C_0-semigroups with a parameter and their applications in kinetic models with spatial transport

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Fragmentation–coagulation processes

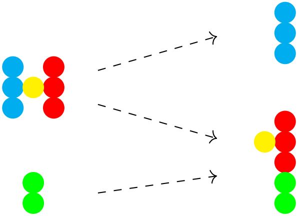

Figure: Pure fragmentation and coagulation

In many applications, such as aerosols or polymers, it makes sense to allow the clusters to be of any size.

Space inhomogeneous fragmentation–coagulation models.

The governing equation is

$$\partial_t u(t,x,m) = [\mathcal{T}_0 u](t,x,m) + [\mathcal{F}]u(t,x,m) + [\mathcal{C}]u(t,x,m), \quad (1)$$

$t > 0,\ x \in \Omega$ and $m \in \mathbb{R}_+$, supplemented with the initial condition

$$u(0,x,m) = \mathring{u}(x,m), \quad (x,m) \in \Omega \times \mathbb{R}_+, \quad (2)$$

where Ω is an open subset of $\mathbb{R}^N$ for some $N \geq 1$, $\mathbb{R}_+ := [0,\infty)$, $\mathcal{T}_0$ is a family of diffusion expression

$$[\mathcal{T}_0 u](x,m) = \nabla_x \cdot (d(x,m)\nabla_x u(x,m)), \quad (3)$$

behaving as

$$(x,m) \mapsto [\mathcal{T}_{0,m} u(\cdot,m)](x). \quad (4)$$

The fragmentation and coagulation processes are described by

$$
\begin{aligned}
[\mathcal{F}]u(t,x,m) &= [\mathcal{A}]u(t,x,m) + [\mathcal{B}]u(t,x,m) \\
&= -a(x,m)u(t,x,m) + \int_m^\infty b(x,m,s)a(x,s)u(t,x,s)\ ds,
\end{aligned}
\tag{5a}
$$

$$
\begin{aligned}
[\mathcal{C}u](t,x,m) &= [\mathcal{C}_1 u](t,x,m) - [\mathcal{C}_2 u](t,x,m) \\
&:= \frac{1}{2}\int_0^m k(x,m-s,s)u(t,x,m-s)u(t,x,s)ds \\
&\quad - u(t,x,m)\int_0^\infty k(x,m,s)u(t,x,s)ds.
\end{aligned}
\tag{5b}
$$

Here, u is the distribution of particles of mass/size $m \in \mathbb{R}_+$ at position $x \in \Omega$ at any time t, $\mathcal{F} = \mathcal{A} + \mathcal{B}$ is the fragmentation operator, split into the so-called loss operator $\mathcal{A}$ and gain operator $\mathcal{B}$, with

- the fragmentation rate a, and
- b describing the distribution of masses m of the daughter particles at position x spawned by splitting of a particle of mass s.

Expression $\mathcal{C}$ describes coagulation operator with k giving the rate of coagulation.

Preliminaries–homogeneous problems.

Difficulties in solving

$$
\begin{aligned}
\partial_t f(t,m) &= \mathcal{F} f(t,m) + \mathcal{C} f(t,m) \ , \qquad (t,m) \in (0,\infty)^2 \ , \\
f(0,m) &= \mathring{f}(m) \ , \qquad m \in (0,\infty) \ , \qquad (6)
\end{aligned}
$$

come from the fact that both the fragmentation rate a and the coagulation rate can be unbounded, for instance at $m = \infty$.

1. *Truncation method.* We construct solutions f_r to the problem with the coefficients a and k modified as follows

$$
a_r(m) = \begin{cases} a(m) & \text{for} \quad m \leq r \\ 0 & \text{for} \quad m > r, \end{cases} \qquad k_r(m,y) = \begin{cases} k(m,y) & \text{for} \quad m + y \leq r \\ 0 & \text{for} \quad m + y > r. \end{cases}
$$

$(f_r)_{r>0}$ is a weakly compact net whose accumulation point is a solution to a suitable weak formulation of (6).

Advantages: possibility to handle very general coagulation coefficients.

Disadvantages: weak solutions, additional work required to prove mass conservation, uniqueness, etc; fragmentation subordinated to coagulation.

Main contributors: J. Ball, J. Carr, I. Stewart, P. Laurençot, D. Wrzosek, P. Escobedo, L. Desvilletes, J. A. Cañizo, A. Giri,..., also A.V. Burobin, P. Dubovskii.

2.*Semigroup method.* Considering (6) as a nonlinear perturbation of the linear dynamics generated by

$$\mathcal{F} = \mathcal{A} + \mathcal{B}.$$

Main contributors: M. Aizenman, T. Bak, A. McBride, W. Lamb, M. Doumic Jauffret, P. Gabriel, É. Bernard... .

Advantages: classical unique mass-conserving solutions.

Disadvantages: The coagulation part subordinated to the fragmentation, typically bounded, but can be extended using the analyticity of the fragmentation operator.

