Conditions for uniform h–dichotomy in terms of uniform non criticality, expansiveness and via generalized Floquet theory

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Short presentation

About our group and this result

Our research group

The Department of Mathematics of the University of Chile gathers a motivated group working on Dichotomies and non autonomous dynamical systems.

This work

This work has been made in collaboration with Heli Elorreaga (Universidad del Bío–Bío), Juan Francisco Peña and David Urrutia (Ph.D Students)

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Exponential Dichotomy (O. Perron, 1930)

Preliminaries & Notation

- Let us consider a linear ODE system in finite dimension:

$$x' = A(t)x$$

- A *fundamental matrix* will be denoted by $\Phi(t)$
- Its corresponding *Cauchy Operator* is $\Phi(t,s) := \Phi(t)\Phi^{-1}(s)$.
- A vector norm will be denoted by $|\cdot|$ and its matrix induced norm is $||\cdot||$.
- The property of dichotomy means that any solution $t \mapsto x(t)$ of the above linear system can be splitted in two classes, namely,

$$x(t) = x_+(t) + x_-(t),$$

where $x_+(\cdot)$ and $x_-(\cdot)$ have a *dichotomous behavior*.

O. Perron: Die Stabilitatsfrage bei Differentialgleichungen, *Mathematische Zeitschrift* 32 (1930) 702–728.

Exponential Dichotomy

Formal definition

Definition

The linear system

$$x' = A(t)x$$

has an exponential dichotomy on $\mathcal{I} \subseteq \mathbb{R}$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & Ke^{-\alpha(t-s)} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)[I-P(s)]|| & \leq & Ke^{-\alpha(s-t)} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

- The projector splits any non trivial solution $t \mapsto x(t,t_0,x_0) = \Phi(t,t_0)x_0$ as:

$$\begin{aligned} x(t,t_0,x_0) &= \Phi(t,t_0)x_0 \\ &= \Phi(t,t_0)[P(t_0) + I - P(t_0)]x_0 \\ &= \underbrace{\Phi(t,t_0)P(t_0)x_0}_{:=x^+(t,t_0,x_0)} + \underbrace{\Phi(t,t_0)[I-P(t_0)]x_0}_{:=x^-(t,t_0,x_0)}. \end{aligned}$$

Exponential Dichotomy

Formal definition

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$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & Ke^{-\alpha(t-s)} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)[I-P(s)]|| & \leq & Ke^{-\alpha(s-t)} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

- The term $x^+(t,t_0,x_0) = \Phi(t,t_0)P(t_0)$ is an *exponential contraction*:

$$|x^+(t,t_0,x_0)| \leq Ke^{-\alpha(t-t_0)}|x_0| \quad \text{for any } t \geq t_0.$$

- The term $x^-(t,t_0,x_0) = \Phi(t,t_0)[I-P(t_0)]$ is an *exponential expansion*:

$$|[I-P(t_0)]x_0|K^{-1}e^{\alpha(t-t_0)} \leq |x^-(t,t_0,x_0)| \quad \text{for any } t \geq t_0.$$

Exponential Dichotomy

Formal definition

Definition

The linear system

$$x' = A(t)x$$

has an exponential dichotomy on $\mathcal{I} \subseteq \mathbb{R}$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & Ke^{-\alpha(t-s)} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| & \leq & Ke^{-\alpha(s-t)} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}, \end{cases}$$

where $Q(t) = [I - P(t)]$.

- Any non–zero initial condition $P(t_0)x_0$ leads to an *exponential contraction*.
- Any non–zero initial condition $Q(t_0)x_0$ leads to an *exponential expansion*.
- The asymptotic behavior is *dichotomous*.
- Contractions/expansions are dominated by an exponential function.
- These features are behind the name *exponential dichotomy*.

Let us consider the linear system:

$$\dot{x} = A(t)x \tag{1}$$

Remark

It is well known that:

- *If $A(t) \equiv A$ is constant, then (1) has an exponential dichotomy on $\mathbb{R}$[a] if and only if*

$$\operatorname{Re}\lambda \neq 0 \quad \forall \lambda \in \sigma(A).$$

- *If $A(t+\omega) = A(t)$ for any $t \in \mathbb{R}$, then (1) has an exponential dichotomy on $\mathbb{R}$[b] if and only if*

$$|\lambda| \neq 1 \quad \forall \lambda \in \sigma(\Phi(\omega)).$$

[a] By this reason, exponential dichotomy is also called as Nonautonomous Hyperbolicity.
[b] The matrix $\Phi(\omega)$ is known as the Monodromy matrix.

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Exponential Dichotomy

Examples

Let us consider the linear system:

$$\dot{x} = A(t)x$$

Remark

Another well known results are:

- *Extension property: If (1) has an Exponential Dichotomy on $[T, +\infty)$ with $T > 0$ then it has an Exponential Dichotomy on $[0, +\infty)$.*
- *Roughness property: if (1) has an Exponential Dichotomy on J and $||B|| < \delta << 1$, then*

$$\dot{x} = [A(t) + B(t)]x$$

  also has an Exponential Dichotomy on J.
- *The Exponential Dichotomy on $J \subseteq \mathbb{R}$ is preserved by kinematical similarity.*

Exponential Dichotomy

Remarks

Definition

The linear system

$$x' = A(t)x$$

has an exponential dichotomy on $\mathcal{I} \subseteq \mathbb{R}$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| \leq Ke^{-\alpha(t-s)} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| \leq Ke^{-\alpha(s-t)} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}, \end{cases}$$

where $Q(t) = [I - P(t)]$.

