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IMDETA

International Meetings on Differential Equations and Their Applications

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Rich dynamics in a model for suspension bridges

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A joint work with Prof. Maurizio Garrione (Politecnico, Milano)

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With many thanks to Prof. Szymańska-Dębowska and the Organizers, for the invitation

In this talk, a model for suspension bridge-type structures with piers is considered. The model encompasses a coupled dynamics involving longitudinal $u(x,t)$ and torsional $\theta(x,t)$ oscillations. Focusing the dynamics on a single specific Fourier component for both the variables, a coupled system of ODEs is obtained. For this latter system, we discuss the occurrence of a possible rich and complex dynamics, including *infinitely many periodic solutions (harmonic and subharmonic)*, for the longitudinal time-component, when the torsional one is small. This goal is achieved by applying a rigorous analytical approach, based on the theory of *linked twist maps*.

1. Subharmonic solutions

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Let us consider a non-autonomous periodic differential systems in $\mathbb{R}^d$

$$(S1) \qquad \underline{x} = \vec{F}(t, \underline{x}),$$

for $\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ **and** $\vec{F} : \mathbb{R} \times \Omega \to \mathbb{R}^d$ **a sufficiently regular vector field which is** T**-periodic in the** t**-variable.**

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Here $\Omega \subseteq \mathbb{R}^d$ is an open domain.

Definition 1 ***By a subharmonic solution of order $m \geq 2$, to system $(S1)$, we mean a mT-periodic solution of the system which is not kT-periodic for all integers $k \in \{1, \ldots, m-1\}$.***

There are different definitions of the concept of "subharmonic solution".
For instance, in
[**J Mawhin, M. Willem**, ***Critical point theory and Hamiltonian systems***, **Applied Mathematical Sciences, 74, Springer-Verlag, New York, 1989**]
a subharmonic is meant in the sense that the solution is mT-periodic for some $m \geq 2$ and its minimal period is strictly greater than T.

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However, in many cases, the two concepts coincide.

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Let $g : J \to \mathbb{R}$ be a locally Lipschitz continuous function defined on an open interval $J \subseteq \mathbb{R}$ and let $p : \mathbb{R} \to \mathbb{R}$ be a continuous [1] and T-periodic function having T as *minimal period*.

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A typical example is the following.

Let $g : J \to \mathbb{R}$ **be a locally Lipschitz continuous function defined on an open interval** $J \subseteq \mathbb{R}$ **and let** $p : \mathbb{R} \to \mathbb{R}$ **be a continuous** [1] **and** T**-periodic function having** T **as** ***minimal period.***
Then, any m**-th order subharmonic of the scalar equation**

$$(D1) \qquad u'' + g(u) = p(t)$$

has mT **as minimal period.**

[1] we could assume weaker regularity conditions on p; indeed, $p \in L^1(0, T)$ would be sufficient.

Indeed, if $\tilde{u}$ is a mT-periodic solution of equation $(D1)$ (or, equivalently, system $x' = y,\ y' = -g(x) + p(t)$), then $\tilde{u}$ has a minimal period, say τ. Hence $\tilde{u}''(t) + g(\tilde{u}(t)) = p(t)$ is τ-periodic and, being T the minimal period of $p(\cdot)$, we find that $\tau = kT$ for some integer $k \geq 1$. By our hypothesis, $\tilde{u}$ is mT-periodic but not kT-periodic for any integer $1 \leq k \leq m - 1$. Hence $\tau = mT$.

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However, the problem of the minimality of the period could be very complicated.

For instance, for $d = 2$, in
[**V.A Pliss**, ***Nonlocal problems of the theory of oscillations***, **Translated from the Russian by Scripta Technica, Inc. Academic Press, New York-London 1966**]
there is an example (attributed to Erugin, 1956) of a system $(S1)$ with $\vec{F}$ T-periodic in the t-variable, possessing a solution which is τ-periodic with $T/\tau \notin \mathbb{Q}$.

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See also
[**A. Cima, A. Gasull, F. Mañosas**, **Periods of solutions of periodic differential equations**, ***Differential Integral Equations*** **29 (2016), 905–922**]
for a more recent study on this topic.

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- ***Topological degree theory / Fixed point index theory in function spaces***
- ***Fixed points / Periodic points concerning the Poincaré map***

Variational methods / critical point theory

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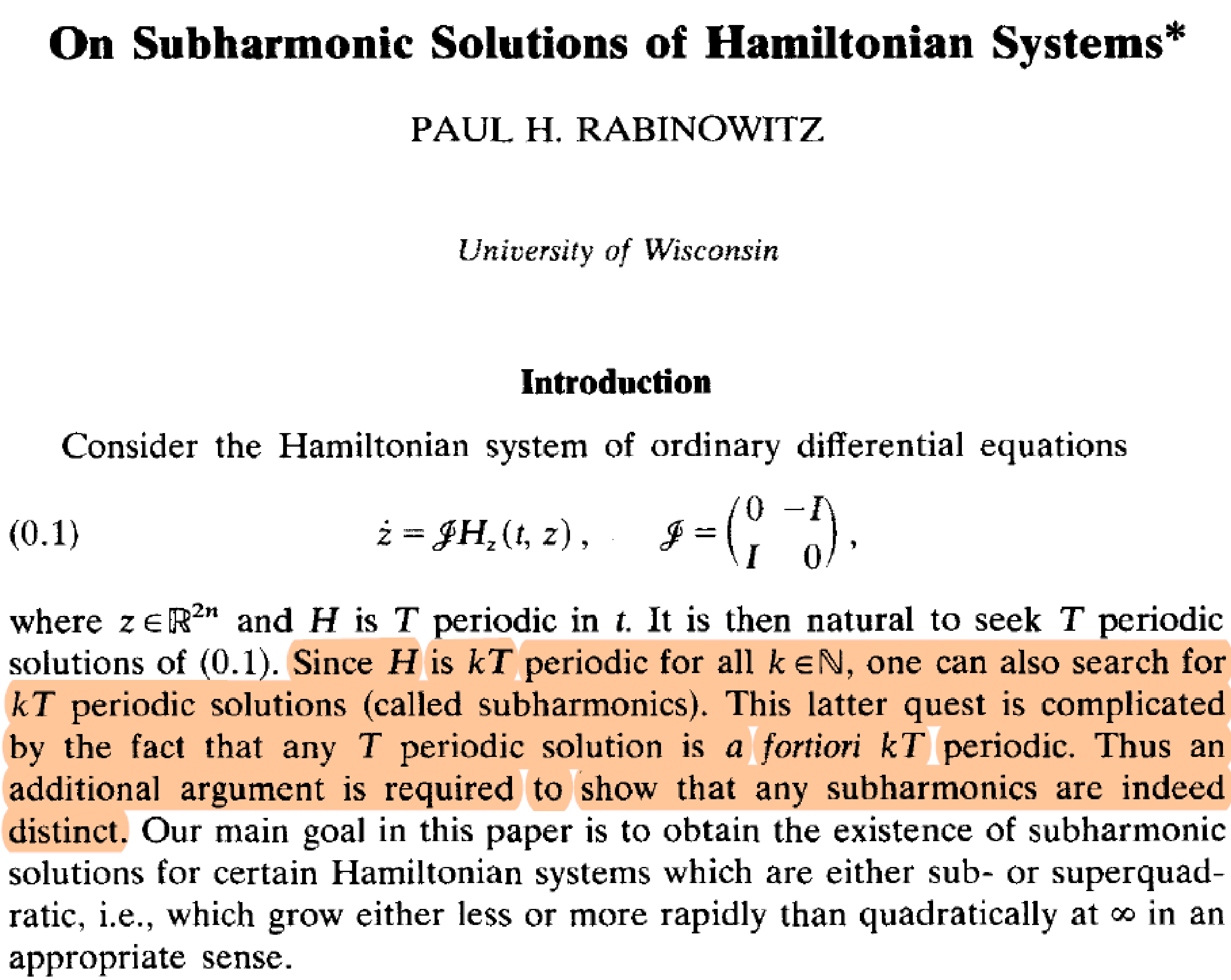
[P.H. Rabinowitz, On subharmonic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* 33 (1980), 609–633]



On Subharmonic Solutions of Hamiltonian Systems*

PAUL H. RABINOWITZ

University of Wisconsin

Introduction

Consider the Hamiltonian system of ordinary differential equations

$$\dot{z} = \mathcal{J} H_z(t, z)\,, \qquad \mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{0.1}$$

where $z \in \mathbb{R}^{2n}$ and H is T periodic in t. It is then natural to seek T periodic solutions of (0.1). Since H is kT periodic for all $k \in \mathbb{N}$, one can also search for kT periodic solutions (called subharmonics). This latter quest is complicated by the fact that any T periodic solution is *a fortiori* kT periodic. Thus an additional argument is required to show that any subharmonics are indeed distinct. Our main goal in this paper is to obtain the existence of subharmonic solutions for certain Hamiltonian systems which are either sub- or superquadratic, i.e., which grow either less or more rapidly than quadratically at ∞ in an appropriate sense.

. . . a huge literature . . .

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[C. Conley, E. Zehnder, Subharmonic solutions and Morse theory, *Phys. A* 124 (1984), 649–657]

[R. Michalek, G. Tarantello, Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems, *J. Differential Equations* 72 (1988), 28–55]

[A. Fonda, M. Ramos, M. Willem, Subharmonic solutions for second order differential equations, *Topol. Methods Nonlinear Anal.* 1 (1993), 49–66]

[E. Serra, M. Tarallo, S. Terracini, Subharmonic solutions to second-order differential equations with periodic nonlinearities, *Nonlinear Anal.* 41 (2000), 649–667]

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Apologies for the missing citations !

