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joint results with

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Turing instability and reaction-diffusion-ODE systems.

Turing Instability

System of ODEs (linear)

$$\frac{d}{dt}u = au + bv$$

$$\frac{d}{dt}v = cu + dv \qquad a, b, c, d \in \mathbb{R}$$

Thm (Exercise)

Assume that

$$\mathrm{Det} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$$

$$\mathrm{Tr} = a + d < 0$$

Then $(0,0)$ is asymptotically stable.

Thm (Alan Turing, 1952)

Consider the system of RD

$$u_t = \gamma \Delta u + au + bv$$

$$v_t = D\Delta v + cu + dv \qquad x \in \Omega \subset \mathbb{R}^n \text{ bounded}$$

$$\frac{\partial u}{\partial \bar{n}} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \bar{n}} = 0 \quad \text{on } \partial\Omega$$

Assume that

$Det > 0$ and $Tr < 0$

$a > 0$ and $d < 0$

Diffusion coefficients

$\gamma > 0$ is small

$D > 0$ is large

The $(0,0)$ is USTABLE.

Proof: we look for a solution in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{\lambda t} \begin{pmatrix} c_1 \varphi_k \\ c_2 \varphi_k \end{pmatrix}$$

φ_k - eigenfunction ~~eigenvalue~~ of Δ with Neumann.

#

Turing for nonlinear systems

$$u_t = \gamma \Delta u + f(u,v)$$
$$v_t = D \Delta v + g(u,v)$$

in Ω

Neumann boundary conditions

Constant solution $(\bar u, \bar v) \in \mathbb{R}^2$

$$f(\bar u, \bar v) = 0 \qquad g(\bar u, \bar v) = 0$$

Linearisation

$$\varphi = u - \bar u \qquad \psi = v - \bar v$$

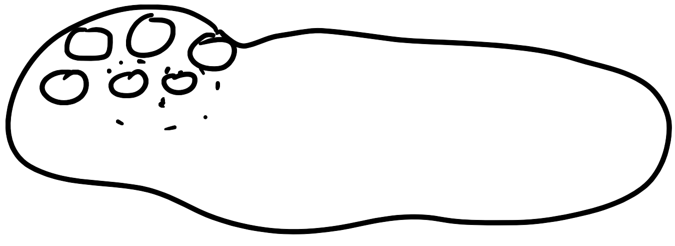
$$\varphi_t = \partial \Delta \varphi + a\varphi + b\psi + R_1(\varphi, \psi)$$

$$\psi_t = D \Delta \psi + c\varphi + d\psi + R_2(\varphi, \psi)$$

Turing Diffusion Driven Instability

Reaction-diffusion-ODE

$$\begin{cases} u_t = f(u,v) \\ v_t = D\Delta v + g(u,v) \\ \partial_n v = 0 \end{cases} \quad \text{in } \Omega \subset \mathbb{R}$$



$u(x,t)$ – density of cells

$v(x,t)$ – density growth factor.

more general

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

$$u_t = f(u,v)$$
$$v_t = \Delta_y v + g(u,v) \quad \text{in } \Omega$$

Constant stationary solutions
$(\bar{u},\bar{v}) \in \mathbb{R}^2$ $\quad f(\bar{u},\bar{v})=0$
$\qquad\qquad\qquad\quad g(\bar{u},\bar{v})=0$

<u>Thm</u> (Turing)

Assume that $f_u(\bar{u},\bar{v}) > 0$ (autocatalysis)

Then $(\bar{u},\bar{v})$ is unstable.

Proof - standard reasoning

Non-constant stationary solution
$\bar{U}(x), \bar{V}(x)$

$$\begin{cases} f(\bar{U}(x), \bar{V}(x)) = 0 & x \in \bar{\Omega} \\ \Delta_y \bar{V} + g(\bar{U},\bar{V}) = 0 \end{cases}$$

<u>Thm</u> (2018)

Assume that $f_u(U(x), V(x)) > 0$
on a set of positive measure

Then $u(x), v(x)$ is unstable!

The Turing mechanism destabilizes ALSO non-constant solutions.

More on stationary solutions (2021)

$$\begin{cases} f(\bar{u}(x), \bar{v}(x)) = 0 & \text{in } \bar{\Omega} \\ \Delta_v \bar{v} + g(\bar{u}, \bar{v}) = 0 \end{cases}$$

Regular stationary solutions

$g, f \in C^2$ $\quad f(u, v) = 0$

$$u = k(v) \qquad k \in C^2$$

$$f(k(v), v) = 0$$

we substitute

$$\Delta_v v + \underbrace{g(k(v), v)}_{h(v)} = 0 \text{ in } \Omega$$

There exist regular stationary solutions.