- The exponential dichotomy *is always related to an interval* $\mathcal{I}$.
- ED can be simultaneously verifies on $\mathbb{R}_0^- = (-\infty, 0]$ and $\mathbb{R}_0^+ := [0, +\infty)$, but this not always implies ED on $\mathbb{R}$.
- If a system has an ED on $\mathbb{R}$, the unique bounded solution is the trivial one.

Exponential Dichotomy

Applications

- Exponential Dichotomy is used in Theoretical Ecology problems

P.E. Kloeden and C. Pötzsche. *Nonautonomous Dynamical Systems in the Life Sciences*, Lecture Notes in Mathematics 2102, Springer, 2013.

- Exponential Dichotomy is used in Control Theory

P. Anh, A. Czornik, T.S. Doan and S. Siegmund, Proportional local assignability of dichotomy spectrum of one-sided continuous time-varying linear systems, J. Differential Equations 309 (2022), 176–195.

P. Anh, A. Czornik, T.S. Doan and N.T.T. Suong, Assignment of spectrum for time-varying linear control systems via kinematic equivalence, Systems Control Lett. 186 (2024), Paper No. 105763, 6 pp.

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2 Equivalences with exponential dichotomy

Equivalences with exponential dichotomy
Context

- In general, there are no algorithm to detect the exponential dichotomy.
- A fruitful approach is to determine qualitative properties which are equivalent to the ED.
- We will focus our study in two ones:
  a) Exponential Expansiveness,
  b) Nonuniform criticality.
- It will necessary to study the bounded growth properties.

Equivalences with exponential dichotomy

Definition

The system (1) has a *bounded growth* on $J \subset \mathbb{R}$ if for each $T > 0$ there exists $C_T \geq 1$ such that any solution $t \mapsto x(t)$ verifies

$$|x(t)| \leq C_T |x(s)| \quad \text{for any } t \in [s, s+T] \cap J.$$

Definition

The system (1) has a *bounded decay* on $J \subset \mathbb{R}$ if for each $T > 0$ there exists $C_T \geq 1$ such that any solution $t \mapsto x(t)$ verifies

$$|x(t)| \leq C_T |x(s)| \quad \text{for any } t \in [s-T, s] \cap J.$$

Definition

The system (1) has a *bounded growth & decay* on $J \subset \mathbb{R}$ if for each $T > 0$ there exists $C_T \geq 1$ such that any solution $t \mapsto x(t)$ verifies

$$|x(t)| \leq C_T |x(s)| \quad \text{for any } t \in [s-T, s+T] \cap J.$$

Lemma (Coppel (1978))

The system (1) has a bounded growth on J if and only if there exists $M>0$ and $\beta\geq 0$ such that

$$||\Phi(t,s)||\leq Me^{\beta(t-s)} \quad \text{for any } t\geq s \text{ with } t,s\in J.$$

$\Leftarrow$) Let $t\mapsto x(t)$ be a solution of (1):

- Let $T>0$ such that $0<t-s<T$ then $t\in[s,s+T]$
- Note that

$$|x(t)|=|\Phi(t,s)x(s)|\leq||\Phi(t,s)||\,|x(s)|\leq Me^{\beta(t-s)}|x(s)|\leq Me^{\beta T}|x(s)|$$

- Then for any $T>0$ there exists $C_T=\max\{1,Me^{\beta T}\}$ such that

$$|x(t)|\leq C_T|x(s)| \quad \text{for any } t\in[s,s+T],$$

and the bounded growth is verified.

Lemma (Coppel (1978))

The system (1) has a bounded growth on J if and only if there exists $M>0$ and $\beta\geq 0$ such that

$$||\Phi(t,s)||\leq Me^{\beta(t-s)} \quad \text{for any } t\geq s \text{ with } t,s\in J.$$

($\Rightarrow$ Let $t\mapsto x(t)$ be a solution of (1):

- Assume that the bounded growth is verified with $T>0$ and $C_T\geq 1$.
- We make a partition of J in intervals of large T such that

$$t\in[s+(n-1)T,s+nT]\cap J \quad \text{for some } n\in\mathbb{N}$$

- The bounded growth helps to deduce that $|x(t)|\leq C_T^n|x(s)|$.
- We can deduce that that $n-1\leq \frac{t-s}{T}\leq n$ and

$$|x(t)|=|\Phi(t,s)x(s)|=C_T^n|x(s)|\leq C_T\,e^{\frac{\ln(C_T)}{T}(t-s)}|x(s)|.$$

- $||\Phi(t,s)||\leq Me^{\beta(t-s)}$ is verified with $M=C_T$ and $\beta=\frac{\ln(C_T)}{T}$.

Equivalences with exponential dichotomy

About the partitions

Lemma

The system (1) has a:

- *Bounded growth on J if and only if there exists $M > 0$ and $\beta \geq 0$ such that*

$$||\Phi(t,s)|| \leq Me^{\beta(t-s)} \quad \text{for any } t \geq s \text{ with } t,s \in J.$$

- *Bounded decay on J if and only if there exists $M > 0$ and $\beta \geq 0$ such that*

$$||\Phi(t,s)|| \leq Me^{\beta(s-t)} \quad \text{for any } s \geq t \text{ with } t,s \in J.$$

- *Bounded growth & decay on J if and only if there exists $M > 0$ and $\beta \geq 0$ such that*

$$||\Phi(t,s)|| \leq Me^{\beta|t-s|} \quad \text{for any } t,s \text{ with } t,s \in J.$$

- These equivalences provide an easy way to detect the bounded growth properties.
- The equivalences were obtained by a tricky use of uniform partitions.

