Long-time asymptotics for nonlinear Schrödinger equations in one space dimension

Jacek Jendrej

IMDETA · Łódź · 4 November 2026

Joint work with Tony Salvi

The nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \sum_{n=1}^{d} \lambda_n |u|^{2n} u, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}, \quad u(t,x) \in \mathbb{C}, \quad \lambda_n \in \mathbb{R}.$$

The equation is Hamiltonian, with conserved energy and mass:

$$E(u) = \int_{\mathbb{R}} \left(\frac{1}{2}|\partial_x u|^2 + \sum_{n=1}^{d} \frac{\lambda_n}{n+1}|u|^{2n+2} \right) dx,$$

$$M(u) = \int_{\mathbb{R}} |u|^2 \, dx.$$

Question

For small, localized initial data, what is the asymptotic behavior as $t \to +\infty$?

Some landmarks

- **Linear equation:** explicit Fourier analysis.
- **Cubic NLS:** complete integrability and inverse scattering.
  Ablowitz–Kaup–Newell–Segur; Zakharov–Manakov; Segur–Ablowitz (1970s).
  Deift–Zhou (1992–1994): rigorous asymptotics, including higher orders.
- **Beyond integrability:** leading-order modified scattering.
  Hayashi–Naumkin (1998); Kato–Pusateri (2011), a simpler Fourier proof.
  Related developments with potentials: Germain–Pusateri–Rousset; Chen–Pusateri.

Jendrej–Salvi (2026)

Higher-order asymptotics for gauge-invariant polynomial nonlinearities.

The linear equation: Fourier representation

Consider $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$, with $u(0) = u_0$.

$$\widehat{u}(t,k) = \int_{\mathbb{R}} u(t,x) e^{-ikx}\, dx, \qquad u(t,x) = \int_{\mathbb{R}} \widehat{u}(t,k) e^{ikx}\, \frac{dk}{2\pi}.$$

The equation becomes

$$i\partial_t \widehat{u} - \frac{k^2}{2}\widehat{u} = 0, \qquad \widehat{u}(t,k) = e^{-itk^2/2}\widehat{u}_0(k).$$

Hence

$$u(t,x) = \int_{\mathbb{R}} e^{i(kx - tk^2/2)}\widehat{u}_0(k)\, \frac{dk}{2\pi}.$$

Frequency-space norms use the measure $dk/(2\pi)$.

Dispersion relation: $\omega(k) = k^2/2$.

The linear equation: the kernel

Substitute the Fourier transform of u_0 and evaluate the Gaussian integral:

$$u(t,x) = \int_{\mathbb{R}} K_t(x-y)u_0(y)\,dy,$$
$$K_t(z) = \int_{\mathbb{R}} e^{-itk^2/2+ikz}\frac{dk}{2\pi} = \frac{1}{\sqrt{2\pi it}}e^{iz^2/(2t)}.$$

Thus, for $t > 0$,

$$u(t,x) = \frac{e^{ix^2/(2t)}}{\sqrt{2\pi it}}\int_{\mathbb{R}} e^{-i(x/t)y}e^{iy^2/(2t)}u_0(y)\,dy.$$

Dispersive decay

$\|u(t)\|_{L^\infty_x} \leq (2\pi t)^{-1/2}\|u_0\|_{L^1_x}$.

Linear asymptotics: expand the kernel

Use

$$e^{iy^2/(2t)} = 1 + \frac{iy^2}{2t} - \frac{y^4}{8t^2} + \cdots, \qquad \widehat{y^2 u_0}(k) = -\partial_k^2 \widehat{u}_0(k).$$

With enough moments of u_0,

$$u(t,x) \sim \frac{e^{ix^2/(2t)}}{\sqrt{2\pi i t}} \left[\widehat{u}_0(x/t) - \frac{i}{2t} \partial_k^2 \widehat{u}_0(x/t) + \cdots \right].$$

