Adjoint first order linear equations with Stieltjes derivatives and Lagrange's identity

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Adjoint linear ODEs

In the context of ODEs, given the linear equation

$$x'(t) = p(t)x(t) + f(t), \tag{LE}$$

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This is because their linear operators,

$$\begin{aligned} Lu(t) &= u'(t) - p(t)u(t), \\ L^{*}v(t) &= v'(t) + p(t)v(t) \end{aligned}$$

are adjoint operators.

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Adjoint dynamic linear equations

Similarly, for a time scale, $\mathbb{T}$, the equations

$$x^{\Delta}(t) = p(t)x(t) + f(t), \tag{DLE}$$

$$x^{\Delta}(t) = -p(t)x(\sigma(t)) + f(t) \tag{ADLE}$$

are called adjoint linear equations, with

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Theorem (Lagrange's identity)

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Adjoint linear in a more general context

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Can we obtain Lagrange's identity in this context?

The Stieltjes derivative

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Specifically, given a nondecreasing and left-continuous function $g : \mathbb{R} \to \mathbb{R}$, we "define" the Stieltjes derivative of f at t_0 as

$$f'_g(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{g(t) - g(t_0)}.$$

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$$f'_g(t_0) = \lim_{t \to t_0} F_{t_0}(t), \qquad F_{t_0}(\cdot) = \frac{f(\cdot) - f(t_0)}{g(\cdot) - g(t_0)}, \qquad \mathrm{Dom}(F_{t_0})?$$

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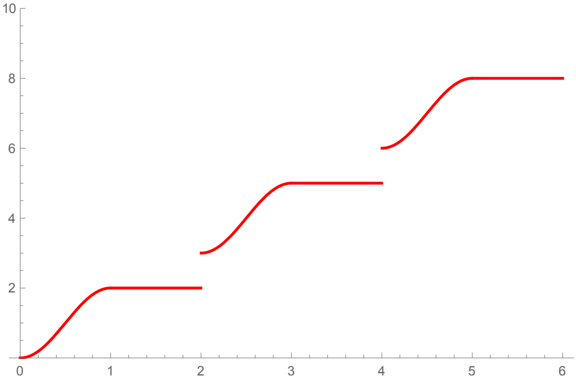

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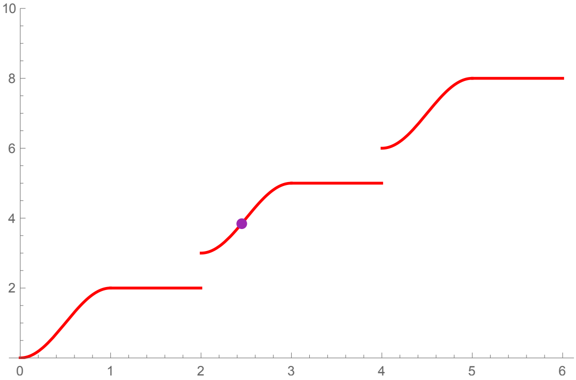

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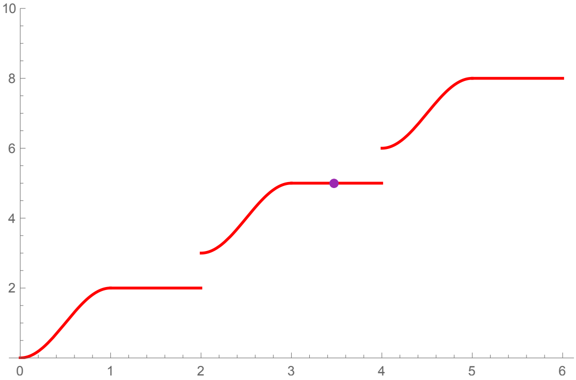