Equivalences with exponential dichotomy

Other qualitative properties related with the exponential dichotomy

Definition (K.J. Palmer, 2006)

The system (1) is *exponentially expansive* on J if for any solution $t \mapsto x(t)$ of (1) and compact interval $[a,b] \subset J$ there exists $L > 0$ and $\beta > 0$ such that:

$$|x(t)| \leq L\left\{e^{-\beta(t-a)}|x(a)| + e^{-\beta(b-t)}|x(b)|\right\} \quad \text{for any } t \in [a,b].$$

Definition (Krasovskii, 1956)

The system (1) is *uniformly non critical* on J if there exists $T > 0$ and $\theta \in (0,1)$ such that any solution $t \mapsto x(t)$ satisfies

$$|x(t)| \leq \theta \sup_{|u-t|\leq T} |x(u)| \quad \text{for all } t \text{ such that } [t-T, t+T] \subset J.$$

- K.J. Palmer: Exponential dichotomy and expansivity, *Annali di Matematica* 185 (2006) S171–S185.
- N.N. Krasovski: On the theory of the second method of A.M. Lyapunov for the investigation of stability, *Mat. Sb. (N.S.)* 82 (1956) 57–64.

Equivalences with exponential dichotomy

Some well known implications

Theorem (Coppel, 1978)

If $J = [0, +\infty)$ and (1) has a bounded growth on J then

$$\text{Uniform Non Criticality} \Rightarrow \text{Exponential Dichotomy}.$$

Theorem (Palmer, 2006)

If $J = [0, +\infty)$ then

$$\text{Exponential Dichotomy} \Rightarrow \text{Exponential Expansiveness} \Rightarrow \text{Uniform Non Criticality}.$$

- W.A. Coppel: Dichotomies in Stability Theory, *Lecture Notes in Mathematics* 629, Springer–Verlag, Berlin (1978).
- K.J. Palmer: Exponential dichotomy and expansivity, *Annali di Matematica* 185 (2006) S171–S185.

Equivalences with exponential dichotomy

Some well known equivalences

Corollary (Palmer,2006)

If (1) has a bounded growth on $J = [0, +\infty)$ then the following properties are equivalent on J:

a) *Exponential Dichotomy on $[0, +\infty)$,*
b) *Exponential Expansiveness on $[0, +\infty)$,*
c) *Non critical Uniformity on $[0, +\infty)$.*

Corollary (Palmer,2006)

If (1) has a bounded decay on $J = (-\infty, 0]$ then the following properties are equivalent on J:

a) *Exponential Dichotomy on $(-\infty, 0]$,*
b) *Exponential Expansiveness on $(-\infty, 0]$,*
c) *Non critical Uniformity on $(-\infty, 0]$.*

The generalization of these equivalences for more general dichotomies encompassing the exponential one has remained elusive. What type of more general dichotomies?.

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3 A generalization: the h–dichotomies

The h–dichotomies

Preliminaires

Definition

Let $h\colon J \to (0,+\infty)$ be an increasing homeomorphism where $J := (a,+\infty)$. The linear system

$$x' = A(t)x$$

has an h–dichotomy on $\mathcal{I} \subseteq J$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| \leq K\left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| \leq K\left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

- R.H. Martin Jr., Conditional stability and separation of solutions to differential equations. J. Differential Equations 13 (1973) 81–105
- J.S. Muldowney, Dichotomies and asymptotic behaviour for linear differential systems. Trans. Amer. Math. Soc. 283 (1984) 465–484.

The h–dichotomies

Preliminaires

Definition

Let $h\colon J \to (0,+\infty)$ be an increasing homeomorphism where $J := (a,+\infty)$. The linear system

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has an h–dichotomy on $\mathcal{I} \subseteq J$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & K\left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| & \leq & K\left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

- Any solution $t \mapsto x(t)$ can be decomposed in an h–contraction and an h–expansion.
- Our goal is to generalize the previous results about equivalences between dichotomy, expansiveness and uniform non criticality.
- The generalization is no trivial and is based in the properties of h.

The h–dichotomies

A baby example

- Let $h\colon J \to (0,+\infty)$ and increasing diffeomorphism.
- If $\alpha > 0$, we will consider:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -\alpha \dfrac{h'(t)}{h(t)} & 0 \\ 0 & \alpha \dfrac{h'(t)}{h(t)} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{for any } t \in J, \tag{2}$$

- Note that

$$\Phi(t) = \begin{bmatrix} \frac{1}{h(t)^\alpha} & 0 \\ 0 & h(t)^\alpha \end{bmatrix} \quad \text{and} \quad P(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for any } t \in J. \tag{3}$$

- We can see that

$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & \left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s,\ t,s \in J, \\ ||\Phi(t,s)Q(s)|| & \leq & \left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t,\ t,s \in J. \end{cases}$$

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The h–dichotomies

A baby example of *polynomial dichotomy*

- Let $h(t) = t$ for any $t \in J = (0, +\infty)$.
- For any $\alpha > 0$, we will consider:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -\alpha\dfrac{1}{t} & 0 \\ 0 & \alpha\dfrac{1}{t} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{for any } t > 0. \tag{4}$$