Topological degree theory / Fixed point index theory in function spaces

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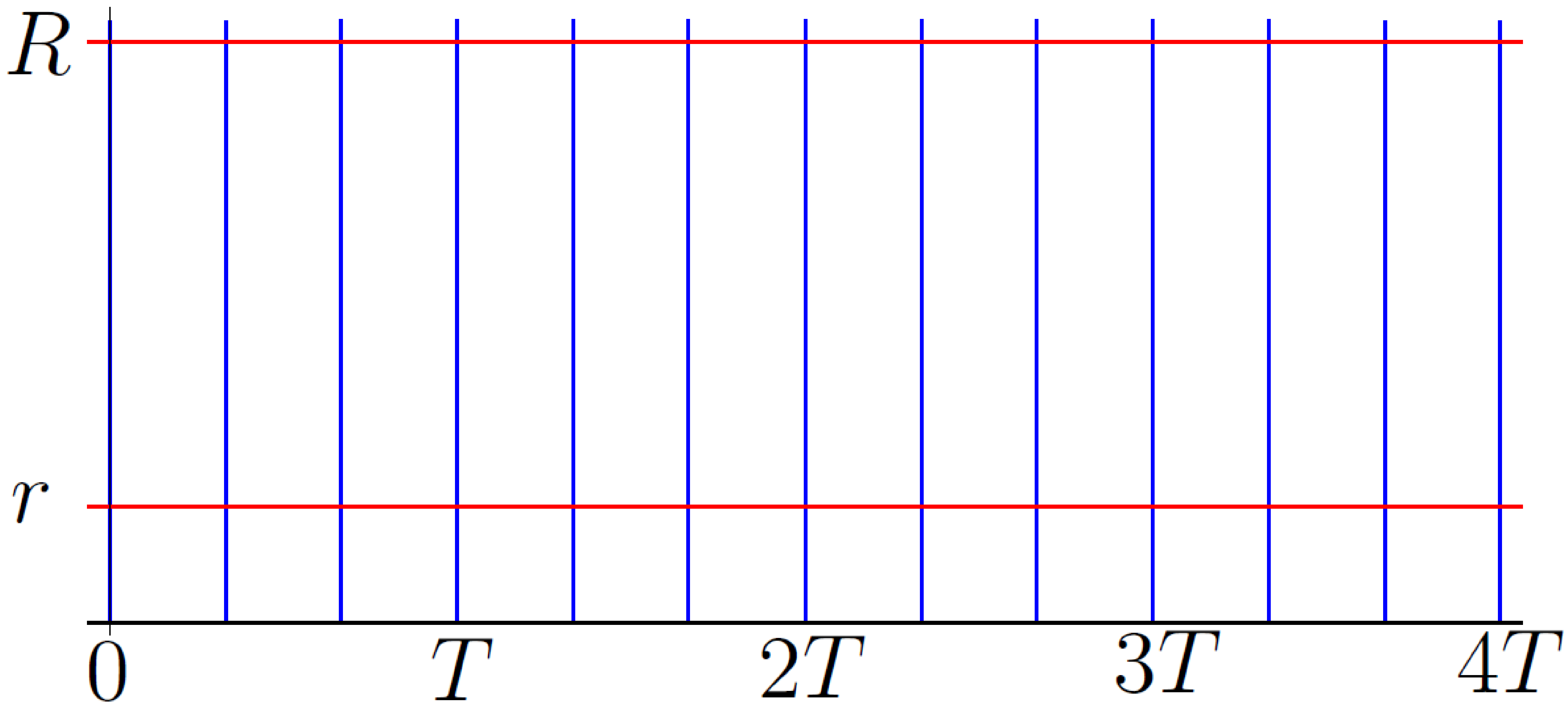
[**G. Feltrin, F.Z.**, Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree, *J. Differential Equations* 262 (2017), 4255–4291]

[**A. Boscaggin, G. Feltrin, F.Z.**, Positive solutions for super-sublinear indefinite problems: high multiplicity results via coincidence degree, *Trans. Amer. Math. Soc.* 370(2018), 791–845]

[**A. Boscaggin, G. Feltrin**, Positive periodic solutions to an indefinite Minkowski-curvature equation, *J. Differential Equations* 269 (2020), 5595–5645]

[**A. Boscaggin, G. Feltrin, E. Sovrano**, High multiplicity and chaos for an indefinite problem arising from genetic models, *Adv. Nonlinear Stud.* 20 (2020), 675–699]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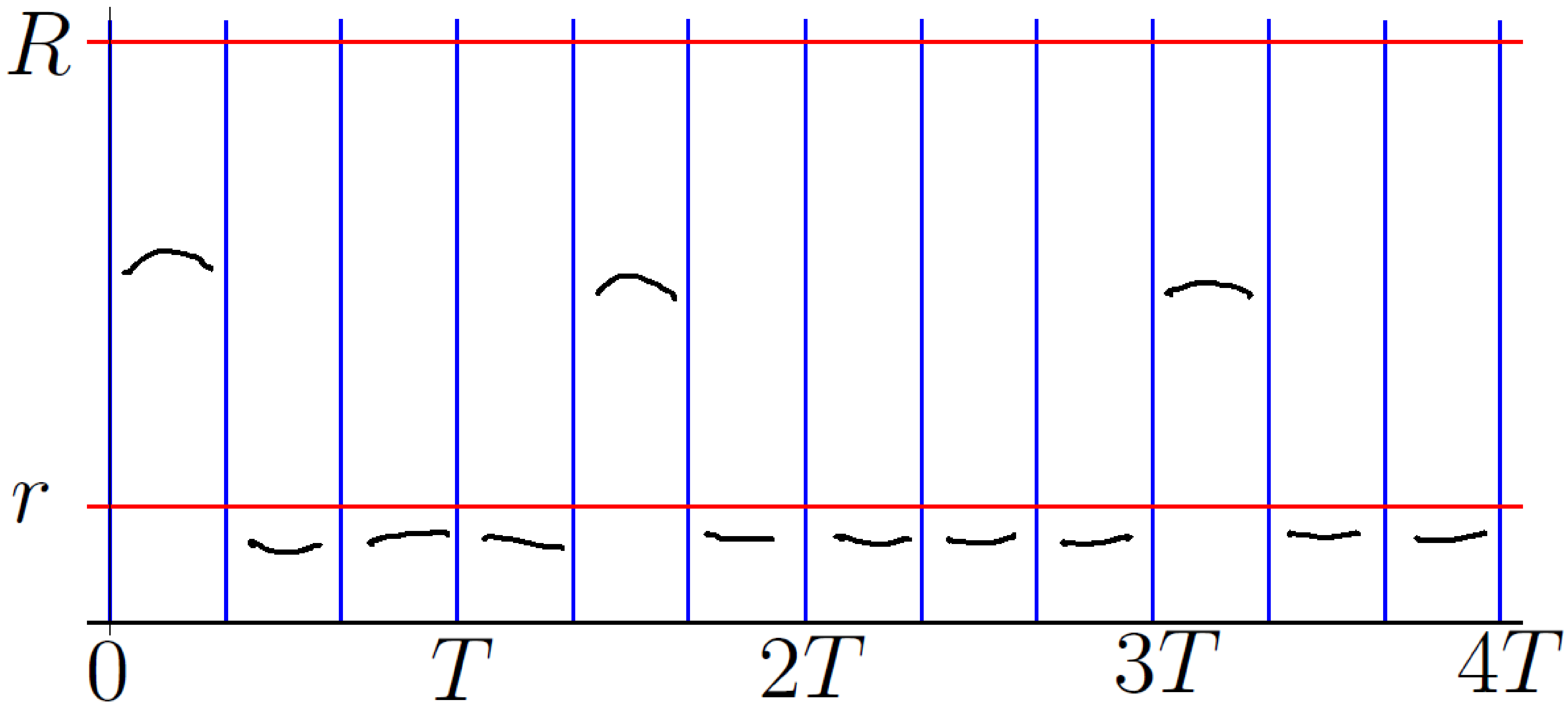
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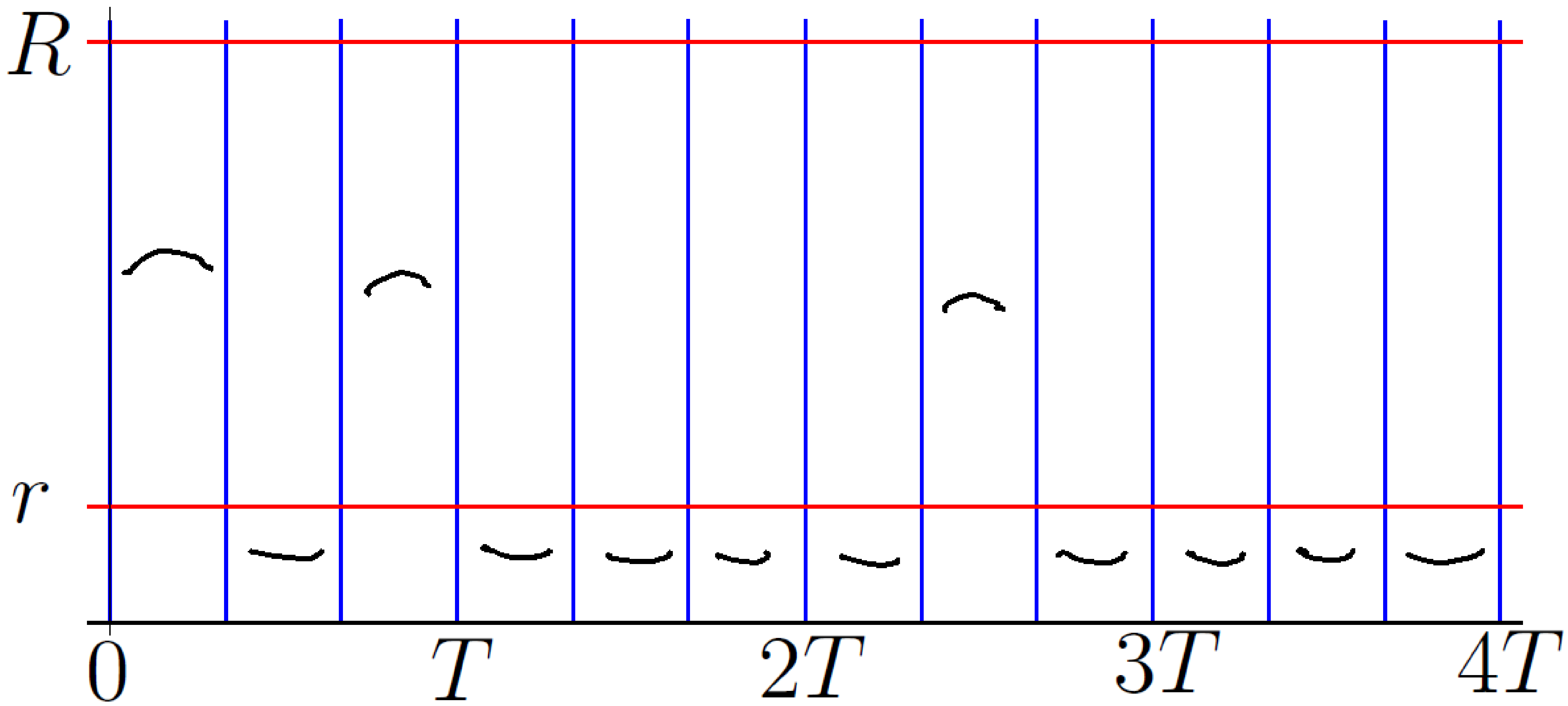
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Amster, Pablo (RA-UBAS); **Benevieri, Pierluigi** (BR-SPL-IMS);
Haddad, Julián [**Haddad, Julián E.**] (BR-FMGS)

Periodic positive solutions of superlinear delay equations via topological degree. (English summary)

Philos. Trans. Roy. Soc. A **379** (2021), *no. 2191, Paper No.* 20190373, 18 *pp.*

In this paper, the authors establish criteria for the existence of positive periodic solutions of nonlinear delay differential equations of the form

$$-u''(t) = f(u(t), u(t-\tau), u'(t)).$$

Their approach is based on Mawhin's coincidence degree theory. The results are then applied to the equation $-u''(t) = a(t)g(u(t), u(t-\tau))$, where g satisfies superlinear growth conditions and a is periodic and sign changing. The work in this paper can be regarded as an extension to that of G. Feltrin and F. Zanolin [Adv. Differential Equations **20** (2015), no. 9-10, 937–982; MR3360396] for the ordinary differential equation $-u''(t) = a(t)g(u(t))$.