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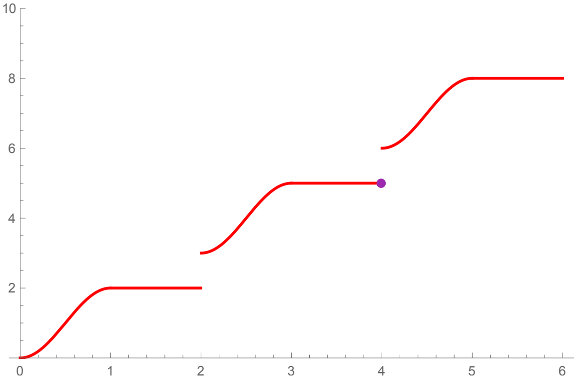

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R. López Pouso, A. Rodríguez, A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives, *Real Anal. Exchange* **40**(2014/15), No. 2, 319–353.

$$C_g := \{t \in \mathbb{R} : g \text{ is constant in } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}$$

$$D_g := \{t \in \mathbb{R} : \Delta g(t) := g(t^+) - g(t) > 0\}$$

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Definition

Define the Stieltjes derivative of $f : \mathbb{R} \to \mathbb{C}$ at $t_0 \in \mathbb{R} \setminus C_g$ as

$$f'_g(t_0) = \begin{cases} \lim\limits_{t \to t_0} \dfrac{f(t) - f(t_0)}{g(t) - g(t_0)}, & t_0 \notin D_g, \\ \lim\limits_{t \to t_0^+} \dfrac{f(t) - f(t_0)}{g(t) - g(t_0)}, & t_0 \in D_g. \end{cases}$$

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- If $\mathrm{Dom}(F_{t_0}) = \mathbb{R} \setminus \{t_0\}$ and $\lim_{t \to t_0} F_{t_0}(t)$ exists, so does $\lim_{t \to t_0^+} F_{t_0}(t)$ and they are equal.

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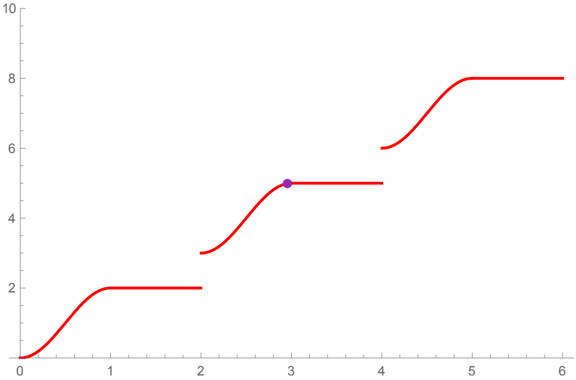
Remark

For $t \in D_g$, $f_g'(t)$ exists if and only if $f(t^+)$ exists and, in that case,

$$f_g'(t) = \frac{f(t^+) - f(t)}{\Delta g(t)}.$$

Properly understanding the Stieltjes derivative

$$f'_g(t_0) = \lim_{t \to t_0} F_{t_0}(t), \qquad F_{t_0}(\cdot) = \frac{f(\cdot) - f(t_0)}{g(\cdot) - g(t_0)}, \qquad \text{Dom}(F_{t_0})?$$



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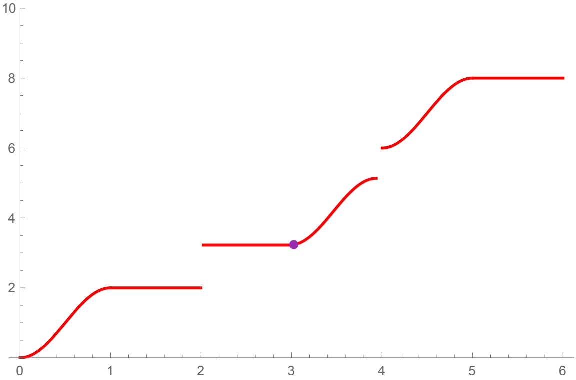

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I. Márquez Albés, Notes on the linear equation with Stieltjes derivatives, *Electron. J. Qual. Theory Differ. Equ.* **42** (2021) 1–18

Since C_g is open, we can write $C_g = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Hence, we can define $N_g^- = \{a_n\}_{n=1}^{\infty} \setminus D_g$ and $N_g^+ = \{b_n\}_{n=1}^{\infty} \setminus D_g$.