- Note that

$$\Phi(t) = \begin{bmatrix} \frac{1}{t^{\alpha}} & 0 \\ 0 & t^{\alpha} \end{bmatrix} \quad \text{and} \quad P(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for any } t > 0. \tag{5}$$

- We can see that

$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & \left(\dfrac{t}{s}\right)^{-\alpha} & \text{for any } t \geq s > 0, \\ ||\Phi(t,s)Q(s)|| & \leq & \left(\dfrac{s}{t}\right)^{-\alpha} & \text{for any } s \geq t > 0. \end{cases}$$

The h–dichotomies

Preliminaires: a group associated to h

Definition

Let $h\colon J \to (0,+\infty)$ be an increasing homeomorphism, where $J := (a,+\infty)$. The linear system

$$x' = A(t)x$$

has an h–dichotomy on $\mathcal{I} \subseteq J$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| & \leq & K\left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| & \leq & K\left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

- As h is an homeomorphism, we define a composition law $h\colon J \times J \to J$ as
$$t \star s = h^{-1}(h(t)h(s)).$$
- The pair $(J,\star)$ is an abelian topologic group.
- The unit element is $e_\star = h^{-1}(1)$ since
$$t \star e_\star = t \iff h^{-1}(h(t)h(e_\star)) = t \iff h(t)h(e_\star) = h(t) \iff h(e_\star) = 1.$$

The h–dichotomies

Preliminaries

Definition

Let $h\colon J \to (0,+\infty)$ be an increasing homeomorphism where $J := (a,+\infty)$. The linear system

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has an h–dichotomy on $\mathcal{I} \subseteq J$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| \leq K\left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| \leq K\left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

- As h is an homeomorphism, we define a composition law $h\colon J \times J \to J$ as
$$t \star s = h^{-1}(h(t)h(s)).$$
- For any $t \in J$ its inverse is $t^{\star -1} := h^{-1}(1/h(t))$ since
$$t \star t^{\star -1} = e_{\star} \iff h^{-1}(h(t)h(t^{\star -1})) = e_{\star} \iff h(t)h(t^{\star -1}) = 1.$$

The h–dichotomies

About the group

- This approach was conceived by M. Pinto (2022–2023)
- A good formalization was carried out by its students J.F. Peña and S. Rivera-Villagrán and has propelled a renewed interest in $h-$dichotomies:

$h(t)$	J	$t \star s$	$e_\star$	J_-
e^t	$\mathbb{R}$	$t+s$	0	$(-\infty,0]$
t	$(0,+\infty)$	ts	1	(0,1]
$\ln(1+t)$	$(0,+\infty)$	$(1+t)^{\ln(1+s)}-1$	$e-1$	$(0,e-1]$
$t+\sqrt{t^2+1}$	$\mathbb{R}$	$t\sqrt{s^2+1}+s\sqrt{t^2+1}$	0	$(-\infty,0]$

Cuadro: The different growth rates and its respective functions.

J.F. Peña and S. Rivera–Villagrán: On uniform asymptotic h–stability for linear non autonomous systems, *Electronic Journal of Qualitative Theory of Differential Equations* (2025) Article 19, 13 pp.

h–dichotomies

The group $(J,\star)$ provides a new perspective for studying the h–dichotomy

- Let us recall that $(J,\star)$ is an abelian group with
$$t\star s = h^{-1}(h(t)h(s)).$$
- Let us recall that he inverse of s is $s^{\star -1} := h^{-1}(1/h(s))$.
- Note that
$$t\star s^{\star-1} = h^{-1}(h(t)h(s^{\star-1})) = h^{-1}\left(\frac{h(t)}{h(s)}\right)$$

Definition

Let $h\colon J\to(0,+\infty)$ be an increasing homeomorphism where $J:=(a,+\infty)$. The linear system $x'=A(t)x$ has an h–dichotomy on $\mathcal{I}\subseteq J$ if there exists $P(\cdot)^2=P(\cdot)$, $K\geq 1$ and $\alpha>0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| \leq K\left(\frac{h(t)}{h(s)}\right)^{-\alpha} = h(t\star s^{\star-1})^{-\alpha} & \text{for any } t\geq s \text{ with } t,s\in\mathcal{I}, \\ ||\Phi(t,s)Q(s)|| \leq K\left(\frac{h(s)}{h(t)}\right)^{-\alpha} = h(s\star t^{\star-1})^{-\alpha} & \text{for any } s\geq t \text{ with } t,s\in\mathcal{I}. \end{cases}$$

h–dichotomies

The group is also orderable

- $(J,\star)$ is a totally ordered group with the order $t \leq_\star s$
$$t \leq_\star s \iff e_\star \leq s \star t^{\star -1}.$$
- We can prove that $t \leq_\star s \iff t \leq s$.
- If $e_\star = h^{-1}(1)$, we can construct an absolute value over J as follows:
$$|\cdot|_\star : J \to [e_\star, +\infty[\quad \text{where} \quad |t|_\star := \begin{cases} t & \text{if } t \geq e_\star, \\ t^{\star -1} & \text{if } t < e_\star. \end{cases}$$
- We can endow the group $(J,\star)$ with a metric $\mathfrak{d} : J \times J \to [e_\star, +\infty[$ by
$$\mathfrak{d}(t,s) := |t \star s^{\star -1}|_\star = \begin{cases} t \star s^{\star -1} & \text{if } t \star s^{\star -1} \geq e_\star \ (\text{or } t \geq s) \\ s \star t^{\star -1} & \text{if } t \star s^{\star -1} < e_\star \ (\text{or } t \leq s) \end{cases} \tag{6}$$