Qingkai Kong

Fixed points / Periodic points concerning the Poincaré map

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For system

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we assume the uniqueness and the global existence the solutions to the associated Cauchy problems.

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Let $\zeta(\cdot; P)$ **be the solution of system** $(S1)$ **with** $\underline{x}(0) = P \in \Omega$.
Then, for any fixed τ, **the map** $\Phi_0^\tau : \Omega \to \Omega$, **with**

$$\Phi_0^\tau(P) := \zeta(\tau; P),$$

is well defined as a homeomorphism.

The map

$$\Phi := \Phi_0^T$$

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A periodic point of period $m \geq 2$ **to** Φ **corresponds to a periodic solution of** $(S1)$ **of period** mT.

General speaking, the search of subharmonic solutions to system $(S1)$ **can be split into two parts.**

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- (1) ***Prove the existence of a*** mT***-periodic solution, or, equivalently, a*** m***-periodic point*** z ***to*** Φ
- (2) ***Prove that*** m ***is minimal in the set*** $\{1, \ldots, m\}$***, or, equivalently, that*** $\Phi^k(z) \neq z$ ***for*** $k = 1, \ldots, m-1$

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To obtain (1), we use a fixed point theorem for the map Φ^m.

To obtain (2), we need some information on the fixed points, in order to distinguish the "truly" periodic points from the fixed points of Φ.

2. The planar case

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We consider now a planar differential system

$$x' = X(t,x,y), \quad y' = Y(t,x,y), \tag{1}$$

where the vector field $\vec{F} : (x,y) \mapsto (X(t,x,y), Y(t,x,y))$ **is** T**-periodic in the** t**-variable. Recall that, by a subharmonic solution of order** $m \geq 2$**, we mean a** mT**-periodic solution to system (1) which is not** kT**-periodic for all integers** $k \in \{1,\dots,m-1\}$**. Such solutions correspond to periodic points of the associated Poincaré map** Φ**, having** m **as a minimal period.**

I will briefly present two different approaches to this problem, analyzing then in more detail some topological methods and their applications to nonlinear second-order ODEs,

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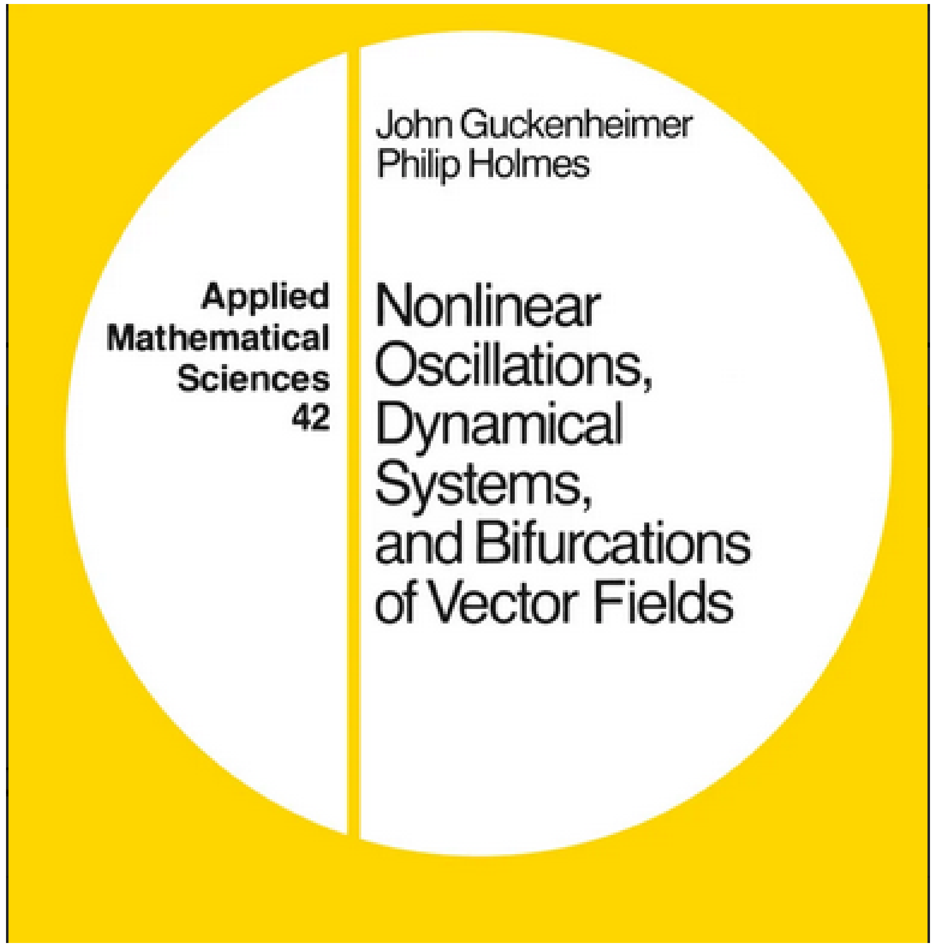
There is also a third approach based on a bifurcation type method and dealing with systems of the form

$$x' = X_0(x, y) + \varepsilon p(t), \qquad y' = Y_0(x, y) + \varepsilon q(t), \tag{2}$$

that is not considered here for time limitations.



[M. Henrard, F.Z., Bifurcation from a periodic orbit in perturbed planar Hamiltonian systems, *J. Math. Anal. Appl.* 277 (2003), 79–103]
[A. Buică, Bifurcations from a normally degenerate cycle in forced planar differential equations, *NoDEA Nonlinear Differential Equations Appl.* 30 (2023), Paper No. 63, 18 pp.]

The Poincaré-Birkhoff fixed point theorem

The Poincaré-Birkhoff fixed point theorem

The Poincaré–Birkhoff fixed point theorem, named also the "twist theorem" or the "Poincaré's last geometric theorem", in the original formulation [Poincaré (1912), Birkhoff (1913)], asserts the existence of at least two fixed points for an area-preserving homeomorphism ϕ of a planar circular annulus

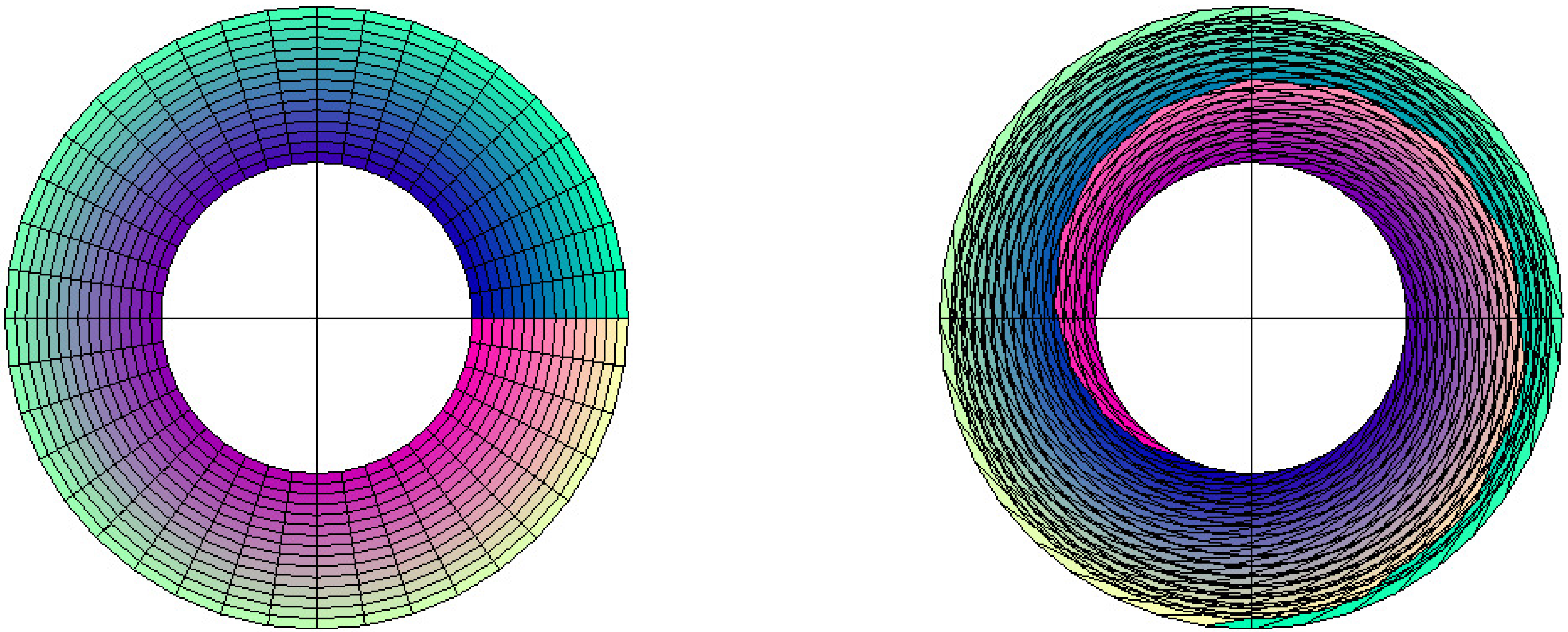
$$\mathcal{A}[a,b] = \{(x,y) : a^2 \leq x^2 + y^2 \leq b^2\}$$

onto itself, such that the points of the inner boundary C_a are advanced along C_a in the clockwise sense and the points of the outer boundary C_b are advanced along C_b in the counter-clockwise sense (or, viceversa).