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Define the g-derivative of $f : \mathbb{R} \to \mathbb{C}$ at $t_0 \in \mathbb{R} \setminus C_g$ as

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The Stieltjes derivative: basic properties

Proposition

If f and h are g-differentiable at $t_0 \in \mathbb{R} \setminus C_g$, then

- *$\alpha f + \beta h$ is g-differentiable at t_0 for any $\alpha, \beta \in \mathbb{R}$ and*

$$(\alpha f + \beta h)'_g(t_0) = \alpha f'_g(t_0) + \beta h'_g(t_0).$$

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$$(f \cdot h)'_g (t_0) = f'_g(t_0)h(t_0) + h'_g(t_0)f(t_0) + f'_g(t_0)h'_g(t_0)\Delta g(t_0).$$

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Proposition

Let f be g-differentiable at $t_0 \in \mathbb{R} \setminus (C_g \cup D_g)$ and let h be differentiable at $f(t_0)$. Then $h \circ f$ is g-differentiable at t_0 and

$$(h \circ f)'_g(t_0) = h'(f(t_0))f'_g(t_0).$$

The Lebesgue-Stieltjes integral

The Lebesgue-Stieltjes integral is defined as

$$\int_A f(s)\,\mathrm{d}\,g(s) := \int_A f(s)\,\mathrm{d}\,\mu_g(s), \quad f \in \mathcal{L}^1_g := \mathcal{L}^1_{\mu_g},$$

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by considering the outer measure

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and its restriction, $\mu_g := \mu_g^*|_{\mathcal{LS}_g}$, to the set

$$\mathcal{LS}_g = \{A \subset \mathbb{R} : \mu_g^*(E) = \mu_g^*(E \cap A) + \mu_g^*(E \cap A^c),\ E \in \mathcal{P}(\mathbb{R})\}$$

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  In particular, for $t \in D_g$,

$$\int_{\{t\}} f_g'(s)\, \mathrm{d}\, g(s) = f_g'(t)\Delta g(t) = \frac{f(t^+) - f(t)}{\Delta g(t)}\Delta g(t) = f(t^+) - f(t).$$

The Fundamental Theorem of Calculus

R. López Pouso, A. Rodríguez, A new unification of continuous, discrete, and impulsive calculus through Stieltjes derivatives, *Real Anal. Exchange* **40**(2014/15), No. 2, 319–353.

Theorem

Let $F : [a,b] \to \mathbb{C}$. Then, the following are equivalent:

(a) $F \in \mathcal{AC}_g([a,b],\mathbb{C})$, *i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ st for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$,*

$$\sum_{n=1}^{m}(g(b_n) - g(a_n)) < \delta \implies \sum_{n=1}^{m}|F(b_n) - F(a_n)| < \varepsilon.$$

(b) $F_g'(t)$ *exists for g-a.a. $t \in [a,b)$, $F_g' \in \mathcal{L}_g^1([a,b),\mathbb{C})$, and*

$$F(t) = F(a) + \int_{[a,t)} F_g'(s)\,\mathrm{d}\,g(s), \quad t \in [a,b].$$

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Theorem

Let $f \in \mathcal{L}^1_g([a,b),\mathbb{C})$. *Consider* $F : [a,b] \to \mathbb{C}$ *given by*

$$F(t) = \int_{[a,t)} f(s)\,\mathrm{d}\,g(s).$$

Then $F \in \mathcal{AC}_g([a,b],\mathbb{C})$ *and* $F'_g(t) = f(t)$ *for* g*-a.a.* $t \in [a,b)$.

Linear equation

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A solution of (SLE) in $[a, b]$ is a function $x \in \mathcal{AC}_g([a, b], \mathbb{C})$ st

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- If $b \in D_g$, x not defined to the right of b, so $\nexists\ x_g'(t)$.
- If $b \notin D_g$, $\mu_g(\{b\}) = 0$, so g-a.e. $[a, b) \iff g$-a.e. $[a, b]$.