For a quantitative leading-order estimate, $0 < \beta \leq 1$ gives

$$u(t,x) = \frac{e^{ix^2/(2t)}}{\sqrt{2\pi i t}} \widehat{u}_0(x/t) + O_{L_x^\infty}\left(t^{-1/2-\beta} \left\| |y|^{2\beta} u_0 \right\|_{L_y^1} \right).$$

Group velocity and local plane waves

Put $x = vt + h$, with h bounded. Then

$$\frac{x^2}{2t} = \frac{v^2 t}{2} + vh + \frac{h^2}{2t}, \qquad \frac{x}{t} = v + \frac{h}{t}.$$

At leading order,

$$u(t, vt + h) \sim \frac{1}{\sqrt{2\pi i t}} \widehat{u}_0(v) e^{iv^2 t/2} e^{ivh}.$$

A local plane wave

The wave number k travels with group velocity

$$v_g(k) = \partial_k \omega(k) = k.$$

The nonlinear equation: the interaction picture

For simplicity, first take the cubic equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u.$$

Define $f(t)$ so that its free evolution to time t equals $u(t)$:

$$e^{it\partial_x^2/2} f(t) = u(t), \qquad \widehat{f}(t,k) = e^{itk^2/2}\widehat{u}(t,k).$$

In other words, if $i\partial_t w + \frac{1}{2}\partial_x^2 w = 0$ with $w(0) = f(t)$, then $w(t) = u(t)$.

- Without the nonlinearity, f would be independent of t.
- If $f(t)$ is localized in space, then linear asymptotics can be applied with $u_0 = f(t)$.
- Propagating spatial localization of f means controlling derivatives of $\widehat{f}$.

$$\|xf\|_{L^2_x} = \left\|\partial_k \widehat{f}\right\|_{L^2_k}, \qquad L^2_k = L^2\left(\mathbb{R}, \frac{dk}{2\pi}\right).$$

Fourier transform of the cubic nonlinearity

Write $u_j = \widehat{u}(t, k_j)$ and $d\mathbf{k} = dk_1\, dk_2\, dk_3$. Then

$$(|u|^2 u)(t,x) = \int_{\mathbb{R}^3} u_1 \overline{u_2} u_3\, e^{i(k_1 - k_2 + k_3)x}\, \frac{d\mathbf{k}}{(2\pi)^3}.$$

Using $\int_{\mathbb{R}} e^{-iqx}\, dx = 2\pi\delta(q)$,

$$\widehat{(|u|^2 u)}(t,k) = \int_{\mathbb{R}^3} u_1 \overline{u_2} u_3\, 2\pi\delta(k - k_1 + k_2 - k_3)\, \frac{d\mathbf{k}}{(2\pi)^3}.$$

Therefore,

$$i\partial_t \widehat{u} - \frac{k^2}{2}\widehat{u} = \lambda \int_{\mathbb{R}^3} u_1 \overline{u_2} u_3\, 2\pi\delta(k - k_1 + k_2 - k_3)\, \frac{d\mathbf{k}}{(2\pi)^3}.$$

The equation for the profile

Set $\widehat{f_j} = \widehat{f}(t, k_j)$ and

$$Q = k - k_1 + k_2 - k_3, \qquad \Phi = \frac{1}{2}(k^2 - k_1^2 + k_2^2 - k_3^2).$$

The linear oscillations cancel, leaving

$$\boxed{\partial_t \widehat{f}(t,k) = -i\lambda \int_{\mathbb{R}^3} e^{it\Phi}\, 2\pi\delta(Q)\widehat{f_1}\overline{\widehat{f_2}}\widehat{f_3}\, \frac{d\mathbf{k}}{(2\pi)^3}.}$$

Recall: localization of f requires control of k-derivatives of $\widehat{f}$, in particular $\left\|\partial_k \widehat{f}\right\|_2 = \|xf\|_2$.

The apparent difficulty

Differentiating $e^{it\Phi}$ with respect to k produces a factor t.