So, first, how to solve

$$\partial_t f = \mathcal{F} f = \mathcal{A} f + \mathcal{B} f \ ?$$

The natural space to analyse the continuous fragmentation - coagulation processes is

$$X = L_1(\mathbb{R}_+, mdm) = \left\{ u:\ \|u\| = \int\limits_0^\infty |u(m)|mdm < +\infty \right\}$$

as for nonnegative u we have $\|u\|_1 = M(u)$, the mass of the ensemble with density u. Best results are obtained in spaces

$$X_r = L_1(\mathbb{R}_+, dm_r) = \left\{ u:\ \|u\|_r := \int\limits_0^\infty |u(m)|(1+m^r)dm < +\infty \right\},$$

$r > 1$.

In X_r, we use the Desch–Miyadera type theorem,

Theorem 1 (W. Desch)

Let $(G_A(t))_{t\geq 0}$ *be a positive* C_0*-semigroup on some* L^1 *space* X *with generator* A *and let* B *be positive on* $D(A)_+$*. If* $\|BR(\lambda, A)\| < 1$ *for large* $\lambda > 0$*, then*

$$K = A + B : D(A) \to X$$

is the generator of a positive C_0*-semigroup on* X*.*

Moreover, if $(G_A(t))_{t\geq 0}$ *is analytic,* $(G_K(t))_{t\geq 0}$ *is also analytic.*

The local conservation of mass requires

$$n_1(s) := \int\limits_0^s b(m,s)m dm = s. \tag{7}$$

Assume that

$$a \text{ is bounded at } 0 \ \ \& \ \ \int\limits_0^s b(m,s) ds =: n_0(s) \leq b_0(1+s^l), \tag{8}$$

where $l \in [0,\infty)$ and $b_0 \geq 1$.

Then, we consider the fragmentation operator

$$F = A + B, \quad \text{on} \quad D(F) = D(A) = \{u \in X_r : \ au \in X_r\}.$$

Crucial role is played by

$$c_r(s) := \frac{n_r(s)}{s^r} := \int\limits_0^s b(m,s)m^r dm \leq 1.$$

Theorem 2

Let a, b *satisfy (8) and for some* $r_0 > 1$ *there are* $c_r < 1$ *and* $s_{r_0} > 0$ *such that for all* $s \geq s_{r_0}$

$$c_r(s) \leq c_r. \tag{9}$$

Then

1. (9) holds for all $r > 1$*;*

2. $F := A + B$ *is the generator of a positive analytic semigroup,* $(G_F(t))_{t\geq 0}$*, on* X_r *for any* $r > \max\{1, l\}$*.*

In the proof of Theorem 2, a crucial role is played by the estimate

$$\|AR(\lambda, A)\|_{X_r} \leq 1 \tag{10}$$

as in the estimate required by the Desch theorem we get

$$\|BR(\lambda, A)\|_{X_r} = \int_0^\infty [BR(\lambda, A)](m)dm \leq \frac{C(s_{r_0})}{\lambda} + c_r\|AR(\lambda, A)\| < 1 \tag{11}$$

for sufficiently large λ.

Benefits of the analyticity of $(G_F(t))_{t\geq 0}$

Using the fact that $D(F) = D(A)$, we can identify the real interpolation space $D_F(\mu, 1)$ with

$$X_r^{(\mu)} := \left\{ f \in X_r : \int_0^\infty |f(m)(1 + a(m))^\mu \, dm_r < \infty \right\}, \qquad (12)$$

and hence, there are constants ω_r, M_r^μ such that for $\mu \in [0, 1]$,

$$\|G_F(t)f\|_{X_r^{(\mu)}} \leq \frac{M_r^{(\mu)} e^{\omega_r t}}{t^\mu} \|f\|_{X_r}, \quad t > 0. \qquad (13)$$

This moment improving property allows for proving the solvability of fragmentation–coagulation equations with unbounded coagulation kernels as long as they are controlled by the fragmentation rates.

Example 3

One of the forms of $b(m,s)$ most often used in applications is

$$b(m,s) = \frac{1}{s} h\left(\frac{m}{s}\right) \tag{14}$$

which is referred to as the homogeneous fragmentation kernel. In this case

$$c_r(s) = \frac{1}{s^{r+1}} \int\limits_0^s h\left(\frac{m}{s}\right) m^r dm = \int\limits_0^1 h(z) z^r dz =: h_r < h_1 = 1$$

Hence, (9) holds.

On the other hand, fragmentation processes in which daughter particles tend to accumulate close both to 0 and to the parent's size may not satisfy (9).

Space inhomogeneous case.