Homogeneous linear equation

A reasonable solution of the homogeneous equation in $[a, b]$ would be

$$x(t) = \exp\left(\int_{[a,t)} p(s)\,\mathrm{d}\,g(s)\right), \quad t \in [a, b].$$

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For $t \notin D_g \cup C_g$,

$$x_g'(t) = \exp\left(\int_{[a,t)} p(s)\,\mathrm{d}\,g(s)\right) \cdot \left(\int_{[a,t)} p(s)\,\mathrm{d}\,g(s)\right)_g' = x(t)p(t).$$

Homogeneous linear equation

A reasonable solution of the homogeneous equation in $[a, b]$ would be

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Homogeneous linear equation: explicit solution

- M. Frigon, R. López Pouso, Theory and applications of first-order systems of Stieltjes differential equations, *Adv. Nonlinear Anal.*
- I. Márquez Albés, Notes on the linear equation with Stieltjes derivatives, *Electron. J. Qual. Theory Differ. Equ.*
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Let $p \in \mathcal{L}^1_g([a,b],\mathbb{C})$ be such that

$$1 + p(t)\Delta g(t) \neq 0, \quad t \in [a,b) \cap D_g. \tag{C}$$

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Define $\widetilde{p} \in \mathcal{L}^1_g([a,b),\mathbb{C})$ as

$$\widetilde{p}(t) = \begin{cases} p(t), & t \in [a,b] \backslash D_g, \\ \dfrac{\log\left(1 + p(t)\Delta g(t)\right)}{\Delta g(t)}, & t \in [a,b] \cap D_g. \end{cases}$$

Homogeneous linear equation: explicit solution

Theorem

Let $p \in \mathcal{L}^1_g([a,b],\mathbb{C})$ *be st* (C) *holds. Then, the map*

$$\exp_g(p,t) := \exp\left(\int_{[a,t)} \widetilde{p}(s)\,\mathrm{d}\,g(s)\right) \quad t \in [a,b],$$

is a solution of the homogeneous problem in $[a,b]$.

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Let $p \in \mathcal{L}^1_g([a,b],\mathbb{C})$ *be st* (C) *holds. Then, the map*

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Theorem

Let $p, f \in \mathcal{L}^1_g([a,b],\mathbb{C})$ *be such that* (C) *holds. Then, the map*

$$x(t) = \exp_g(p,t)\left(1 + \int_{[a,t)} \frac{f(s)}{\exp_g(p,s)(1+p(s)\Delta g(s))}\,\mathrm{d}\,g(s)\right)$$

is a solution of the (SLE) *in* $[a,b]$.

Adjoint linear equation

Given the equation linear equation with Stieltjes derivatives

$$x'_g(t) = p(t)x(t) + f(t), \tag{SLE}$$

we define the adjoint linear equation with Stieltjes derivatives as

$$x'_g(t) = \frac{-p(t)}{1 + p(t)\Delta g(t)} x(t) + \frac{f(t)}{1 + p(t)\Delta g(t)} \tag{ASLE}$$

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- requires condition (C) to be well-defined.

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An equivalent formulation is

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Observe that (ASLE):

- requires condition (C) to be well-defined.
- can be regarded as a particular case of (SLE).

Why that definition of adjoint equation?

The adjoint equation of (ASLE) is

$$x_g'(t) = P(t)x(t) + F(t)$$

where

$$P(t) = \frac{-\frac{-p(t)}{1+p(t)\Delta g(t)}}{1 + \frac{-p(t)}{1+p(t)\Delta g(t)}\Delta g(t)}$$

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Naturally, this is not enough to justify calling (ASLE) the adjoint equation as other equations satify that (e.g. $x'_g = -px + h$).