The next figure shows the effect of the twist on the annulus $\mathcal{A}[1,2]$ under the action of the dynamical system

$$\begin{cases} x' = 2py(x^2+y^2)^{p-1} \\ y' = -2px(x^2+y^2)^{p-1} \end{cases}$$

for $p>1$, on a short time interval.



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[G. D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc., New York, 1927]

A complete description of Birkhoff's approach, with also the explanation how to obtain a second fixed point, by suitably modifying Birkhoff's argument, can be found in the expository article by Brown and Neumann

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[M. Brown, W. D. Neumann, Proof of the Poincaré–Birkhoff fixed point theorem, *Michigan Math. J.* 24 (1977), 21–31]

The history of the “twist” theorem and its generalizations and developments is quite interesting but impossible to summarize in few lines (or in a short talk). After about hundred years of studies on this topic, some controversial “proofs” of its extensions have been settled only recently.

[F. Dalbono, C. Rebelo: Poincaré–Birkhoff fixed point theorem and periodic solutions of asymptotically linear planar Hamiltonian systems, Turin Fortnight Lectures on Nonlinear Analysis (2001), *Rend. Sem. Mat. Univ. Politec. Torino* 60 (2002), 233–263 (2003)]

[P. Le Calvez, J. Wang, Some remarks on the Poincaré-Birkhoff theorem, *Proc. Amer. Math. Soc.* 138 (2010), 703–715]

[A. Fonda, A.J. Ureña A higher dimensional Poincaré-Birkhoff theorem for Hamiltonian flows, *Ann. Inst. H. Poincaré C Anal. Non Linéaire 34* (2017), 679–698]

[A. Fonda, *Playing around resonance. An invitation to the search of periodic solutions for second order ordinary differential equations*, Birkhäuser/Springer, Cham, 2016]

A version for planar Hamiltonian systems

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We consider a planar Hamiltonian system

$$x' = \frac{\partial}{\partial y}\mathcal{H}(t,x,y), \qquad y' = -\frac{\partial}{\partial x}\mathcal{H}(t,x,y) \tag{3}$$

and suppose that the vector field $\vec{F}(t,z) := (\frac{\partial}{\partial y}\mathcal{H}(t,x,y), -\frac{\partial}{\partial x}\mathcal{H}(t,x,y))$, **for** $z=(x,y)$, **is continuous and** T**-periodic in the** t**-variable.**

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We also assume the uniqueness (the uniqueness hypothesis is not necessary, if we apply Fonda-Ureña version of the theorem) and the global continuability for the solutions of the initial value problems.

For any initial point $w \in \mathbb{R}^2$ **let** $\zeta(\cdot, w) = (\zeta_1(\cdot, w), \zeta_2(\cdot, w))$ **be the solution of** (3) **with** $\zeta(0, w) = w$.

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If

$$\vec{F}(t, 0) \equiv 0, \tag{4}$$

we have that $\zeta(t, w) \neq 0$ **for every** t, **provided that** $w \neq 0$. **Hence we can pass to polar coordinates and determine the normalized angular displacement of the solution** $\zeta(\cdot, w)$ **on a time interval** $[0, \tau]$ **as**

$$\mathrm{Rot}_w(\tau) := \frac{1}{2\pi} \int_0^\tau \frac{\frac{\partial}{\partial y}\mathcal{H}(t, \zeta(t, w))\zeta_2(t, w) + \frac{\partial}{\partial x}\mathcal{H}(t, \zeta(t, w))\zeta_1(t, w)}{|\zeta(t, w)|^2}\, dt,$$

where $|\cdot|$ **denotes the Euclidean norm of a vector in the plane.**

The *rotation number* $\mathrm{Rot}_w(\tau)$ is the algebraic count of the clockwise turns of the solution $\zeta(t,w)$ around the origin in the time interval $[0,\tau]$.

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Using the fact that (as a consequence of Liouville's theorem), for every $\tau > 0$, the mapping $w \mapsto \zeta(\tau, w)$ is an area preserving homeomorphism of the plane onto itself, we can apply a consequence of the Poincaré-Birkhoff fixed point theorem [2], which reads as follows.

[2] in the version of W.-Y. Ding [Acta Mathematatica Sinica (1982)] for the case of a standard annulus or, respectively, in the version of Ding–Rebelo

Theorem 1 ***Assume*** (4) ***and let*** $m \geq 1$ ***be a fixed integer. Suppose that there are*** $0 < r < R$ ***and a positive integer*** j ***such that***

$$\mathrm{Rot}_w(mT) > j,\ \forall\,|w| = r \quad \textbf{\textit{and}} \quad \mathrm{Rot}_w(mT) < j,\ \forall\,|w| = R. \tag{5}$$

Then there exist at least two initial points $w_1 \neq w_2$, ***with***

$$r < |w_1|,\ |w_2| < R$$

such that

$$\mathrm{Rot}_{w_1}(mT) = \mathrm{Rot}_{w_2}(mT) = j \tag{6}$$

and the solutions $\zeta(\cdot, w_1)$ ***and*** $\zeta(\cdot, w_2)$ ***of*** (3) ***are*** mT***-periodic.***

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and the solutions $\zeta(\cdot, w_1)$ ***and*** $\zeta(\cdot, w_2)$ ***of*** (3) ***are*** mT***-periodic.***

Instead of the circumferences of radius r and R we can take two *strictly star-shaped curves* around the origin bounding a *topological annulus*. The fixed points belong to the *interior* of such annulus.

The points w_1 and w_2 are obtained as fixed points of the m-th iterate of the Poincaré map $\Phi : w \mapsto \zeta(T, w)$. Hence, they determine two orbits

$$\mathcal{O}_1 := \{w_1, \Phi(w_1), \ldots, \Phi^{m-1}(w_1)\} \quad \textbf{and} \quad \mathcal{O}_2 := \{w_2, \Phi(w_2), \ldots, \Phi^{m-1}(w_2)\}$$

of fixed points of Φ^m. By a remark in [W.D. Neumann (1977)] we know that it is always possible to choose w_1 and w_2 so that their orbits are disjoint. This means that the corresponding mT-periodic solutions are not in the same periodicity class.

In the special case of

$$u'' + f(t,u) = 0 \tag{7}$$

which writes as a planar Hamiltonian system of the form

$$x' = y, \qquad y' = -f(t,x)$$

and if we assume

$$f(t,0) \equiv 0,$$

the condition (6) corresponds to the fact that the solutions $v^{(i)}(\cdot) := \zeta_1(\cdot, w_i)$ **(**$i = 1,2$**) of (7) possess exactly** $2j$ **zeros in the interval** $[0, mT[$ **. From the relationships between** m **and** j **it is easy to produce conditions ensuring the minimality of the period, and therefore to obtain subharmonic solutions with a given minimal period.**

The same remark extends to equation

$$(\varphi(u'))' + f(t,u) = 0 \tag{8}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ **is an increasing homeomorphism with** $\varphi(0) = 0$, **passing to the planar Hamiltonian system**

$$x' = \varphi^{-1}(y), \qquad y' = -f(t,x).$$

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There is a large and growing literature on the applications of the Poincaré-Birkhoff theorem.

For a recent application of this technique to planar Hamiltonian systems, see

[D. Qian, P.J. Torres, P. Wang, Periodic solutions of second order equations via rotation numbers, *J. Differential Equations* 266 (2019), 4746–4768]

3. Linked Twist Mappings

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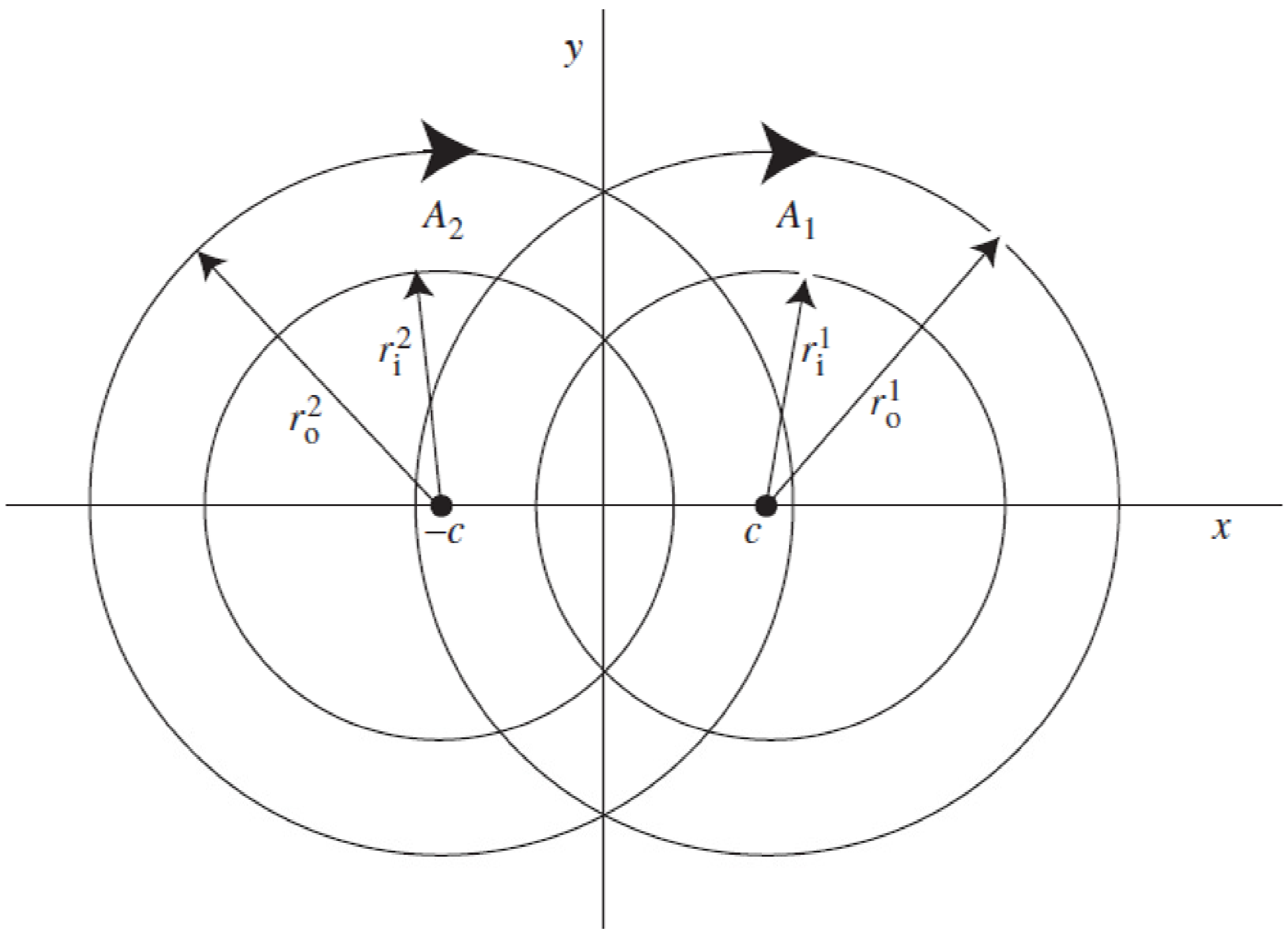
4 S. Wiggins and J. M. Ottino