Thm. (2021)

Assume $f, g \in C^2$, $\Omega \subset \mathbb{R}^n$ bounded with $\partial\Omega$ smooth

Ω is convex

Then every nonconstan is UNSTABLE.

Jump - discontinuos stationary solutions

$$f(u, v) = 0 \quad \text{in } \Omega$$

$$\Delta_v V + g(u, v) = 0$$

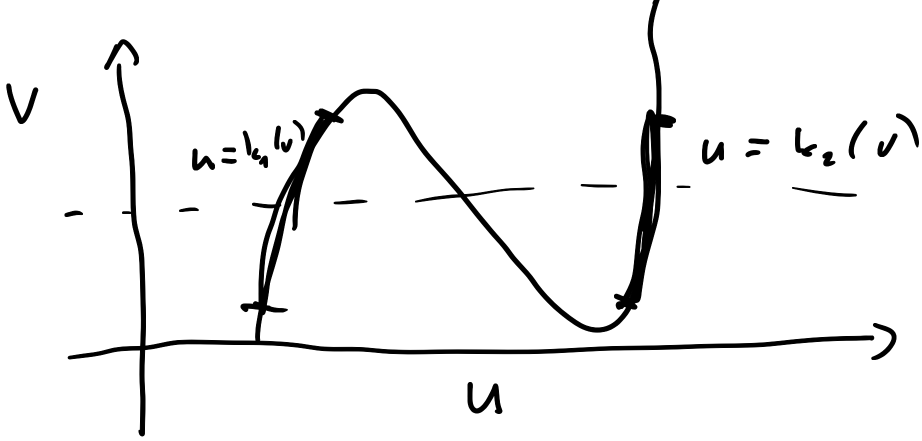
we assume that there are two funtion k_1, k_2 such that

$$f(k_1(v), v) = 0$$

$$f(k(v), v) = 0$$

Example

$$v - p(u) = 0 \qquad p - \text{polynom.}$$



Thm.

Assumptions

- There is a constant station solution $(\bar{u}, \bar{v}) \in \mathbb{R}^2$
- we have two branches of solutions to eqution $f(u,v)=0$
- $\Omega = \Omega_1 \cup \Omega_2$ arbitrary subsets

Then

For sufficiently small $\varepsilon > 0$

and small $|\Omega_2| > 0$

there is a constan solution

$$(u,v) \in L^\infty(\Omega) \times W_0^{2,p}(\Omega) \quad p > 1$$

$$\begin{cases} u(x) = \begin{cases} k_1(v(x)) & x \in \Omega_1 \\ k_2(v(x)) & x \in \Omega_2 \end{cases} \\ \Delta_v v + g(u,v) = 0 & x \in \Omega \end{cases}$$

where

$$\| v - \bar{v} \|_\infty < \varepsilon$$

$$\| u - \bar{u} \|_{L^\infty(\Omega_1)} < \varepsilon$$

$$\| u - k_1(\bar{v}) \|_{L^\infty(\Omega_2)} < \varepsilon$$

Thm (2021).

We have sufficient conditions for asymptotic stability of jump-discontinuous stationary solutions.