The “magic identity”

$$\partial_t \widehat{f}(t,k) = -i\lambda \int_{\mathbb{R}^3} e^{it\Phi}\, 2\pi\delta(Q)\widehat{f}_1\widehat{\overline{f}}_2\widehat{f}_3\, \frac{d\mathsf{k}}{(2\pi)^3}.$$

Introduce

$$D = \partial_k + \partial_{k_1} + \partial_{k_2} + \partial_{k_3}.$$

Since $DQ = 0$ and $D\Phi = Q$, we have, in the sense of distributions,

$$\boxed{D\left(e^{it\Phi}\delta(Q)\right) = itQe^{it\Phi}\delta(Q) = 0.}$$

Thus an “output” derivative ∂_k can be replaced by “input” derivatives ∂_{k_j}:

$$\partial_k\left(e^{it\Phi}\delta(Q)\right) = -(\partial_{k_1} + \partial_{k_2} + \partial_{k_3})\left(e^{it\Phi}\delta(Q)\right).$$

Integrate by parts: the derivatives fall on the profiles $\widehat{f}_j$.

Propagation of one spatial moment

Integrating by parts gives

$$\partial_t \partial_k \widehat{f}(t,k) = -i\lambda \int_{\mathbb{R}^3} e^{it\Phi}\, 2\pi\delta(Q) \sum_{j=1}^{3} \partial_{k_j} \left(\widehat{f}_1 \overline{\widehat{f}_2} \widehat{f}_3\right) \frac{d\mathbf{k}}{(2\pi)^3}.$$

Return to physical space and use Plancherel:

$$\left\| \partial_t \partial_k \widehat{f}(t) \right\|_2 \lesssim |\lambda|\, \|u(t)\|_\infty^2 \left\| \partial_k \widehat{f}(t) \right\|_2.$$

For $t \geq 1$, if $\|u(t)\|_\infty \lesssim \varepsilon t^{-1/2}$, Grönwall gives

$$\|xf(t)\|_2 \lesssim \|xf(1)\|_2\, t^{C\varepsilon^2} \lesssim \varepsilon t^{C\varepsilon^2}.$$

(By local well-posedness, u_0 and $f(1)$ are equivalent in any reasonable norm.)

A physical-space formula for the profile equation

We also need a pointwise estimate for $\partial_t \widehat{f}(t,k)$. Substituting the Fourier transforms into the profile equation gives

$$\partial_t \widehat{f}(t,k) = \int_{\mathbb{R}^3} K(t,\mathbf{x},k) f(t,x_1) \overline{f(t,x_2)} f(t,x_3)\, d\mathbf{x},$$

with

$$K(t,\mathbf{x},k) = \frac{-i\lambda}{2\pi t} \exp\left[-ik(x_1 - x_2 + x_3) - \frac{i(x_1 - x_2)(x_3 - x_2)}{t}\right].$$

Extract the resonant term and estimate the error

Replace $e^{-i(x_1-x_2)(x_3-x_2)/t}$ by 1. The integral factorizes. Set $\gamma = \lambda/(2\pi)$:

$$\boxed{\partial_t \widehat{f} = -\frac{i\gamma}{t}\widehat{|f|^2 f} + R.}$$

For $0 < \beta \leq 1$,

$$\|R(t)\|_\infty \lesssim |\lambda| t^{-1-\beta} \|f(t)\|_1^2 \left\| |x|^{2\beta} f(t) \right\|_1 .$$

If $0 < \beta < 1/4$, Cauchy–Schwarz gives

$$\|f\|_1 + \left\| |x|^{2\beta} f \right\|_1 \lesssim_\beta \|\langle x \rangle f\|_2 .$$

For sufficiently slow growth of $\|\langle x \rangle f(t)\|_2$, the remainder is integrable on $[1, \infty)$.