H. Elorreaga, J.F. Peña and G. Robledo: Uniform h-dichotomies: noncritical uniformity and expansivity, *Mathematische Annalen* 393 (2025) 1769–1795.

h–dichotomies

The orderability of $(J,\star)$ provides another perspective for studying the h–dichotomy

- If $e_\star = h^{-1}(1)$, we can construct an absolute value over J as follows:

$$|\cdot|_\star : J \to [e_\star, +\infty[\quad \text{where} \quad |t|_\star := \begin{cases} t & \text{if } t \geq e_\star, \\ t^{\star -1} & \text{if } t < e_\star. \end{cases}$$

- We can endow the group $(J,\star)$ with a metric $\mathfrak{d} : J \times J \to [e_\star, +\infty[$ by

$$\mathfrak{d}(t,s) := |t \star s^{\star -1}|_\star = \begin{cases} t \star s^{\star -1} & \text{if } t \star s^{\star -1} \geq e_\star \\ s \star t^{\star -1} & \text{if } t \star s^{\star -1} < e_\star \end{cases}$$

Definition

Let $h\colon J \to (0,+\infty)$ be an increasing homeomorphism where $J := (a,+\infty)$. The linear system $x' = A(t)x$ has an h–dichotomy on $\mathcal{I} \subseteq J$ if there exists $P(\cdot)^2 = P(\cdot)$, $K \geq 1$ and $\alpha > 0$ such that

$$\begin{cases} ||\Phi(t,s)P(s)|| \leq K\left(\frac{h(t)}{h(s)}\right)^{-\alpha} = h(t \star s^{\star -1})^{-\alpha} = h(\mathfrak{d}(t,s))^{-\alpha} & \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}, \\ ||\Phi(t,s)Q(s)|| \leq K\left(\frac{h(s)}{h(t)}\right)^{-\alpha} = h(s \star t^{\star -1})^{-\alpha} = h(\mathfrak{d}(t,s))^{-\alpha} & \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}. \end{cases}$$

h–dichotomies

More details

- Any abelian group is a $\mathbb{Z}$–module by considering the external composition law

$$\begin{array}{rcl} \mathbb{Z} \times J & \to & J \\ (k,t) & \mapsto & t^{\star k} := \begin{cases} \underbrace{t \star \cdots \star t}_{k-\text{times}} & \text{if} \quad k > 0 \\ e_{\star} & \text{if} \quad k = 0 \\ \underbrace{t^{\star -1} \star \cdots \star t^{\star -1}}_{k-\text{times}} & \text{if} \quad k < 0. \end{cases} \end{array}$$

- By using the fact that $h(\cdot)$ is an homeomorphism we can see that

$$t^{\star k} = h^{-1}(h(t^{\star k})) = h^{-1}(h(t)^k) \quad \text{for any } k \in \mathbb{Z},$$

- The above external composition law can be revisited as

$$t^{\star k} = h^{-1}\left(h(t)^k\right). \tag{7}$$

- We have the useful identity: $h(t^{\star k}) = h(t)^k$.

Equivalences with h–dichotomy

The bounded h–growth properties

Definition

The system (1) has a ***bounded h–growth*** on $\mathcal{I} \subset J$ if for each $T > e_\star$ there exists $C_T \geq 1$ such that any solution $t \mapsto x(t)$ verifies

$$|x(t)| \leq C_T |x(s)| \quad \text{for any } t \in [s, s \star T] \cap \mathcal{I}.$$

Definition

The system (1) has a ***bounded h–decay*** on $\mathcal{I} \subset J$ if for each $T > e_\star$ there exists $C_T \geq 1$ such that any solution $t \mapsto x(t)$ verifies

$$|x(t)| \leq C_T |x(s)| \quad \text{for any } t \in [s \star T^{\star -1}, s] \cap J.$$

Definition

The system (1) has a ***bounded h–growth & h–decay*** on $\mathcal{I} \subset J$ if for each $T > e_\star$ there exists $C_T \geq 1$ such that any solution $t \mapsto x(t)$ verifies

$$|x(t)| \leq C_T |x(s)| \quad \text{for any } t \in [s \star T^{\star -1}, s \star T] \cap J.$$

Equivalences with h–dichotomy

About the partitions

Lemma (Elorreaga, R., Peña – 2025)

The system (1) has a:

- *Bounded growth on $\mathcal{I} \subset J$ if and only if there exists $M > 0$ and $\beta \geq 0$ such that*

$$||\Phi(t,s)|| \leq M \left(\frac{h(t)}{h(s)}\right)^{\beta} \quad \text{for any } t \geq s \text{ with } t,s \in \mathcal{I}.$$

- *Bounded h–decay on $\mathcal{I}$ if and only if there exists $M > 0$ and $\beta \geq 0$ such that*

$$||\Phi(t,s)|| \leq M \left(\frac{h(s)}{h(t)}\right) e^{\beta} \quad \text{for any } s \geq t \text{ with } t,s \in \mathcal{I}.$$

- *Bounded h–growth & h–decay on $\mathcal{I}$ if and only if there exists $M > 0$ and $\beta \geq 0$ such that*

$$||\Phi(t,s)|| \leq M h(\mathfrak{d}(t,s))^{\beta} \quad \text{for any } t,s \text{ with } t,s \in J.$$

- The equivalences were obtained by a tricky use of uniform partitions.
- But was is a uniform partition in this context?