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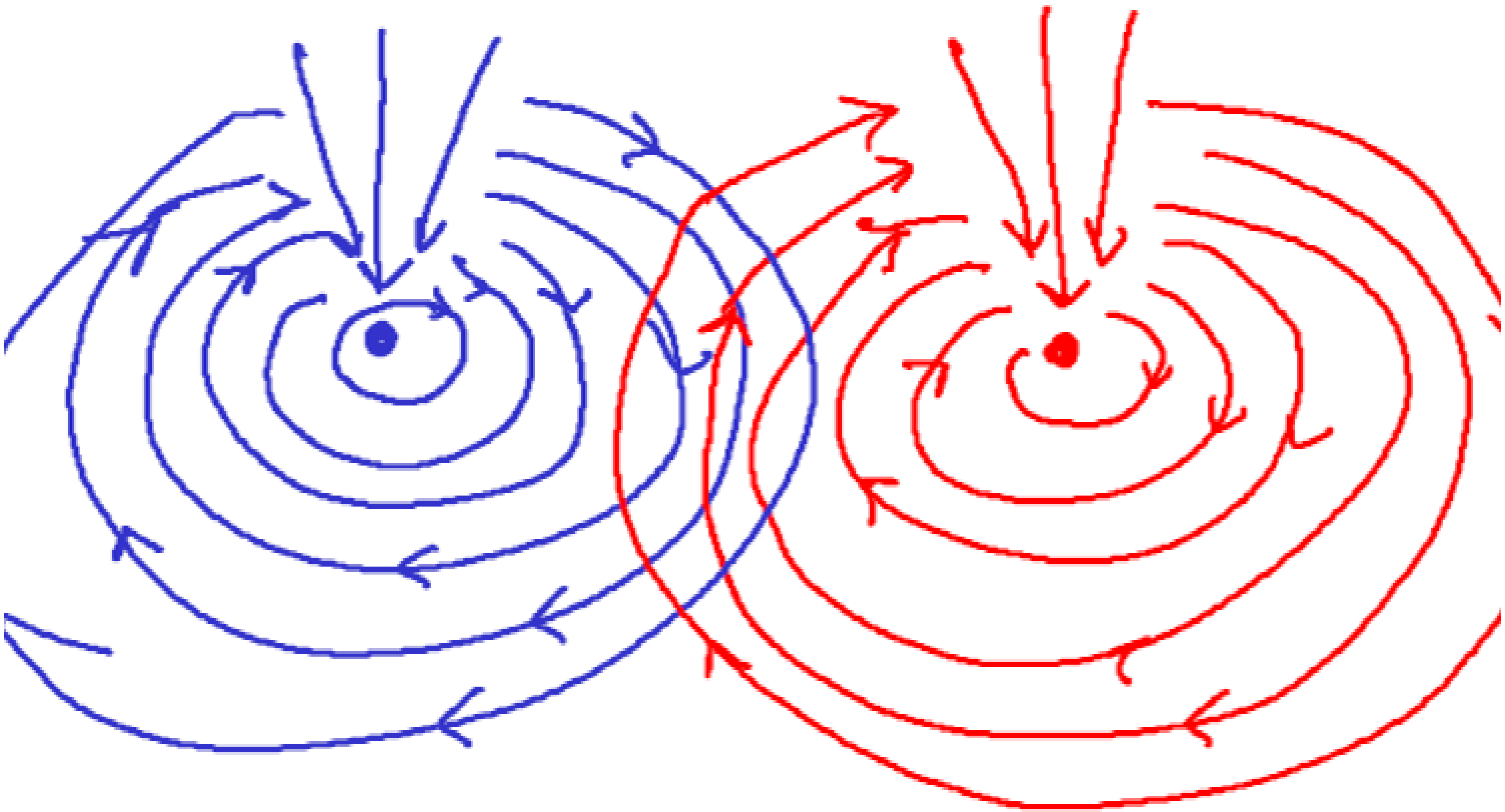