Closing the bootstrap

Local theory controls $0 \le t \le 1$. For $t \ge 1$, assume $\|u(t)\|_\infty \le M\varepsilon t^{-1/2}$ and $\|\langle x\rangle f(t)\|_2 \le M\varepsilon t^{\alpha}$, with $0 < 3\alpha < \beta < 1/4$.

1. **Propagate localization.** The weighted differential inequality gives $\|xf(t)\|_2 \lesssim \varepsilon t^{C_1 M^2 \varepsilon^2}$. Choose ε so that $C_1 M^2 \varepsilon^2 < \alpha$.
2. **Bound the remainder.** The weighted bound in the preceding kernel estimate gives $\|R(t)\|_\infty \le C_M \varepsilon^3 t^{-1-\beta+3\alpha}$.
3. **Control the amplitude.** The resonant term only changes the phase:

$$\left\|\widehat{f}(t)\right\|_\infty \le \left\|\widehat{f}(1)\right\|_\infty + \int_1^t \|R(s)\|_\infty \, ds \lesssim \varepsilon.$$

The linear asymptotic estimate applied to $f(t)$ then gives

$$\|u(t)\|_\infty \le (2\pi t)^{-1/2} \left\|\widehat{f}(t)\right\|_\infty + C_* \varepsilon t^{-1/2-\beta+\alpha} \le \tfrac{1}{2} M\varepsilon t^{-1/2}.$$

The resonant ODE and modified scattering

Without R, the equation $\partial_t z = -i\gamma |z|^2 z/t$ has solution

$$z(t) = z(1) e^{-i\gamma |z(1)|^2 \log t}, \qquad t \geq 1.$$

For the actual profile, starting at $t = 1$, set

$$\Theta(t,k) = \gamma \int_1^t \frac{|\widehat{f}(s,k)|^2}{s}\, ds, \qquad W(t,k) = e^{i\Theta(t,k)} \widehat{f}(t,k).$$

Then $\partial_t W = e^{i\Theta} R$ is integrable. Consequently,

$$W(t) = W_\infty + O_{L^\infty}(t^{-\delta}), \qquad \Theta(t) = \gamma |W_\infty|^2 \log t + \phi + O_{L^\infty}(t^{-\delta}).$$

The amplitude converges; the phase keeps rotating.

The leading asymptotic formula

With $A = e^{-i\phi} W_\infty$,

$$u(t,x) = \frac{e^{ix^2/(2t)}}{\sqrt{2\pi i t}} A(x/t) e^{-i\gamma |A(x/t)|^2 \log t} + O_{L^\infty_x}(t^{-1/2-\delta}).$$

- The same $t^{-1/2}$ decay as in the linear problem.
- A logarithmic phase shift caused by resonant self-interaction.
- For the polynomial equation, γ is replaced by $\gamma_1 = \lambda_1/(2\pi)$; terms of degree at least five give integrable contributions to the profile equation.

Higher-order asymptotics

Apply the magic identity repeatedly

Write $\mathcal{K} = e^{it\Phi}\, 2\pi\delta(Q)$ and $S = \partial_{k_1} + \partial_{k_2} + \partial_{k_3}$. Since $\partial_k \mathcal{K} = -S\mathcal{K}$,

$$\partial_k^m \mathcal{K} = (-S)^m \mathcal{K}.$$

After m integrations by parts,

$$\partial_t \partial_k^m \widehat{f} = -i\lambda \sum_{a+b+c=m} \frac{m!}{a!b!c!} \int_{\mathbb{R}^3} \mathcal{K}\, (\partial_{k_1}^a \widehat{f}_1)\overline{(\partial_{k_2}^b \widehat{f}_2)}(\partial_{k_3}^c \widehat{f}_3)\, \frac{d\mathbf{k}}{(2\pi)^3}.$$

No powers of t are created by differentiation

The highest derivatives enter linearly; the remaining terms involve lower derivatives. This provides a hierarchy of estimates for $\|x^m f(t)\|_2$.