Equivalences with h–dichotomy

Notation

It always difficult to get adapted to a new notation :(

Definition

Given $(J, \star)$ with $J := (a, +\infty)$:

- The interval $J_- := (a, e_\star]$ will be denoted as the negative half–line.
- The interval $J_+ := [e_\star, +\infty)$ will be denoted as the positive half–line.
- The interval $J := (a, +\infty) = J_- \cup J_+$ will be denoted as the full line.

We will be focused on h–dichotomies on J, J_- and J_+.

Equivalences with h–dichotomy

An example of uniform partition

- Let us consider $h(t) = t$, $J = (0, +\infty)$ and $T = 2 > e_\star = 1$.
- We define the intervals $J_n = [2^{\star n}, 2^{\star(n+1)})$ for any $n \in \mathbb{Z}$.
- That is

$$J_+ := [e_\star, +\infty) = [1, 2) \cup [2, 2^2) \cup \cdots \cup [2^k, 2^{k+1}) \cup \cdots$$

- In addition

$$J_- = (0, 1] = \cdots \cup \left[\frac{1}{2^3}, \frac{1}{2^2}\right) \cup \left[\frac{1}{2^2}, \frac{1}{2}\right) \cup \left[\frac{1}{2}, 1\right)$$

- All the intervals have large $T = 2$ by considering the distance $\mathfrak{d}$ induced by $(J, \star)$
- This is counterintuitive with the classical additive perspective where the length of these intervals contracts in $(0, 1)$ and expands in $(1, +\infty)$.
- We think that this fact illustrates the difficulty of working on this problem, but at the same time it shows its surprising and intricate nature.

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4 Main results

Main results

New properties in an h–context

Definition

The system (1) is *h–expansive* on $\mathcal{I} \subset J$ if for any solution $t \mapsto x(t)$ of (1) and compact interval $[a,b] \subset \mathcal{I}$ there exists $L > 0$ and $\beta > 0$ such that:

$$|x(t)| \leq L\left\{h(\mathfrak{d}(t,a))^{-\beta}|x(a)| + h(\mathfrak{d}(b,t))^{-\beta}|x(b)|\right\} \quad \text{for any } t \in [a,b].$$

Definition

The system (1) is *uniformly h–non critical* on $\mathcal{I}$ if if there exists $T > e_*$ and $\theta \in (0,1)$ such that any solution $t \mapsto x(t)$ satisfies

$$\begin{aligned} |x(t)| &\leq \theta \sup_{|u \star t^{\star-1}|_\star \leq T} |x(u)| \\ &\leq \theta \sup\left\{|x(u)| \colon \frac{1}{h(T)} < \frac{h(u)}{h(t)} \leq h(T)\right\} \quad \text{for all } t \text{ such that } [t \star T^{\star-1}, t \star T] \subset \mathcal{I}. \end{aligned}$$

H. Elorreaga, J.F. Peña and G. Robledo: Uniform h-dichotomies: noncritical uniformity and expansivity, *Mathematische Annalen* 393 (2025) 1769–1795.

Main results

Some equivalences on the half lines $J_- := (a, e_\star]$ and $J_+ := [e_\star, +\infty)$.

Theorem (Elorreaga, Peña, R. – 2025)

If the linear system (1) *has a* **bounded h–growth** *on* $[e_\star, +\infty)$. *Then the following three statements are equivalent:*

(i) *The system* (1) *has a h–dichotomy on* $[e_\star, +\infty)$.

(ii) *The system* (1) *is h–expansive on* $[e_\star, +\infty)$.

(iii) *The system* (1) *is uniformly h–noncritical on* $[e_\star, +\infty)$.

Moreover, without the assumption of bounded h–growth, it is still true that: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

Corollary

If the linear system (1) *has a* **bounded h–decay** *on* $(a, e_\star]$. *Then the following statements are equivalent:*

(i) *The system* (1) *has a uniform h–dichotomy on* $(a, e_\star]$.

(ii) *The system* (1) *is h–expansive on* $(a, e_\star]$.

(iii) *The system* (1) *is uniformly h–noncritical on* $(a, e_\star]$.

Moreover, without the assumption of bounded h–decay, it is still true that: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

Main results

Some equivalences on $J = (a, +\infty)$

Theorem (Elorreaga, R., Urrutia – 2026)

If (1) *has a* **bounded h–growth** *on* $J^+ := [e_\star, +\infty)$ *and a* **bounded h–decay** *on* $J^- := (a, e_\star]$. *Then the following three statements are equivalent:*

(i) *The system* (1) *has uniform* h*–dichotomies on* $[e_\star, +\infty)$ *and* $(a, e_\star]$[a]. *Moreover, the unique globally bounded solution is the trivial.*

(ii) *The system* (1) *is* h*–expansive on* J.

(iii) *The system* (1) *is uniformly* h*–noncritical on* J.

Moreover, without the assumption of bounded h*–growth and* h*–decay, it is still true that:* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

[a] Note that this not implies h–dichotomy on J.

H. Elorreaga, G. Robledo, D. Urrutia. Conditions for uniform –dichotomy in terms of uniform non criticality, expansiveness and via generalized Floquet theory. ArXiv preprint arXiv:2603.27425

- Can we obtain necessary and sufficient conditions for h–dichotomy on J?

Main results

Necessary and sufficient conditions for h–dichotomy on the full line J.