Linear operators

The equation (SLE) can be rewritten in the form $Lx = f$ for the linear operator $L : \mathcal{AC}_g([a,b],\mathbb{C}) \to \mathcal{L}^1_g([a,b],\mathbb{C})$ defined as

$$Lu(t) = u'_g(t) - p(t)u(t), \quad g\text{-a.a. } t \in [a,b).$$

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$$\widehat{L}v(t) = v'_g(t) + \frac{p(t)}{1+p(t)\Delta g(t)}v(t), \quad g\text{-a.a. } t \in [a,b).$$

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If we want to rewrite it as $L^*x = f$, we must consider the map $L^* : \mathcal{AC}_g([a,b],\mathbb{C}) \to \mathcal{L}^1_g([a,b],\mathbb{C})$ defined as

$$L^*v(t) = v_g'(t)(1+p(t)\Delta g(t)) + p(t)v(t), \quad g\text{-a.a. } t \in [a,b).$$

Lagrange's identity

I. Márquez Albés, A. Slavík, M. Tvrdý, Duality for Stieltjes differential and integral equations, submitted for publication.

Theorem (Lagrange's identity)

Given $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$, *we have that for* g*-a.a.* $t \in [a,b)$

$$(x \cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta g(t)\right) Lx(t) + x(t)L^* y(t).$$

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Remark

Since $L^* = (1 + p\Delta g)\widehat{L}$, we also have that for g-a.a. $t \in [a,b)$

$$(x \cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta g(t)\right)Lx(t) + (1 + p(t)\Delta g(t))x(t)\widehat{L}y(t).$$

Lagrange's identity: proof

Proof.

Let $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$. There exists $N \subset [a,b)$ st $\mu_g(N) = 0$ and $x'_g(t), y'_g(t)$ exist for $t \in [a,b) \setminus N$.

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$$\begin{aligned}(xy)'_g(t) &= x'_g(t)y(t) + y'_g(t)x(t) + x'_g(t)y'_g(t)\Delta g(t) \\ &= x'_g(t)\left(y(t) + y'_g(t)\Delta g(t)\right) + y'_g(t)x(t)\end{aligned}$$

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Adding and subtracting $p(t)x(t)(y(t) + y_g'(t)\Delta g(t))$,

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□

Lagrange's identity: consequences I

Theorem

If $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$ *are st* $Lx = L^*y = 0$ *g-a.e. in* $[a,b)$*, then*

$$x(t)y(t) = x(a)y(a), \quad t \in [a,b].$$

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Proof.

We can find $N \subset [a,b)$ st $\mu_g(N) = 0$ and, for $t \in [a,b) \setminus N$,

$$\begin{aligned} Lx(t) = L^*y(t) &= 0, \\ (x \cdot y)'_g(t) &= \left(y(t) + y'_g(t)\Delta g(t)\right) Lx(t) + x(t)L^*y(t). \end{aligned}$$

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Thus, $(x \cdot y)'_g(t) = 0$ g-a.e. in $[a,b)$.

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$$\begin{aligned} Lx(t) &= L^*y(t) = 0, \\ (x\cdot y)'_g(t) &= \left(y(t) + y'_g(t)\Delta g(t)\right) Lx(t) + x(t)L^*y(t). \end{aligned}$$

Thus, $(x\cdot y)'_g(t) = 0$ g-a.e. in $[a,b)$. Since $x\cdot y \in \mathcal{AC}_g([a,b],\mathbb{C})$,

$$x(t)y(t) = x(a)y(a) + \int_{[a,t)} (x\cdot y)'_g(t)\,\mathrm{d}\,g(s) = x(a)y(a). \qquad \square$$

Lagrange's identity: consequences II

Theorem

Let $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$ *be st* $xy = \alpha \in \mathbb{C} \setminus \{0\}$ *in* $[a,b]$.

- *If* $Lx(t) = 0$ *for* g*-a.a.* $t \in [a,b)$, *then*

$$L^* y(t) = 0 \quad g\text{-a.a. } t \in [a,b).$$

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$$Lx(t) = 0 \quad g\text{-a.a. } t \in [a,b).$$

Proof.

Since $x(t)y(t) = \alpha$, $(x\cdot y)'_g(t) = 0$. Hence, by Lagrange's identity,
$$0 = (x\cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta g(t)\right) Lx(t) + x(t)L^*y(t),$$

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Let $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$ be st $xy = \alpha \in \mathbb{C}\setminus\{0\}$ in $[a,b]$.