To solve

$$\partial_t \boldsymbol{u} = \boldsymbol{T}_0\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{C}\boldsymbol{u}, \tag{15}$$

we try to mimic the space homogeneous case, that is,

1. Solve the parametric version of

   $$\partial_t u(t,x,m) = [\mathcal{T}_0 u(t,\cdot,m)](x),$$

   to get the generator $\boldsymbol{T}$ of the solution semigroup.
2. Show that $\boldsymbol{T} = \boldsymbol{T}_0 + \boldsymbol{A}$ generates a substochastic semigroup.
3. Use the Desch perturbation theorem to get the generation by

   $$\boldsymbol{u}_t = \boldsymbol{K}\boldsymbol{u} = \boldsymbol{T}_0\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u}. \tag{16}$$

4. Solve (15) as a semilinear perturbation of $\boldsymbol{u}_t = \boldsymbol{K}\boldsymbol{u}$.

Recall that

$$[\mathcal{C}u](t,x,m) = \frac{1}{2}\int_0^m k(x,m-s,s)u(t,x,m-s)u(t,x,s)ds \\ - u(t,x,m)\int_0^\infty k(x,m,s)u(t,x,s)ds.$$

For this, we will need two spaces in the x variable,

$$X_x^1 := L_1(\Omega, dx) \text{ or } X_x^0 = C(\overline{\Omega}).$$

For the mass variable, as before we let

$$dm_r := w_r(m)dm := (1+m^r)dm, r \geq 0,$$

and define

$$X_{m,r} := L_1(\mathbb{R}_+, w_r(m)dm).$$

Then, for $i = 0, 1$,

$$\mathcal{X}_r^i = L_1(\mathbb{R}_+, X_x^i, dm_r) \tag{17a}$$

endowed with the norm

$$||u||_{\mathcal{X}_r^i} := \int_0^\infty \|\boldsymbol{u}(m)\|_{X_x^i} dm_r \tag{17b}$$

We have

$$\mathcal{X}_r^1 = L_1(\Omega \times \mathbb{R}_+, dxdm_r) = L_1(\Omega, X_{m,r}, dx) = L_1(\mathbb{R}_+, X_x, dm_r). \tag{18}$$

We identify scalar functions $(t, x, m) \mapsto u(t, x, m)$ with $\mathcal{X}_r^i-$valued functions $t \to \boldsymbol{u}(t) := u(t, \cdot, \cdot)$.

Gluing semigroups. Suppose that we are given a family of operators $\{(T_m, D(A_m))\}_{m\in\mathbb{R}_+}$ in X_x and assume that for $m \in \mathbb{R}_+$, $(T_m, D(T_m))$ generates a C_0-semigroup $(G_m(t))_{t\geq 0}$ in X_x. Then, in particular, there exist constants ω_m and $M_m \geq 1$ such that

$$||G_{T_m}(t)||_{L(X_x)} \leq M_m e^{\omega_m t}, \quad m \in \mathbb{R}_+. \tag{19}$$

On $\mathcal{X}_r$ we define the extensions

$$[\boldsymbol{G}_{\boldsymbol{T}}(t)\boldsymbol{u}](m) = G_m(t)u(m), \quad \boldsymbol{u} \in \mathcal{X}_r,\, m \in \mathbb{R}_+, \tag{20}$$

$$[\boldsymbol{T}\boldsymbol{u}](m) = T_m u(m), m \in \mathbb{R}_+, \tag{21}$$

on $D(\boldsymbol{T}) := \{\boldsymbol{u} \in \mathcal{X}_r :\ \boldsymbol{u}(m) \in D(T_m),\, m \in \mathbb{R}_+, \boldsymbol{T}\boldsymbol{u} \in \mathcal{X}_r\}$

Proposition 1

Assume that there are M and ω such that for all $m \in \mathbb{R}_+$, we have $M_m \leq M$ and $\omega_m \leq \omega$. If $m \to R(\lambda, T_m)\boldsymbol{f}(m)$ is measurable for any $\lambda > \omega$ and $\boldsymbol{f} \in \mathcal{X}_r$, then the operator $\boldsymbol{T}$ generates a semigroup $(\boldsymbol{G}_{\boldsymbol{T}}(t))_{t\geq 0}$ satisfying (20) and its resolvent is given by

$$[R(\lambda, \boldsymbol{T})\boldsymbol{f}](m) = R(\lambda, T_m)\boldsymbol{f}(m), \quad m \in \mathbb{R}_+. \tag{22}$$

.

Proof.

The measurability of $\boldsymbol{u}(m) := R(\lambda, T_m)\boldsymbol{f}(m)$ together with the uniform bound on M_m ensure that $\boldsymbol{u} \in \mathcal{X}$ and hence, by induction, $m \mapsto R^n(\lambda, T_m)\boldsymbol{f}(m)$ is measurable for any n. Then, for $\lambda > \omega$,

$$\|R(\lambda, \boldsymbol{T})\boldsymbol{f}\|_{\mathcal{X}_r} = \int_0^\infty \|R^n(\lambda, T_m)\boldsymbol{f}(m)\|_{X_x} dm_r \leq \frac{M}{(\lambda-\omega)^n}\|\boldsymbol{f}\|_{\mathcal{X}_r},$$

that is, $\boldsymbol{T}$ is a Hille-Yosida operator (and hence closed). To show that it is densely defined, we use that $\mathcal{X}_r = L_1(\mathbb{R}_+, X_x) \simeq L_1(\mathbb{R}_+)\widehat{\otimes}_\pi X_x$ to show that

$$S_0 = \{\boldsymbol{u} : \ \boldsymbol{u}(m) = \phi(m)u^m, \phi \in L_1(\Omega), u^m \in D(T_m)\}$$

is linearly dense in $\mathcal{X}_r$. □

Proposition 2

Consider $((T_m, D))_{m\in\mathbb{R}_+}$ on a common domain D.