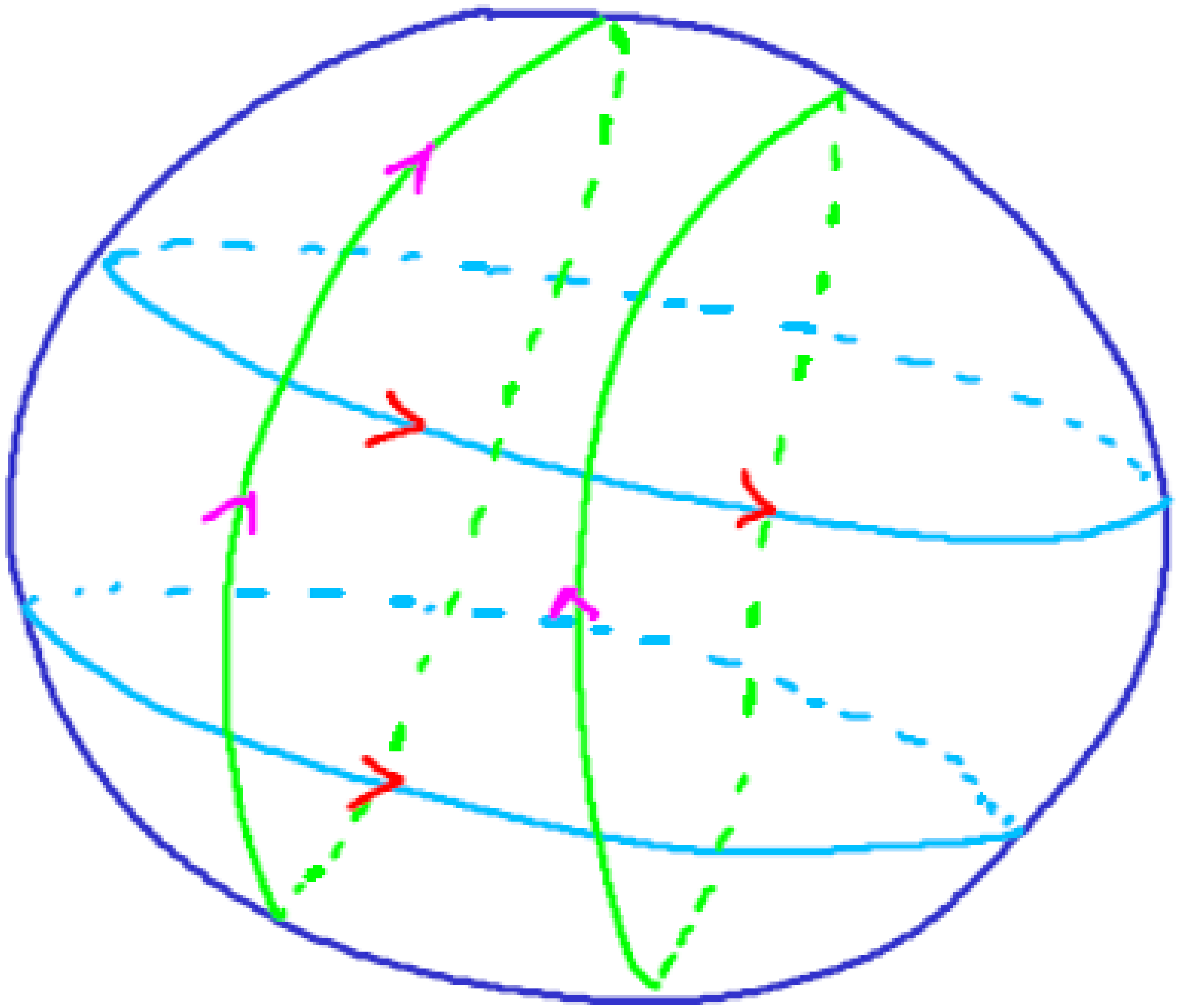

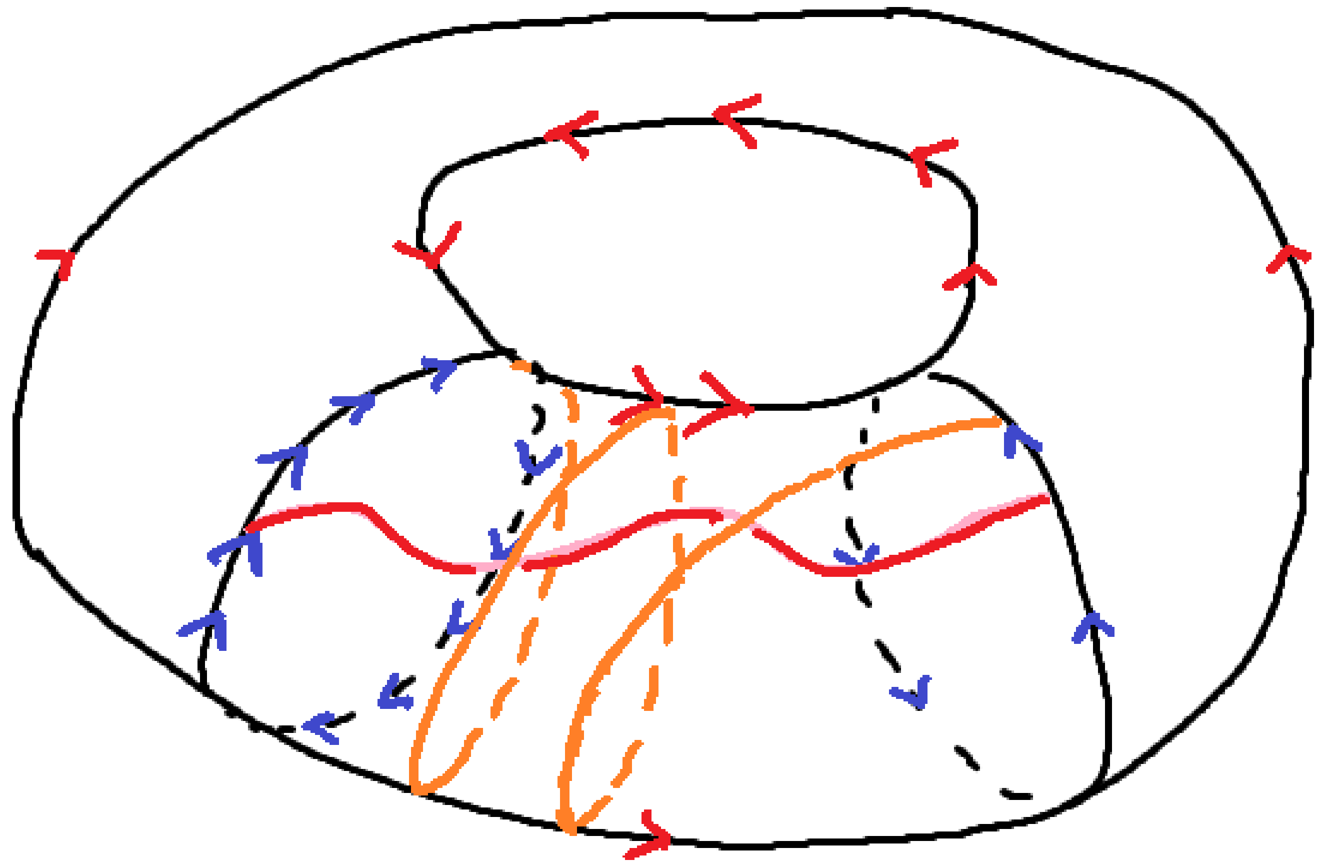

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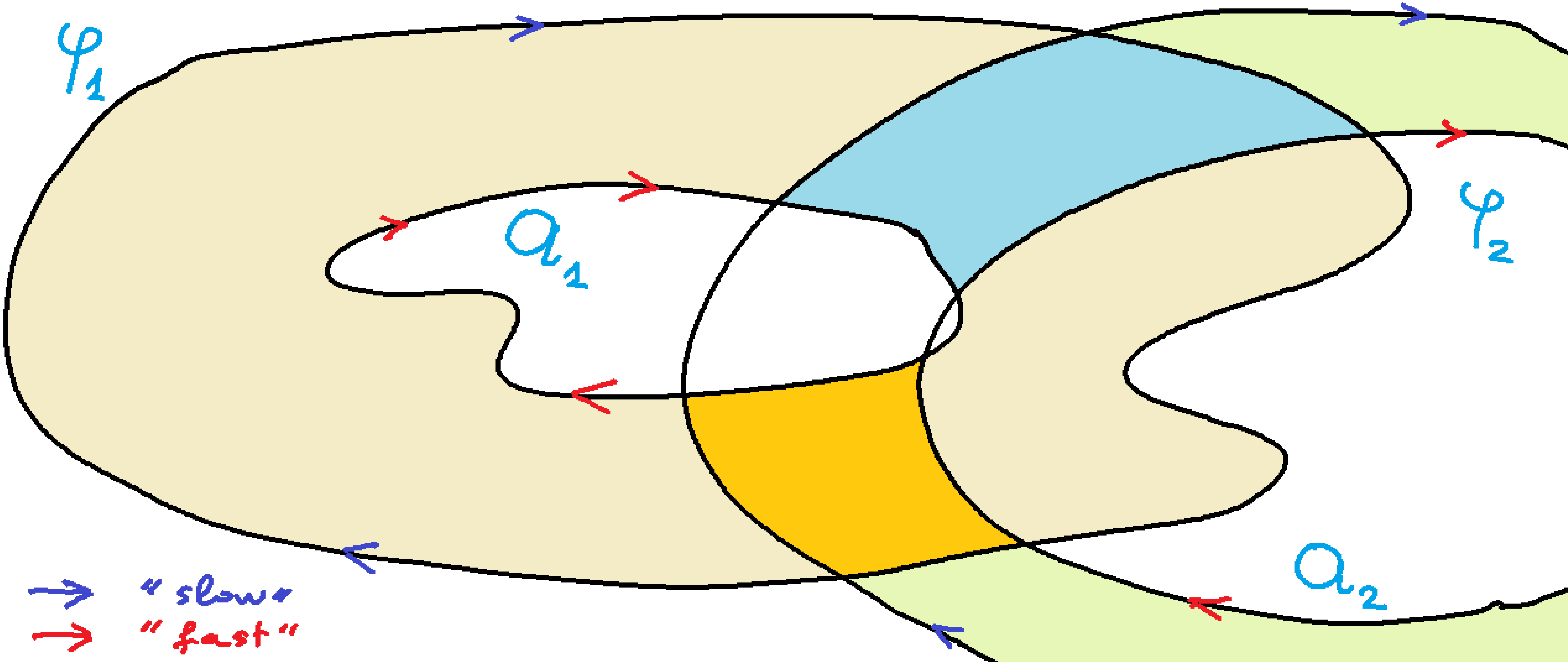
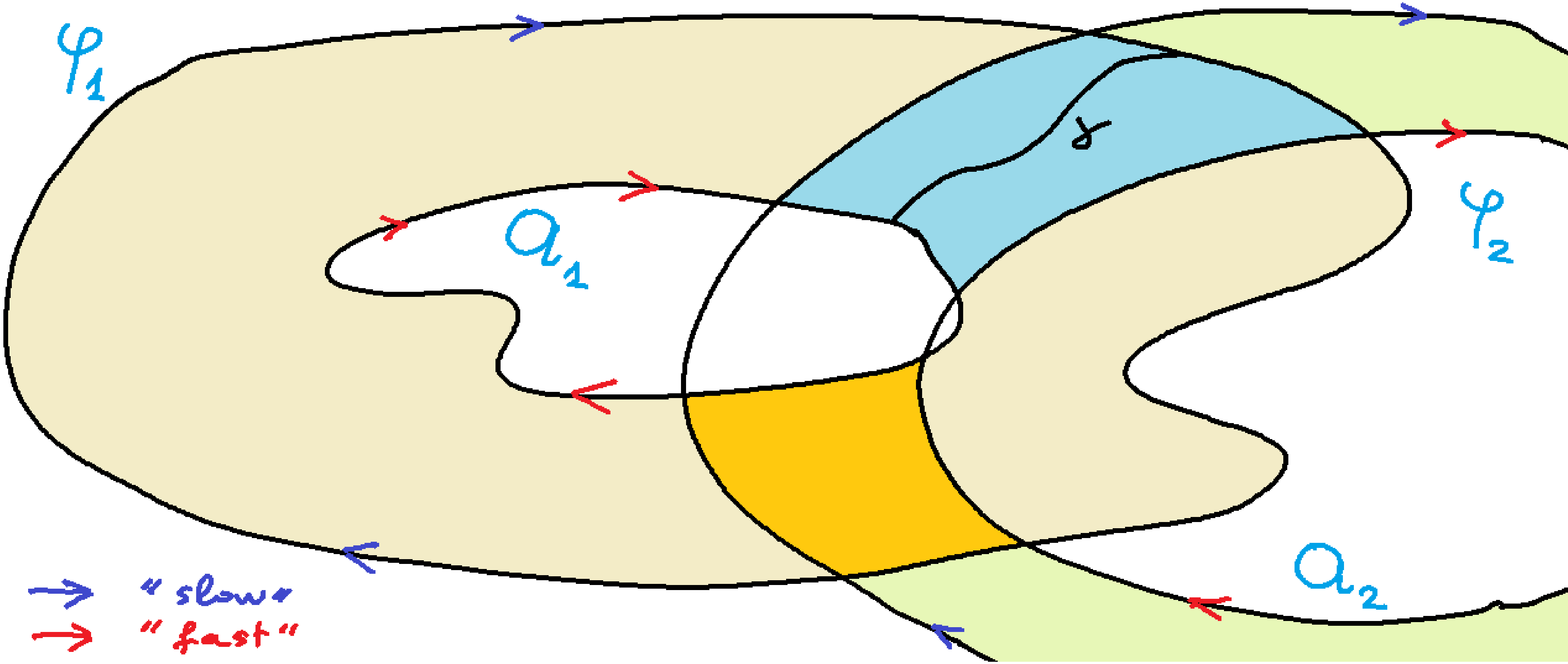
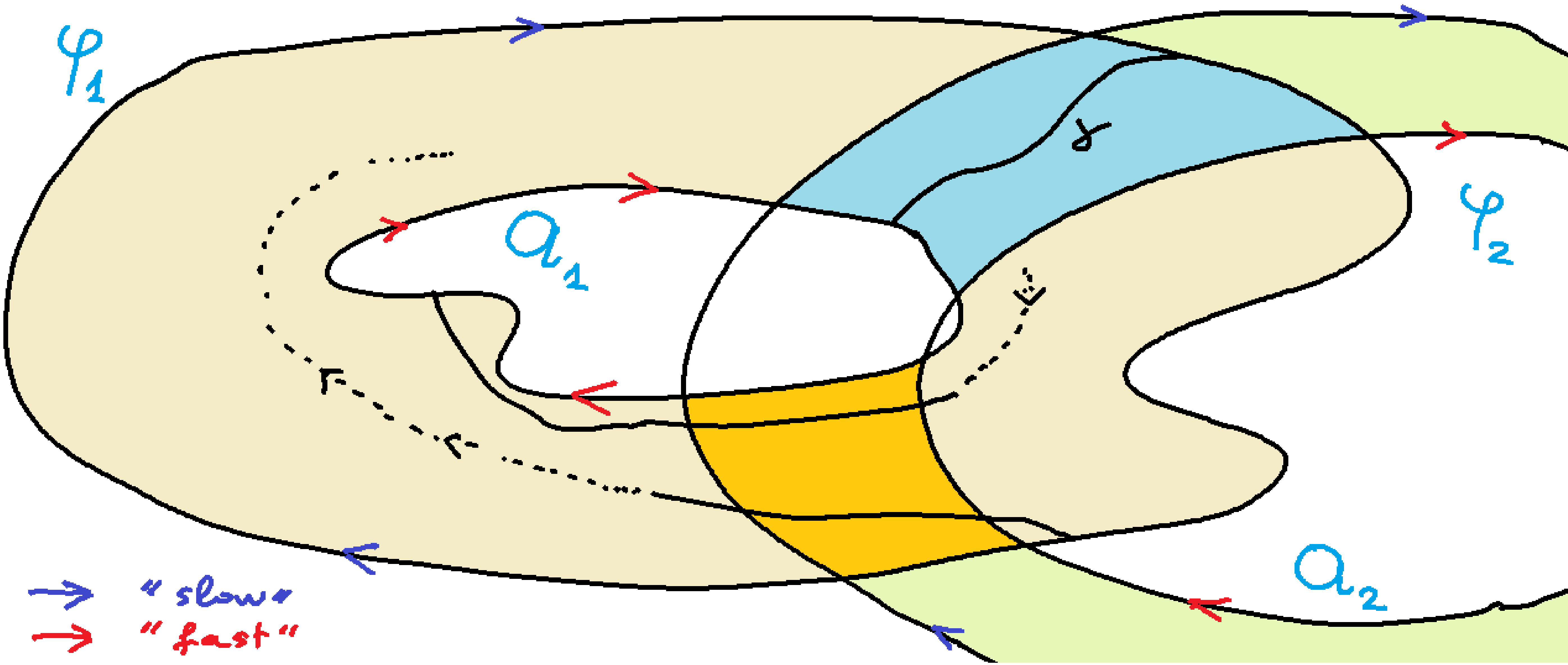