Theorem (Elorreaga, R., Urrutia – 2026)

The system (1) *has a uniform h–dichotomy on J if and only if:*

a) *The system has h–dichotomies on $J^- := (a, e_\star]$ and $J^+ := [e_\star, +\infty)$ with projections $P^-(\cdot)$ and $P^+(\cdot)$ respectively,*

b) *The system has no nontrivial bounded solutions,*

c) *The index of the linear system is zero.*

$$i(A) = \dim \mathcal{R}P^+(\cdot) + \dim \mathcal{N}P^-(\cdot) - n = 0.$$

Main results

A generalization of Floquet Theory

Definition (Burton & Muldowney – 1968)

The linear system (1) is a Generalized Floquet System (GFS) with respect to f if f is an absolutely continuous function on $J := (a, +\infty)$ such that

$$f'(t)A(f(t)) = A(t) \tag{8}$$

for almost all $t > a$ and $f(t) > t$ for all $t > a$.

T.A. Burton, J.S. Muldowney. A generalized Floquet Theory, *Arch. Math.* 19 (1968) 188–194.

- When $f(t) = t + \omega$ and $A(t+\omega) = A(t)$ we recover the classical ω–periodicity.
- We will consider the particular case $f(t) = t \star T$ with $T > e_\star$.
- We will assume that $A(\cdot)$ fulfills the following property:

$$(t \star T)' A(t \star T) = \frac{h'(t)h(T)}{h'(t \star T)} A(t \star T) = A(t) \quad \text{for any } t \in J. \tag{9}$$

If we assume that (1) is a GFS with $f(t) = t \star T$ we have that:

- The biperiodicity property $\Phi(t \star T, s \star T) = \Phi(t, s)$ for any $t, s \in J$.
- The property $\Phi(t \star T^{\star n}) = \Phi(t)V^n$ for any $n \in \mathbb{N}$.
- If $\Phi(e_\star) = I$, then we have $\Phi(T) = V$, which is called the monodromy matrix.

Theorem

If the linear system (1) is a generalized Floquet system, the following properties are equivalent:

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*

ii) *The linear system (1) has a uniform h–dichotomy on J.*

Main Results

Sketch of proof ($\Rightarrow$

Theorem

If the linear system (1) *is a generalized Floquet system, the following properties are equivalent:*

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*
ii) *The linear system* (1) *has a uniform h–dichotomy on J.*

- Let us consider the difference equation $x_{n+1} = Vx_n$.
- As the eigenvalues of V are not in the unit circle, the above system has an exponential dichotomy on $\mathbb{Z}$: there exists $P^2 = P$ and $K_0 > 0$, $\alpha > 0$ such that:

$$\begin{cases} ||V^n P[V^k]^{-1}|| \leq K_0 e^{-\alpha(n-k)} & \text{for any } n \geq k \\ ||V^n (I-P)[V^k]^{-1}|| \leq K_0 e^{-\alpha(k-n)} & \text{for any } k \geq n. \end{cases} \tag{10}$$

- We construct a uniform partition for J:

$$J = \bigcup J_n \quad \text{where } J_n = [T^{*(n-1)}, T^{*n}).$$

- If $t > s$, there exists n, k with $n \geq k$: $s \in [T^{*(k-1)}, T^{*k})$ and $t \in [T^{*(n-1)}, T^{*n})$.

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Main Results

Sketch of proof ($\Rightarrow$)

Theorem

If the linear system (1) *is a generalized Floquet system, the following properties are equivalent:*

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*

ii) *The linear system* (1) *has a uniform h–dichotomy on J.*

- As $h(\cdot)$ is strictly increasing, we have

$$s \in [T^{*(k-1)}, T^{*k}) \iff h(T^{*(k-1)}) \leq h(s) < h(T^{*k}) \iff h(T)^{k-1} \leq h(s) < h(T)^{k},$$

- Then we can deduce that

$$k-1 \leq \frac{\ln(h(s))}{\ln(h(T))} < k \quad \text{and} \quad n-1 \leq \frac{\ln(h(t))}{\ln(h(T))} < n, \tag{11}$$

- As $\alpha > 0$, the above estimations leads to

$$\alpha(k-n) < \alpha + \frac{\alpha}{\ln(h(T))} \ln\left(\frac{h(s)}{h(t)}\right) = \alpha + \ln\left(\frac{h(t)}{h(s)}\right)^{-\frac{\alpha}{\ln(h(T))}}. \tag{12}$$

Main Results

Sketch of proof ($\Rightarrow$

Theorem

If the linear system (1) *is a generalized Floquet system, the following properties are equivalent:*

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*

ii) *The linear system* (1) *has a uniform h–dichotomy on J.*

- Then we can deduce that

$$s \in [T^{*(k-1)}, T^{*k}) \Rightarrow s = T^{\star(k-1)} \star \tilde{s} \quad \text{for some } \tilde{s} \in [e_\star, T). \tag{13}$$

$$t \in [T^{*(n-1)}, T^{*n}) \Rightarrow t = T^{\star(n-1)} \star \tilde{t} \quad \text{for some } \tilde{t} \in [e_\star, T). \tag{14}$$

- We also can deduce that

$$\Phi(t) = \Phi(\tilde{t} \star T^{\star(n-1)}) = \Phi(\tilde{t})V^{n-1} \quad \text{and} \quad \Phi(s) = \Phi(\tilde{s} \star T^{\star(k-1)}) = \Phi(\tilde{s})V^{k-1},$$

- Let

$$K_1 := \max_{u \in [0, e_\star]} ||\Phi(u)|| \quad \text{and} \quad K_2 := \max_{u \in [0, e_\star]} ||\Phi^{-1}(u)||$$