Assume that for a.a. $m \in \mathbb{R}_+$, there is a sequence $((T_{m,n}, D))_{n\in\mathbb{N}}$ such that for each $u \in D$,

a) $m \mapsto T_{m,n}u, n \in \mathbb{N}$, are continuous functions,

b) $\lim_{n\to\infty} T_{m,n}u = T_m u$.

Assume that $R(\lambda, T_m), R(\lambda, T_{m,n})$, $m \in \mathbb{R}_+, n \in \mathbb{N}$, are defined for $\lambda > \lambda_0$ are uniformly bounded by some C for $m \in \mathbb{R}_+, n \in \mathbb{N}, \lambda > \lambda_0$. Then, for any $\boldsymbol{f} \in \mathcal{X}_r$,

$$\mathbb{R}_+ \ni m \mapsto R(\lambda, T_n)\boldsymbol{f}(m) \in X_x$$

is Bochner measurable.

Step 1. First, we consider $((T_m, D))_{m\in\mathbb{R}_+}$ of operators continuous on $u \in D$. From

$$\lambda \boldsymbol{u}(m) - T_m \boldsymbol{u}(m) = \boldsymbol{f}(m)$$
$$\lambda \boldsymbol{u}(m+h) - T_{m+h} \boldsymbol{u}(m+h) = \boldsymbol{f}(m+h),$$

we get

$$\begin{aligned}
&\|\boldsymbol{u}(m+h) - \boldsymbol{u}(m)\|_{X_x} \\
&\leq \|R(\lambda, T_{m+h})(T_{m+h} - T_m)\boldsymbol{u}(m)\|_{X_x} \\
&+ \|R(\lambda, T_{m+h})(\boldsymbol{f}(m+h) - \boldsymbol{f}(m))\|_{X_x} \\
&\leq \|(T_{m+h} - T_m)\boldsymbol{u}(m)\|_{X_x} + \|\boldsymbol{f}(m+h) - \boldsymbol{f}(m)\|_{X_x},
\end{aligned}$$

which gives the continuity of $\boldsymbol{u}$.

Step 2. If $\boldsymbol{f} \in \mathcal{X}_r$ is not continuous, there is a sequence $(\boldsymbol{f}_n)_{n\in N}$ of continuous functions converging to $\boldsymbol{f}$ in $\mathcal{X}_r$. Letting

$$\boldsymbol{u}_n(m) = R(\lambda, T_m)\boldsymbol{f}_n(m), \quad n \in \mathbb{N},$$

there is $\boldsymbol{u} \in \mathcal{X}_r$ such that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^\infty \|\boldsymbol{u}(m) - \boldsymbol{u}_n(m)\|_{X_x} dm_r \\ &= \lim_{n\to\infty}\int_0^\infty \|\boldsymbol{u}(m) - R(\lambda, T_m)\boldsymbol{f}_n(m)\|_{X_x} dm_r = 0.\end{aligned}$$

Passing to subsequences, we can prove that for a.e. m,

$$\boldsymbol{u}(m) = R(\lambda, T_m)\boldsymbol{f}(m),$$

hence $R(\lambda, \boldsymbol{T})\boldsymbol{f} \in \mathcal{X}_r$.

Step 3. Finally, we drop the assumption of the continuity of T_m and consider

$$\lambda \boldsymbol{u}(m) - T_m \boldsymbol{u}(m) = \boldsymbol{f}(m)$$

$$\lambda \boldsymbol{u}_n(m) - T_{m,n} \boldsymbol{u}_n(m) = \boldsymbol{f}(m),$$

where $T_{m,n}$ are as in the assumptions of the proposition. Thus, as above,

$$\|\boldsymbol{u}_n(m) - \boldsymbol{u}(m)\|_{X_x} \leq C\|(T_{m,n} - T_m)\boldsymbol{u}(m)\|_{X_x},$$

hence, by assumption, for almost any m, $\boldsymbol{u}(m) = \lim_{n\to\infty} \boldsymbol{u}_n(m)$ in X_x. Since $\boldsymbol{u}_n$ are Bochner measurable by the first part of the proof, $\boldsymbol{u}$ is also Bochner measurable.

Application to the diffusion operator.

We consider

$$\begin{aligned}\partial_t u(t,x,m) &= [\mathcal{T}_0 u](t,x,m) = \nabla_x \cdot (d(x,m)\nabla_x u(t,x,m)),\\ \frac{\partial u}{\partial \boldsymbol{n}}(t,x,m) &= 0, \quad x \in \partial\Omega, \\ u(0,x,m) &= \mathring{u}(x,m).\end{aligned} \tag{23}$$

in $\Lambda = \Omega \times \mathbb{R}_+$, where $\Omega \subseteq \mathbb{R}^N$ is an open set with sufficiently smooth boundary.