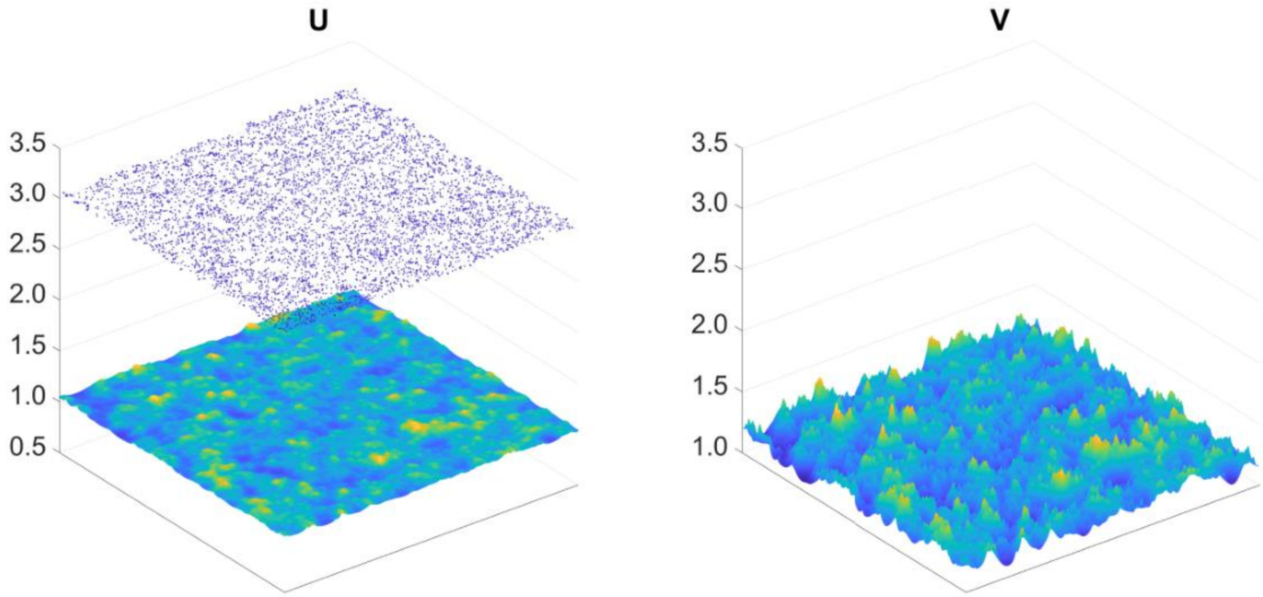
$$\begin{cases} u_t = f(u, v) \\ v_t = \Delta_v v + g(u, v) \end{cases}$$

Discontinuous stationary solution of the model

$$u_t = v - p(u)$$

$$v_t = \Delta v + \alpha u - \beta v$$

Ω_2 is random



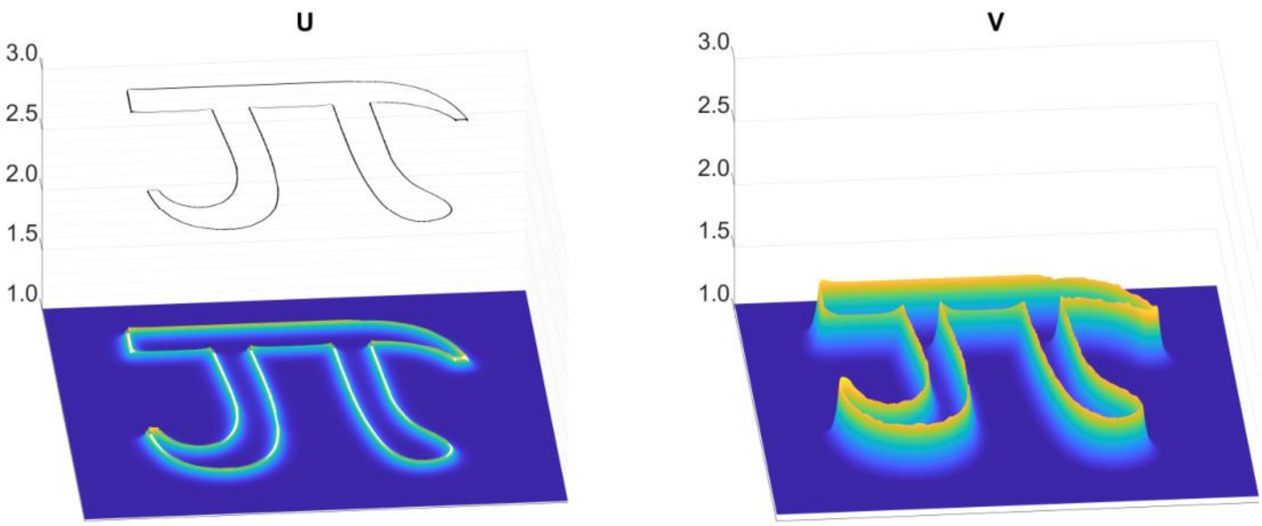
This solution has been obtained by numerical simulations for large $t > 0$.

Discontinuous stationary solution of

$$u_t = v - p(u)$$

$$v_t = \Delta v + \alpha u - \beta v$$

with π-shape Ω_2



The paper with the proof that all regular pattern are unstable : arXiv: 2105.05023

The paper about discontinuous patterns will appear soon.