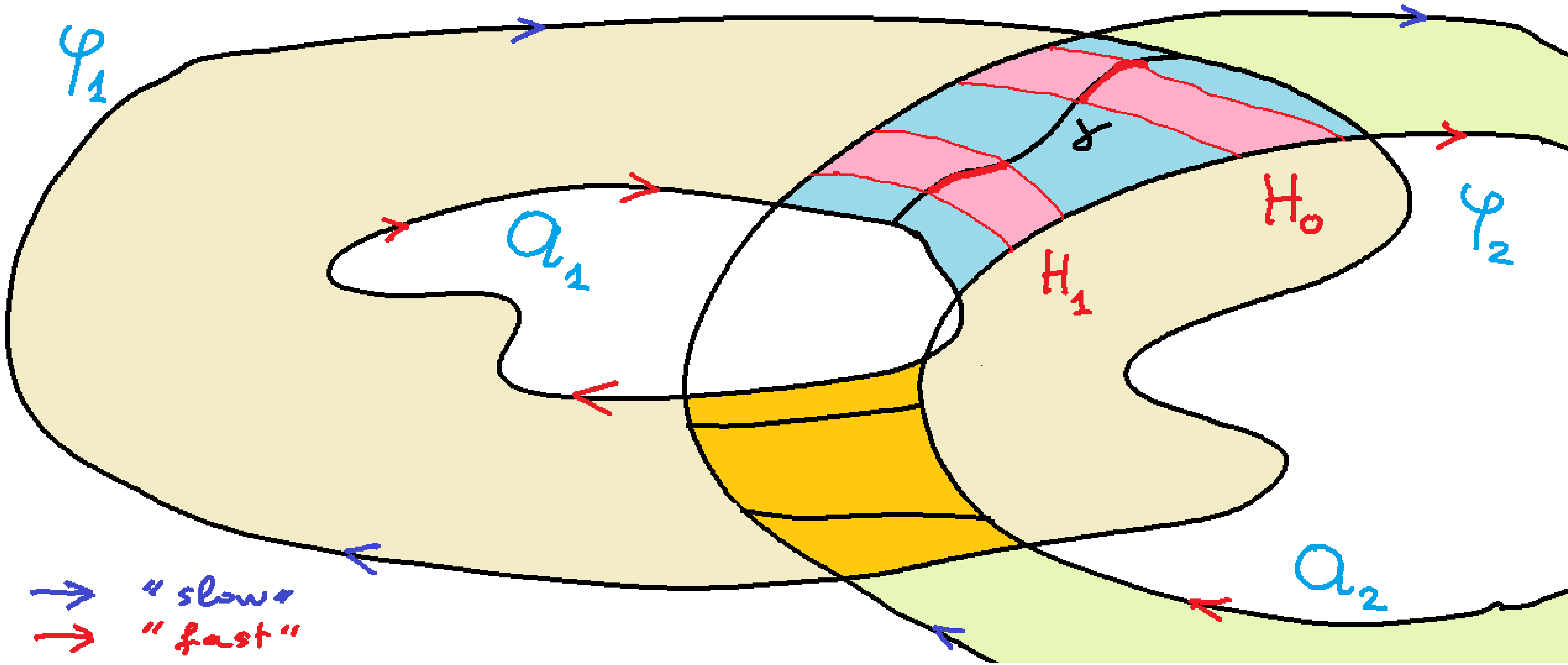
φ1
a1
φ2
a2
"slow"
"fast"



φ1
a1
γ
φ2
a2
"slow"
"fast"

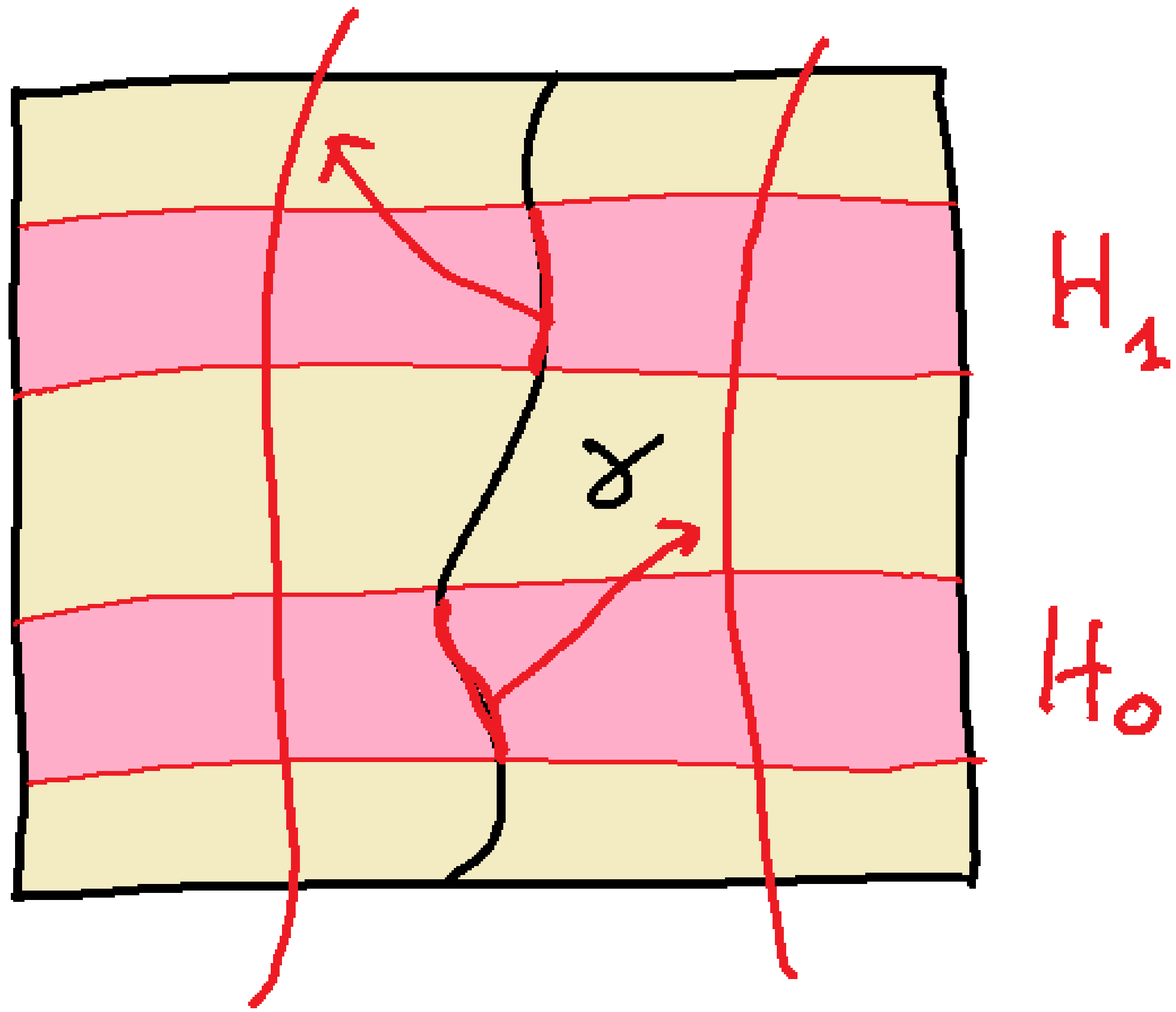


φ_1
a_1
γ
φ_2
a_2
"slow"
"fast"