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$$0 = (x\cdot y)'_g(t) = \left(y(t) + y'_g(t)\Delta g(t)\right) Lx(t) + x(t)L^*y(t),$$

so if $Lx(t) = 0$ and $x(t) \neq 0$, we must have that $L^*y(t) = 0$. □

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Proof.

If $L^*y(t) = 0$, we get $(y(t) + y_g'(t)\Delta g(t))Lx(t) = 0$.

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Lagrange's identity: consequences III

Given that $L^* = (1 + p\Delta g)\widehat{L}$, we can obtain the following result.

Theorem

Let $x, y \in \mathcal{AC}_g([a, b], \mathbb{C})$. Then, the following hold:

- *If $Lx = \widehat{L}y = 0$ g-a.e. in $[a, b)$, then*

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- *If* $Lx(t) = 0$ *for g-a.a.* $t \in [a,b)$*, then*

$$\widehat{L}y(t) = 0, \quad g\text{-a.a. } t \in [a,b].$$

Lagrange's identity: consequences III

Given that $L^* = (1 + p\Delta g)\widehat{L}$, we can obtain the following result.

Theorem

Let $x, y \in \mathcal{AC}_g([a,b],\mathbb{C})$. Then, the following hold:

- *If $Lx = \widehat{L}y = 0$ g-a.e. in $[a,b)$, then*

$$x(t)y(t) = x(a)y(a), \quad g\text{-a.a. } t \in [a,b).$$

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Lagrange's identity: consequences IV

From Lipschitz's uniqueness criterion for IVP, we obtain:

Proposition

Let $x_a, y_a \in \mathbb{C}$. The unique solution of $Lx = 0$, $x(a) = x_a$, is

$$x(t) = x_a \exp_g(p, t), \quad t \in [a, b].$$

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Corollary

For g-a.a. $t \in [a, b]$,

$$\exp_g(p, t)^{-1} = \exp_g\left(-\frac{p}{1 + p\Delta g}, t\right).$$

Relations with other adjoint problems I

The pair of equations

$$x_g'(t) = p(t)x(t) + f(t), \qquad \text{(SLE)}$$

$$x_g'(t) = \frac{-p(t)}{1 + p(t)\Delta g(t)}x(t) + \frac{f(t)}{1 + p(t)\Delta g(t)} \qquad \text{(ASLE)}$$

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is a generalization of the usual pair of adjoint linear ODEs. Indeed, for $g(t) = t$, we have that $\Delta g(t) = 0$, $t \in \mathbb{R}$, so we get

$$x'(t) = p(t)x(t) + f(t),$$
$$x'(t) = -p(t)x(t) + f(t)$$

Relations with other adjoint problems II

The next natural question is: are (SLE)-(ASLE) a generalization of the corresponding dynamic equations?

$$x^{\Delta}(t) = p(t)x(t) + f(t) \qquad \text{(DLE)}$$

$$y^{\Delta}(t) = -p(t)y(\sigma(t)) + f(t) \qquad \text{(ADLE)}$$

with $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$.

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with $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. This is partially covered in

- M. Frigon, R. López Pouso, Theory and applications of first-order systems of Stieltjes differential equations, *Adv. Nonlinear Anal.* **6**(2017), No. 1, 13–36.
- I. Márquez Albés, Differential problems with Stieltjes derivatives and applications, Ph.D. thesis, Universidade de Santiago de Compostela, 2021.

Relations with other adjoint problems III

Time scales and Stieltjes calculus are equivalent for

$$g(t) = \inf\{s \in \mathbb{T} : s \geq t\},$$

in which case, the two derivatives are equal. Thus, (SLE) and (DLE) are equivalent.

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$$x_g'(t)(1 + p(t)\Delta g(t)) = x_g'(t) + p(t)(x(t^+) - x(t)),$$

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$$x'_g(t)(1 + p(t)\Delta g(t)) = x'_g(t) + p(t)(x(t^+) - x(t)),$$

we see that (ASLE) can be rewritten as

$$x'_g(t) = -p(t)x(t^+) + f(t),$$

from which it is easy to check that it is equivalent to (ADLE).

Adjoint first order linear equations with Stieltjes derivatives and Lagrange's identity

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