C_T

Remarks and questions

- **open** : give an elementary proof of the degree lemma
- CMZ lemma shows that $d_L[L - \gamma_0, \Omega]$, **blind to non constant periodic orbits of autonomous systems** only detects **equilibria**
- a TPS of a T-periodically forced autonomous system obtained by degree 'reduce' to some equilibria when the forcing tends to zero
- CMZ lemma kills an old **wrong *a priori* idea** :
  - to prove the existence of limit cycles by degree in C_T in order to move beyond planar problems and phase plane methods
  - the hope was proving the existence of periodic motions for a shell model of pulsating star (a system of *N* second order ode's (motion) coupled with *N* first order ode's (heat transfer)
  - it could have been the topics of my PhD thesis (some 55 years ago) and is still open

Thank you for your kind attention !

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