We consider both cases, $X_x = L_1(\Omega)$ and $X_x = C(\overline{\Omega})$, that is, we work in $\mathcal{X}^i$, $i = 0$ or $i = 1$.

For a moment, we drop the dependence on m. We assume that

$$d \in C^1(\overline{\Omega}) \quad \text{and} \quad 0 < d_{\min} \leq d(x) \leq d_{\max} < \infty, \tag{24}$$

for all $x \in \Omega$. By [Fattorini, Theorems 4.8.11 & 4.8.3], the closure of the restriction of $\mathscr{T}_0$ to $\{u \in C^2(\overline{\Omega});\ \partial_{\boldsymbol{n}} u = 0 \text{ on } \partial\Omega\}$, denoted by T_0, generates a substochastic semigroup on X^i_x and we can prove that $D(T_0) = D(\Delta)$ is independent of m.

Now, we return to the dependance of d on m assuming that for almost any $m \in \mathbb{R}_+$, $d(\cdot, m) \in C(\overline{\Omega})$ and there exist $d_{\min}(m) > 0$ and $d_{\max} < \infty$ such that

$$d_{\min}(m) \leq d(x, m) \leq d_{\max}(m). \tag{25}$$

We denote by $T_{0,m}$ the realization of the expression $\nabla_x \cdot (d(x, m)\nabla_x \cdot)$ which generates a semigroup in $L_1(\Omega)$, as outlined above, so that $D(T_{0,m}) = D(\Delta)$ for almost any m. We can prove the following theorem.

Theorem 4

Let $i = 0$ or $i = 1$.

$$(0,\infty) \ni m \to d(\cdot, m) \in C^1(\overline{\Omega}) \textit{ be Bochner measurable.} \tag{26}$$

(a) For almost every m, the operator $(T_{0,m}, D(\Delta))$ generates a substochastic semigroup $(G_{T_{0,m}}(t))_{t\geq 0}$ in X^i_x.

(b) The operator $\boldsymbol{T}_0$, defined by (3), with the domain

$$D(\boldsymbol{T}_0)=\{\boldsymbol{u} \in \mathcal{X}^i_r : \ \boldsymbol{u}(\cdot, m) \in D(\Delta), (x, m)\to[\mathcal{T}_0\boldsymbol{u}(\cdot, m)](x) \in \mathcal{X}^i_r\} \tag{27}$$

generates a substochastic semigroup $(\boldsymbol{G}_{\boldsymbol{T}_0}(t))_{t\geq 0}$ on $\mathcal{X}^i_r$.

$$D(\boldsymbol{A}) = \{\boldsymbol{u} \in \mathcal{X}_r^i :\ au \in \mathcal{X}_r^i\}.$$

We assume that there are $\alpha_1, \alpha_2 \in L_{\infty,loc}(\mathbb{R}_+)$ such that

$$\alpha_1(m) \le a(x,m) \le \alpha_2(m), \quad x \in \Omega, \tag{28}$$

with $\alpha_2(m) \le M\alpha_1(m)$ for some $M < \infty$.

Theorem 5

Assume that a satisfies (28). Then the operator $(\boldsymbol{T}^i, D(\boldsymbol{T}^i)) := (\boldsymbol{T}_0^i + \boldsymbol{A}^i, D(\boldsymbol{T}_0^i) \cap D(\boldsymbol{A}^i))$, *is the generator of a substochastic semigroup, say* $(\boldsymbol{G}_{\boldsymbol{T}}(t))_{t\ge 0}$ *in* $\mathcal{X}_r^i$.

Application of the Desch theorem

In $\mathcal{X}_r^1$, the calculations can be done as in homogeneous case, to show that

$$\boldsymbol{K}^1 = \boldsymbol{T}_0^1 + \boldsymbol{A}^1 + \boldsymbol{B}^1$$

with $D(\boldsymbol{K}^1) = D(\boldsymbol{T}_0^1) \cap D(\boldsymbol{A}^1)$ generates a positive (diffusion-fragmentation) semigroup.

In $\mathcal{X}_r^0$, however, instead of (10), we only get

$$\|AR(\lambda, A)\| \leq M \tag{29}$$

and (39) is not available under (9).

We introduce a more general problem. Let (28) be satisfied, and let $\beta(m,s) \geq 0$ be a measurable function. Then, we consider

$$\begin{aligned} \partial_t u(t,x,m) &= \mathcal{T}_0 u(t,x,m) - \alpha_1(m)u(t,x,m) \\ &\quad + \int_m^\infty \alpha_2(m)\beta(m,s)u(t,x,m)ds. \end{aligned} \tag{30}$$

Observe, that β need not satisfy the local mass conservation equation (7) and the fragmentation part no longer is conservative in $L_1(\mathbb{R}_+, mdm)$.