The same cancellation for every gauge-invariant monomial

For $|u|^{2n}u$, define

$$Q_n = k - \sum_{j=1}^{2n+1} (-1)^{j+1} k_j, \qquad \Phi_n = \frac{1}{2}\left(k^2 - \sum_{j=1}^{2n+1} (-1)^{j+1} k_j^2 \right).$$

Because $\sum_{j=1}^{2n+1} (-1)^{j+1} = 1$,

$$D_n = \partial_k + \sum_{j=1}^{2n+1} \partial_{k_j} \quad \Longrightarrow \quad D_n Q_n = 0, \quad D_n \Phi_n = Q_n.$$

Thus $D_n(e^{it\Phi_n}\delta(Q_n)) = 0$, again at every derivative order.

The theorem: the expansion to arbitrary order

Jendrej–Salvi (2026). Let $N \in \{0, 1, \dots\}$. If

$$\|u_0\|_{H^1} + \left\|\langle x\rangle^{2N+1} u_0\right\|_2 < \varepsilon_N \qquad \text{with } \varepsilon_N > 0 \text{ sufficiently small,}$$

then the solution is global and, as $t \to \infty$, with $v = x/t$ and $\gamma_1 = \lambda_1/(2\pi)$,

$$u(t,x) = \frac{e^{ix^2/(2t) - i\gamma_1 |u_{0,0}(v)|^2 \log t - i\phi(v)}}{\sqrt{2\pi i t}} \sum_{p=0}^{N} \frac{1}{t^p} \sum_{k=0}^{2p} (\log t)^k u_{p,k}(v) + u_{\mathrm{err}}(t,x).$$

$$\|u_{\mathrm{err}}(t)\|_{L^\infty_x} = O(t^{-N-1/2-\delta}), \qquad \delta > 0,$$

$$u_{p,k} \in W^{2(N-p),\infty}(\mathbb{R}), \qquad \phi \in W^{2N,\infty}(\mathbb{R}; \mathbb{R}).$$

Why logarithms appear in the corrections

The leading profile is $\widehat{f}(t,k) \approx A(k)e^{-ib(k)\log t}$, where $b = \gamma_1|A|^2$. Thus

$$\partial_k\left(Ae^{-ib\log t}\right) = e^{-ib\log t}\left(\partial_k A - i(\partial_k b)A\log t\right).$$

Each additional frequency derivative can add one power of $\log t$. The linear stationary-phase expansion starts with

$$e^{it\partial_x^2/2}g(x) \sim \frac{e^{ix^2/(2t)}}{\sqrt{2\pi it}}\left[\widehat{g}(v) - \frac{i}{2t}(\partial_k^2\widehat{g})(v) + \cdots\right].$$

The mechanism

Two derivatives per inverse power of t produce up to two extra logarithms.

Which nonlinearities enter each correction?

At the level of the resonant profile equation,

$$|u|^{2n}u \quad \rightsquigarrow \quad t^{-n}|\widehat{f}|^{2n}\widehat{f}.$$

For $n \geq 2$, integrating a tail produces a term of size $t^{-(n-1)}$.

order in the bracket	explicit nonlinear contributions
1	cubic logarithmic phase
t^{-1}	cubic corrections and quintic term
t^{-2}	also the septic term

Selected references

- P. Deift and X. Zhou (1994), Long-time asymptotics for integrable systems. Higher order theory, *Commun. Math. Phys.* **165**, 175–191.
- N. Hayashi and P. I. Naumkin (1998), Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations, *Amer. J. Math.* **120**, 369–389.
- J. Kato and F. Pusateri (2011), A new proof of long-range scattering for critical nonlinear Schrödinger equations, *Differential Integral Equations* **24**, 923–940.
- J. Jendrej and T. Salvi (2026), High-order long-time asymptotics for small solutions to the one-dimensional nonlinear Schrödinger equation, *Nonlinearity* **39**, 095007.