- Then we have:

Main Results

Sketch of proof ($\Rightarrow$

Theorem

If the linear system (1) *is a generalized Floquet system, the following properties are equivalent:*

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*
ii) *The linear system* (1) *has a uniform h–dichotomy on J.*

- Then we have (when $t \geq s$):

$$\begin{aligned} ||\Phi(t)P\Phi^{-1}(s)|| &= ||\Phi(\tilde{t})V^{n-1}P[V^{k-1}]^{-1}\Phi^{-1}(\tilde{s})|| \\ &\leq ||\Phi(\tilde{t})||\,||V^{n-1}P[V^{k-1}]^{-1}||\,||\Phi^{-1}(\tilde{s})|| \\ &\leq K_1K_2\,||V^{n-1}P[V^{k-1}]^{-1}|| \end{aligned}$$

Since

a) $\Phi(t) = \Phi(T^{\star(n-1)} \star \tilde{t}) = \Phi(\tilde{t})V^{(n-1)}$ and $\Phi^{-1}(s) = V^{(k-1)}\Phi^{-1}(\tilde{s})$,
b) $||\Phi(\tilde{t})|| \leq K_1 := \max\limits_{u\in[0,e_\star]} ||\Phi(u)||$ and $||\Phi^{-1}(\tilde{s})|| \leq K_2 := \max\limits_{u\in[0,e_\star]} ||\Phi^{-1}(u)||$.

Main Results

Sketch of proof ($\Rightarrow$

Theorem

If the linear system (1) *is a generalized Floquet system, the following properties are equivalent:*

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*

ii) *The linear system* (1) *has a uniform h–dichotomy on J.*

- By recalling that

$$\begin{cases} ||V^n P[V^k]^{-1}|| & \leq & K_0 e^{-\alpha(n-k)} & \text{for any } n \geq k \\ ||V^n (I-P)[V^k]^{-1}|| & \leq & K_0 e^{-\alpha(k-n)} & \text{for any } k \geq n. \end{cases}$$

we can conclude that

$$\begin{aligned} ||\Phi(t)P\Phi^{-1}(s)|| &= ||\Phi(\tilde{t})V^{n-1}P[V^{k-1}]^{-1}\Phi^{-1}(\tilde{s})|| \\ &\leq ||\Phi(\tilde{t})||\,||V^{n-1}P[V^{k-1}]^{-1}||\,||\Phi^{-1}(\tilde{s})|| \\ &\leq K_1K_2\,||V^{n-1}P[V^{k-1}]^{-1}|| \\ &\leq K_1K_2K_0e^{-\alpha(n-k)} \end{aligned}$$

Main Results

Sketch of proof ($\Rightarrow$

Theorem

If the linear system (1) *is a* *generalized Floquet system*, *the following properties are equivalent:*

i) *The eigenvalues of the monodromy matrix V are not in the unit circle,*

ii) *The linear system* (1) *has a uniform h–dichotomy on J.*

- We have deduced that

$$||\Phi(t)P\Phi^{-1}(s)|| \quad \leq \quad K_1K_2K_0e^{-\alpha(n-k)}.$$

- By recalling that

$$\alpha(k-n) < \alpha + \frac{\alpha}{\ln(h(T))}\ln\left(\frac{h(s)}{h(t)}\right) = \alpha + \ln\left(\frac{h(t)}{h(s)}\right)^{-\frac{\alpha}{\ln(h(T))}}.$$

- We conclude that (when $t \geq s$)

$$||\Phi(t)P\Phi^{-1}(s)|| \quad \leq \quad K_1K_2K_0e^{\alpha}\left(\frac{h(t)}{h(s)}\right)^{-\frac{\alpha}{\ln(h(T))}}$$

- An estimation for $||\Phi(t)[I-P]\Phi^{-1}(s)||$ can be done similarly $\square$.

Main result

A byproduct

Corollary

If the linear system (1) *is a* GFS *and* $t \mapsto x(t, t_0, x_0)$ *is a solution of* (1) *passing trough* $x_0 \neq 0$ *at* $t = t_0 \in J$*, then*

a) *If all* $\lambda \in \sigma(\Phi(T)) = \sigma(V)$ *are inside the unit circle then the origin is uniformly* h*–stable, that is, there exists* $K > 0$ *and* $\alpha > 0$ *such that*

$$|x(t, t_0, x_0)| \leq K \left(\frac{h(t)}{h(t_0)} \right)^{-\alpha} |x_0| \quad \text{for any } t \geq t_0. \tag{15}$$

b) *If all* $\lambda \in \sigma(\Phi(T)) = \sigma(V)$ *are outside the unit circle then the origin is uniformly* h*–unstable, that is, there exists* $K > 0$ *and* $\alpha > 0$ *such that*

$$|x(t, t_0, x_0)| \geq \left(\frac{h(t)}{h(t_0)} \right)^{\alpha} \frac{|x_0|}{K} \quad \text{for any } t \geq t_0. \tag{16}$$

Main results

Conclusions

- The relation between h–dichotomy and noncritical h–uniformity have been addressed in several previous results. Our group approach manages to give a good emulation of the classical results.
- We improve the Burton $\&$ Muldowney results since its Floquet's theory result didn't allow to prove the existence of a dichotomy.
- Our final Corollary also improves the Burton $\&$ Muldowney results since we provide a specific rate of contraction/expansion.

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Conditions for uniform h–dichotomy in terms of uniform non criticality, expansiveness and via generalized Floquet theory

Thank you!!!