By $\mathfrak{A}^i$ and $\mathfrak{B}^i$ we denote the equivalents for (30) of $\boldsymbol{A}^i$ and $\boldsymbol{B}^i$, respectively, in $\mathcal{X}_r^i$, where, as usual, $i \in \{0,1\}$. By (28),

$$D(\mathfrak{A}^i) = D(\boldsymbol{A}^i). \tag{31}$$

Under the assumptions of this section, $\mathfrak{B}^i$ is a well-defined operator on $D(\mathfrak{A}^i)$ for sufficiently large r. Then, we define the operator $\mathfrak{K}^i = \boldsymbol{T}_0^i + \mathfrak{A}^i + \mathfrak{B}^i = \mathfrak{T}^i + \mathfrak{B}^i$ to be the restriction of the expression on the right-hand side of (30) to

$$D(\mathfrak{K}^i) = D(\boldsymbol{K}^i) = D(\boldsymbol{T}_0^i) \cap D(\mathfrak{A}^i) = D(\boldsymbol{T}^i) = D(\mathfrak{T}^i).$$

Let $z = \frac{m}{s}$, $0 \leq z \leq 1$, and define the normalized moments of β (whenever they exist) by

$$c_r(s) := \frac{n_r(s)}{s^r} := \frac{1}{s^r} \int_0^s m^r \beta(m,s)dm = s \int_0^1 z^r \beta(zs,s)dz.$$

As for b, we assume that $n_0(s)$ exists and there is $l \geq 0$ such that

$$n_0(s) \leq \beta_0(1 + s^l) \tag{32}$$

for any $s \geq 0$. By the Dominated Convergence Theorem,

$$\lim_{r\to\infty} c_r(s) = 0. \tag{33}$$

For our purpose, we need this limit to be uniform in s, which is not always the case.

A bounded set $\mathcal{E} \in L_1(\Theta, d\mu)$, where $\mu(\Theta) < \infty$ is called uniformly integrable (or equi-integrable) if for any $\epsilon > 0$ there is $\delta > 0$ such that for any measurable $\Theta_0 \subset \Theta$ with $\mu(\Theta_0) < \delta$ we have

$$\sup_{f \in \mathcal{E}} \int_{\Theta_0} |f| d\mu < \epsilon. \tag{34}$$

Proposition 3

Assume that there are $r_0 \geq 0$ and $s_0 \geq 0$ such that the set

$$\mathcal{E}_{r_0} := \{[0,1] \ni z \mapsto sz^{r_0}\beta(zs, s)\}_{s \geq s_0} \tag{35}$$

is equi-integrable. Then the limit (33) is uniform in $s \geq s_0$.

The usefulness of (30) follows from

Proposition 4

Assume that β satisfies (35). If, for some $r \geq r_0$ and $\lambda > 0$,

$$\|\mathfrak{B}^i R(\lambda, \mathfrak{T}^i)\|_{L(\mathcal{X}_r^i)} < 1, \tag{36}$$

then $\mathfrak{K}^i$ generates a positive semigroup, say, $(\boldsymbol{G}_{\mathfrak{K}^i}(t))_{t\geq 0}$ solving (30) in $\mathcal{X}_r^i$. If

$$b(x,m,s) \leq \beta(m,s), \quad \text{for a.e. } x \in \Omega, \tag{37}$$

then $\boldsymbol{K}^i$ generates a positive semigroup, say, $(\boldsymbol{G}_{\boldsymbol{K}^i}(t))_{t\geq 0}$, solving (16) in $\mathcal{X}_r^i$. Moreover,

$$\boldsymbol{G}_{\boldsymbol{K}^i}(t) \leq \boldsymbol{G}_{\mathfrak{K}^i}(t). \tag{38}$$

The main result of this section is

Theorem 6

Let (32) *and* (35) *be satisfied. Then there exists* $r_1 > \max\{l, r_0\}$ *such that for any* $r \geq r_1$, $(\mathfrak{K}^i, D(\boldsymbol{T}^i))$ *generates a positive* C_0*-semigroup, say* $(\boldsymbol{G}^i_{\mathfrak{K}}(t))_{t\geq 0}$, *on* $\mathcal{X}^i_r$. *If* (37) *holds, also* $(\boldsymbol{K}^i, D(\boldsymbol{T}^i))$ *generates a positive* C_0*-semigroup on* $\mathcal{X}^i_r$.

In the proof, (39) is replaced by

$$\|BR(\lambda, \mathfrak{A}^i)\| \leq \frac{C(s_0)}{\lambda} + c_r(s_0)M < 1 \qquad (39)$$

for sufficiently large λ and r.

The semigroup $(\boldsymbol{G}_{\boldsymbol{K}^i}(t))_{t\geq 0}$ is anlaytic semigroup. Unfortunately, the identification of interpolation spaces between $D(\boldsymbol{K}^i)$ and $\mathcal{X}_r^i$, needed for the moment regularisation (13), is far from obvious. We can, however, use the analyticity of the semigroup $(\boldsymbol{G}_{\mathfrak{F}^i}(t))_{t\geq 0}$ generated by the fragmentation operator $\mathfrak{F}^i = \mathfrak{A}^i + \mathfrak{B}^i$ to prove necessary estimates if $\mathcal{T}_0^i$ is independent of m. Then, defining

$$\mathcal{X}_r^{(i,\mu)} := \left\{ \boldsymbol{f} \in \mathcal{X}_r^i : \int_0^\infty \|\boldsymbol{f}(m)\|_{X_x^i} (\omega + \alpha_1(m))^\mu \, dm_r < \infty \right\}, \tag{40}$$

for some ω_r, M_r^μ we get

$$\|\boldsymbol{G}_{\mathfrak{F}}(t)\boldsymbol{f}\|_{\mathcal{X}_r^{(i,\mu)}} \leq \frac{M_r^{(\mu)} e^{\omega_r t}}{t^\mu} \|\boldsymbol{f}\|_{\mathcal{X}_r^i}, \quad t > 0, \mu \in [0,1]. \tag{41}$$

Theorem 7

Let $\mathcal{T}_0$ be independent of m and for some a_0, m_0,

$$\alpha_1(m) \geq a_0 m^{\gamma}, \qquad m \geq m_0. \tag{42}$$

Then, for any $r \geq r_1$ and $\mu \in [0,1]$, there are constants $M_r^{(\mu)}$ and ω_r such that for any $t > 0, \boldsymbol{f} \in \mathcal{X}_r^i$, then for $q := \mu\gamma \leq \gamma$

$$\begin{aligned} \|\boldsymbol{G}_{\boldsymbol{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}^i_{r+q}} &\leq \|\boldsymbol{G}_{\mathfrak{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}^i_{r+q}} \leq C_1\|\boldsymbol{G}_{\mathfrak{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}_r^{(i,\mu)}} \\ &\leq C_1 \frac{M_r^{(\mu)}}{t^{\mu}} e^{\omega_r t}\|\boldsymbol{f}\|_{\mathcal{X}^i_r}, \quad \boldsymbol{f} \in \mathcal{X}^i_r. \end{aligned} \tag{43}$$

We have

$$\boldsymbol{G}_{\mathfrak{K}^i}(t)\boldsymbol{f} = \boldsymbol{G}_{\boldsymbol{T}_0^i}(t)\boldsymbol{G}_{\mathfrak{F}^i}(t)\boldsymbol{f} = \boldsymbol{G}_{\mathfrak{F}^i}(t)\boldsymbol{G}_{\boldsymbol{T}_0^i}(t)\boldsymbol{f}, \quad \boldsymbol{f} \in \mathcal{X}_r^i.$$

Analyticity of $(\boldsymbol{G}_{\mathfrak{F}^i}(t))_{t\geq 0}$ yields for $\boldsymbol{f} \in \mathcal{X}_r^i$ and $\mu \in [0,1]$,

$$\begin{aligned}\|\boldsymbol{G}_{\boldsymbol{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}_r^{(i,\mu)}} &\leq \|\boldsymbol{G}_{\mathfrak{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}_r^{(i,\mu)}} = \|\boldsymbol{G}_{\mathfrak{F}^i}(t)\boldsymbol{G}_{T_0^i}(t)\boldsymbol{f}\|_{\mathcal{X}_r^{(i,\mu)}} \\ &\leq \frac{M_r^{(\mu)}}{t^\mu}e^{\omega_r t}\|\boldsymbol{G}_{\boldsymbol{T}_0^i}(t)\boldsymbol{f}\|_{\mathcal{X}_r^i} = \frac{M_r^{(\mu)}}{t^\mu}e^{\omega_r t}\|\boldsymbol{f}\|_{\mathcal{X}_r^i},\end{aligned}$$

where we used the fact that $(\boldsymbol{G}_{\boldsymbol{T}_0^i}(t))_{t\geq 0}$ is contractive. Since

$$\int_0^\infty \|\boldsymbol{f}(m)\|_{X_x}\, dm_q \leq C_1\|\boldsymbol{f}\|_{\mathcal{X}_r^{\left(\frac{q}{\gamma}\right)}},$$

for some C_1, and $\mu = \frac{q}{\gamma}$, we get

$$\|\boldsymbol{G}_{\boldsymbol{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}_{r+q}^i} \leq C_1\|\boldsymbol{G}_{\boldsymbol{K}^i}(t)\boldsymbol{f}\|_{\mathcal{X}_r^{(i,\mu)}} \leq C_1\frac{M_r^{(\mu)}}{t^\mu}e^{\omega_r t}\|\boldsymbol{f}\|_{\mathcal{X}_r^i}, \quad \boldsymbol{f} \in \mathcal{X}_r^i.$$

Diffusion–fragmentation–coagulation problem. Assume that there are $q < \gamma$ and k_0 such that for $x \in \Omega, m, s \in \mathbb{R}_+$ we have

$$0 \leq k(x, m, s) \leq k_0(1 + m^q)(1 + s^q), \tag{44}$$

For a given $r \geq r_0$, define $p = r + q$.

We work in $\mathcal{X}_r := \mathcal{X}_r^0 \subset \mathcal{X}_r^1$ and try to find a mild solution to

$$\partial_t \boldsymbol{u} = \boldsymbol{T}_0\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{C}\boldsymbol{u}, \tag{45}$$

where

$$\begin{aligned}\boldsymbol{C}\boldsymbol{u}(m) &= \boldsymbol{C}_1\boldsymbol{u}(m) - \boldsymbol{C}_2\boldsymbol{u}(m)\\ &= \frac{1}{2}\int_0^m k(\cdot, m-s, s)\boldsymbol{u}(m-s)\boldsymbol{u}(s)ds - \boldsymbol{u}(m)\int_0^\infty k(\cdot, m, s)\boldsymbol{u}(s)ds.\end{aligned}$$

Making the nonlinearity nonnegative: with

$\boldsymbol{A}_q\boldsymbol{u}(m) := -a_q(1+m^q)\boldsymbol{u}(m)$, we consider $\boldsymbol{C}_q = \boldsymbol{C} - \boldsymbol{A}_q$

$$\partial_t \boldsymbol{u} = \boldsymbol{T}_0\boldsymbol{u} + (\boldsymbol{A}+\boldsymbol{A}_q)\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{C}_q = \boldsymbol{K}_q\boldsymbol{u} + \boldsymbol{C}_q\boldsymbol{u}, \tag{46}$$

Proposition 5

For any fixed $b > 0$, define

$$\mathcal{U}_b := \{f \in \mathcal{X}_{r,+} : \ \|f\|_{\mathcal{X}_r} \leq b\} \tag{47}$$

and let $a_q := 2k_0 b$. The operator $\boldsymbol{C}_q : \mathcal{X}_r \to \mathcal{X}_p$ is positive, bounded, globally Lipschitz continuous on $\mathcal{U}_b$ and continuously Fréchet differentiable as a function from $\mathcal{X}_r$ to $\mathcal{X}_p$.

We consider the problem, that arises as the mild formulation of (1),

$$\boldsymbol{f}(t) = \mathbb{F}\boldsymbol{f}(t) := \boldsymbol{G}_{\boldsymbol{K}_q}(t)\mathring{\boldsymbol{u}} + \int\limits_0^t \boldsymbol{G}_{\boldsymbol{K}_q}(t-\tau)\boldsymbol{C}_q\boldsymbol{f}(\tau)d\tau. \qquad (48)$$

in the space $Y_r = C([0,T], \mathcal{U}_b)$, where T is to be determined so that $\mathbb{F}$ is a contraction on Y_r.

Theorem 8

Let $p \geq r_1$ *(and hence* $r = p + q > r_1$*). Under the assumptions of this section, for any* $\mathring{\boldsymbol{u}} \in \mathcal{X}_{r,+}$ *there is a mild solution to* (45) *in* $\mathcal{X}_{r,+}$ *defined on a maximal time interval* $I_{\max} := [0, T_{\mathring{\boldsymbol{u}}})$, *and if* $T_{\mathring{\boldsymbol{u}}} < 0$, *then* $\limsup_{t \to T_{\mathring{\boldsymbol{u}}}} \|\boldsymbol{u}(t)\|_{\mathcal{X}_r} = \infty$.

Let $D_p(\boldsymbol{T})$ *be the domain of* $\boldsymbol{T}$ *in* $\mathcal{X}_p \supset \mathcal{X}_r$*. For any* $\mathring{\boldsymbol{u}} \in \mathcal{X}_r \cap D_p(\boldsymbol{T})$, *the mild solution is in* $C(I_{\max}, \mathcal{X}_r) \cap C^1(\mathring{I}_{\max}, \mathcal{X}_r) \cap C((0, T_{\mathring{\boldsymbol{u}}}), D_p(\boldsymbol{T}))$, *where* $\mathring{I}_{\max} = (0, T_{\mathring{\boldsymbol{u}}})$, *and it is a classical solution to* (45) *in* $\mathcal{X}_p$.

A crucial role is played by the integrability of the integrand in (48). Here, for any function $\boldsymbol{u} \in Y_r$ and for any $\sigma > 0$, $0 \leq \tau \leq T$, and $q' := \frac{q}{\gamma}$, we have, by (43) and Proposition 5,

$$\left\| G_{\boldsymbol{K}_q}(\sigma)\boldsymbol{C}_q\boldsymbol{u}(\tau)\right\|_{\mathcal{X}_r} \leq \frac{M_p e^{\omega_p\sigma}}{\sigma^{q'}} \left\|\boldsymbol{C}_q\boldsymbol{u}(\tau)\right\|_{\mathcal{X}_p} \leq \frac{M_p e^{\omega_p\sigma}}{\sigma^{q'}} K(\mathcal{U}_b).$$

The restrictive assumption that the transport part $\mathscr{T}_0$ is independent of m was only needed for the availability of (43), necessary to prove that $G_{\boldsymbol{K}_q}(t)\boldsymbol{C}_q, t \geq 0$, are well-defined operators on $\mathcal{X}_r$ despite $\boldsymbol{C}_q$ being unbounded there. Thus, we immediately obtain analogous theorems if

- $\boldsymbol{C}$ is bounded;
- the gain term $\mathscr{B}$ is absent, that is, for diffusion–absorption–coagulation problems.

Further, the general theory works also if diffusion is replaced by an advection operator.