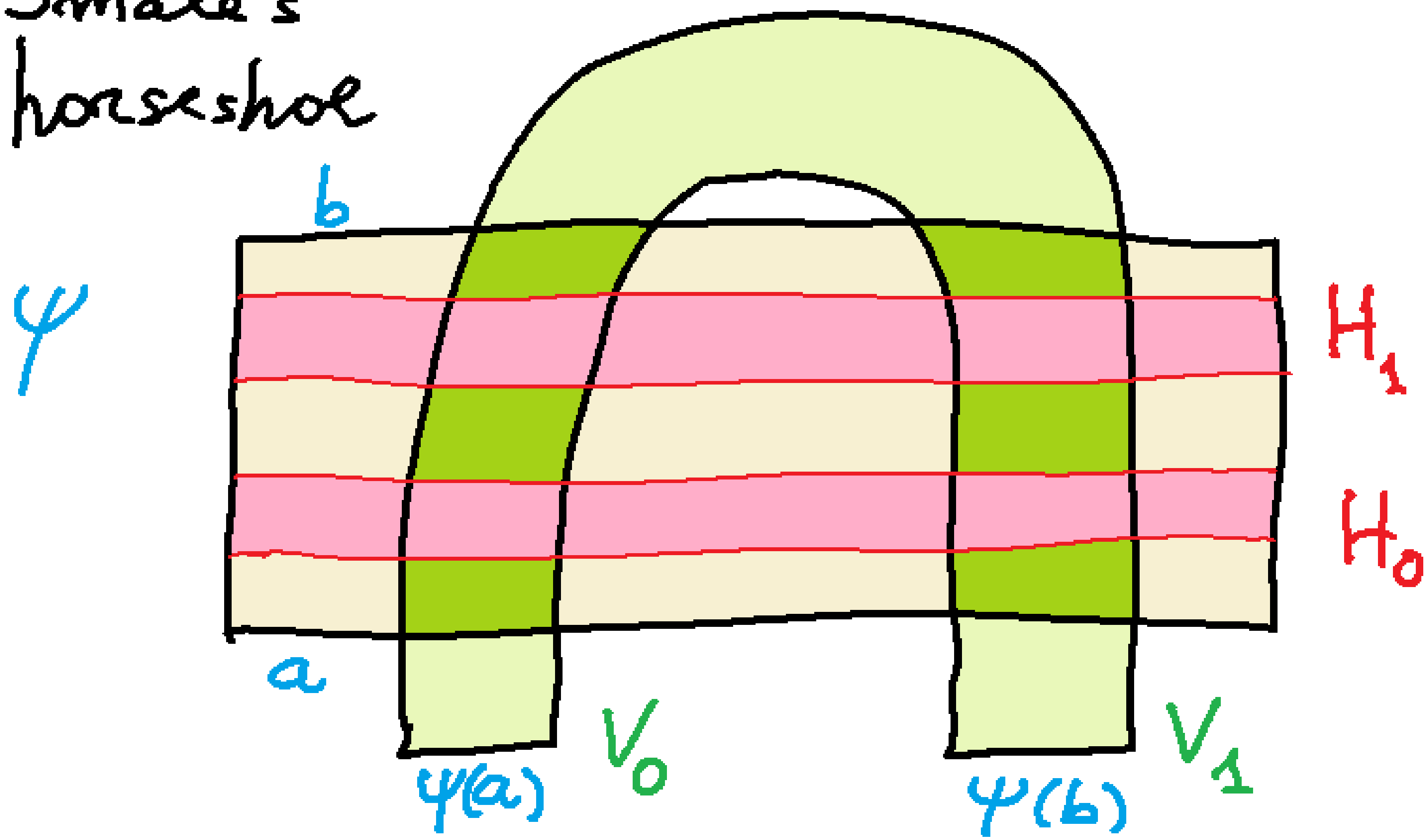


φ1
φ2
a1
a2
H0
H1
γ
"slow"
"fast"

$\varphi_2 \circ \varphi_1$



Smale's horseshoe



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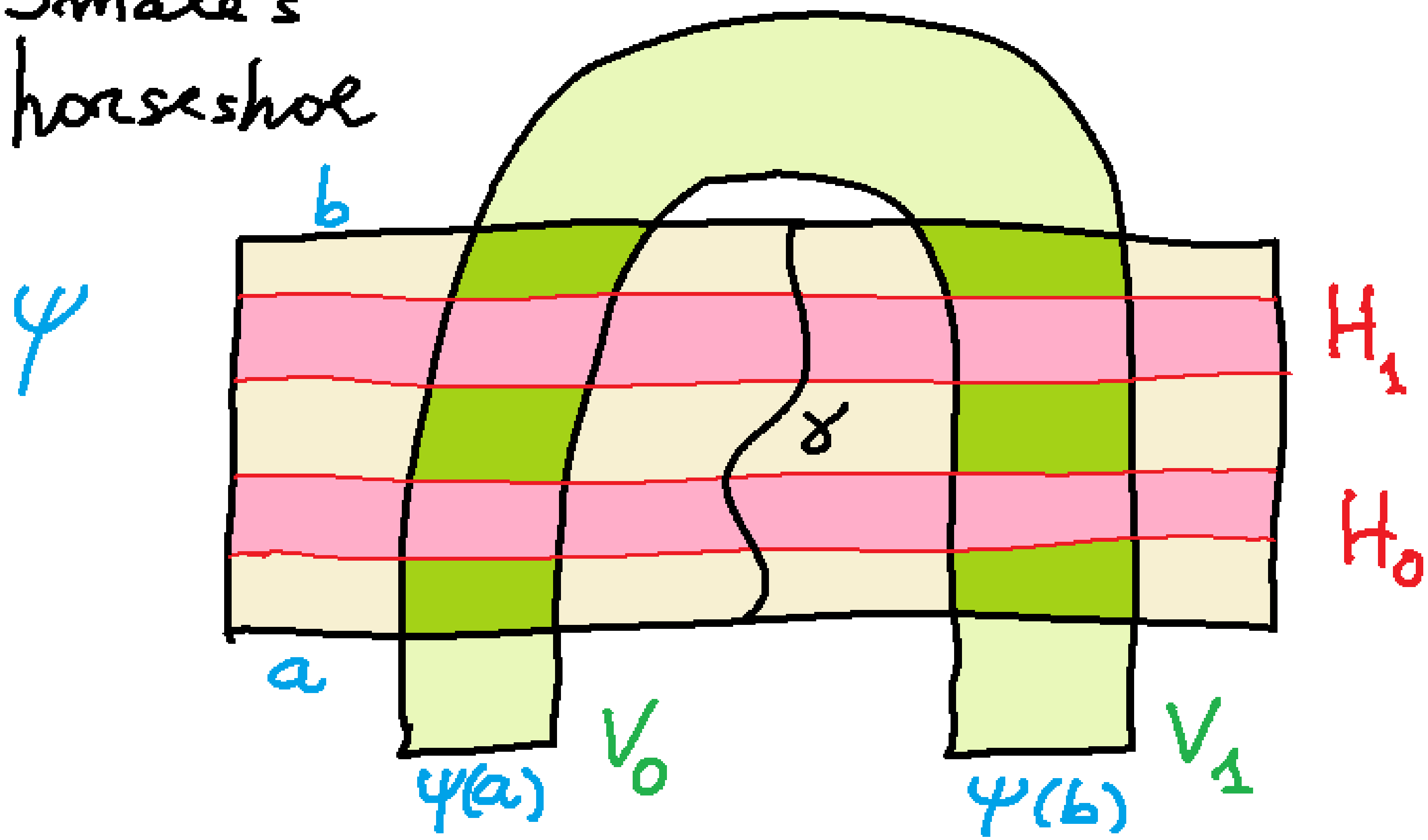
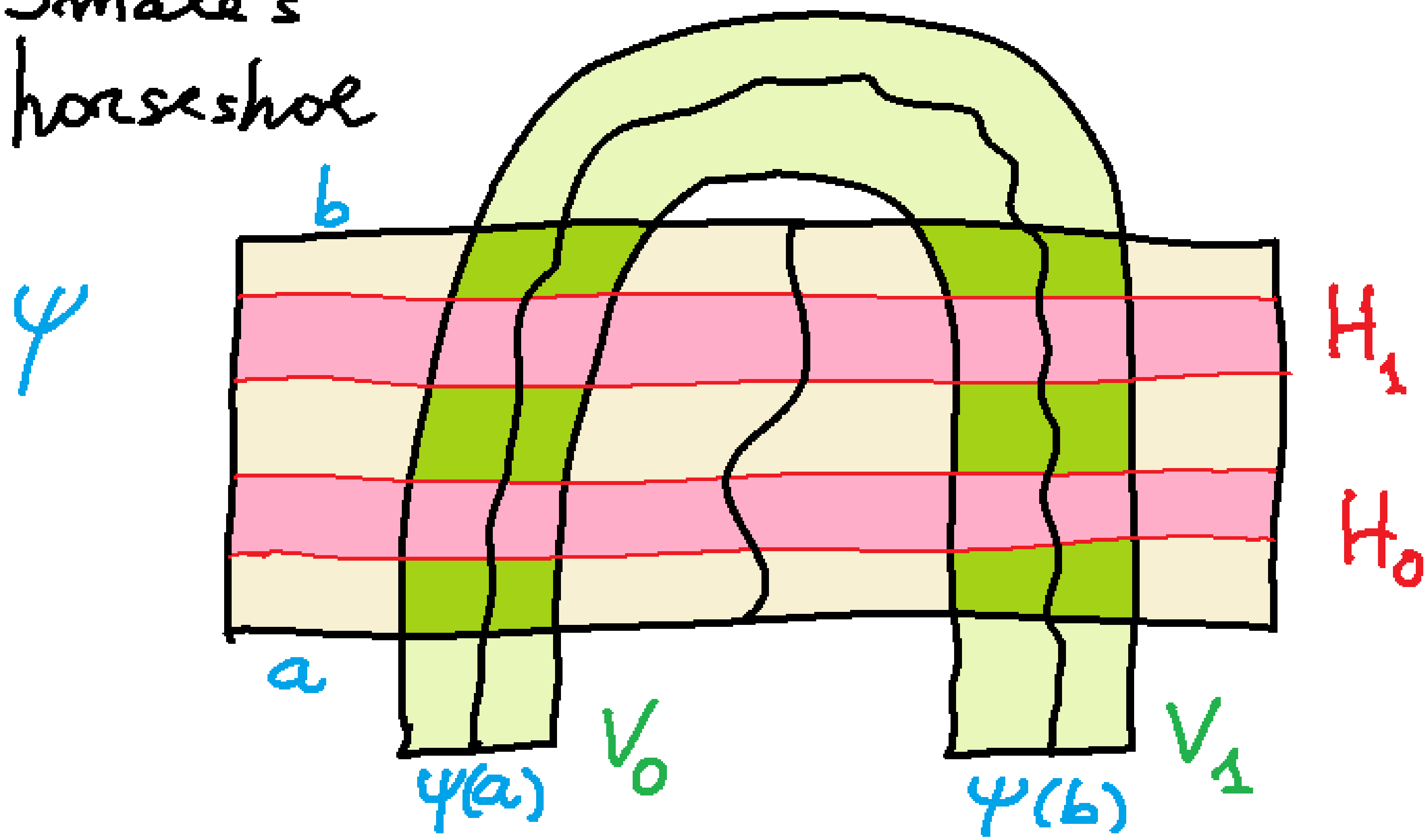
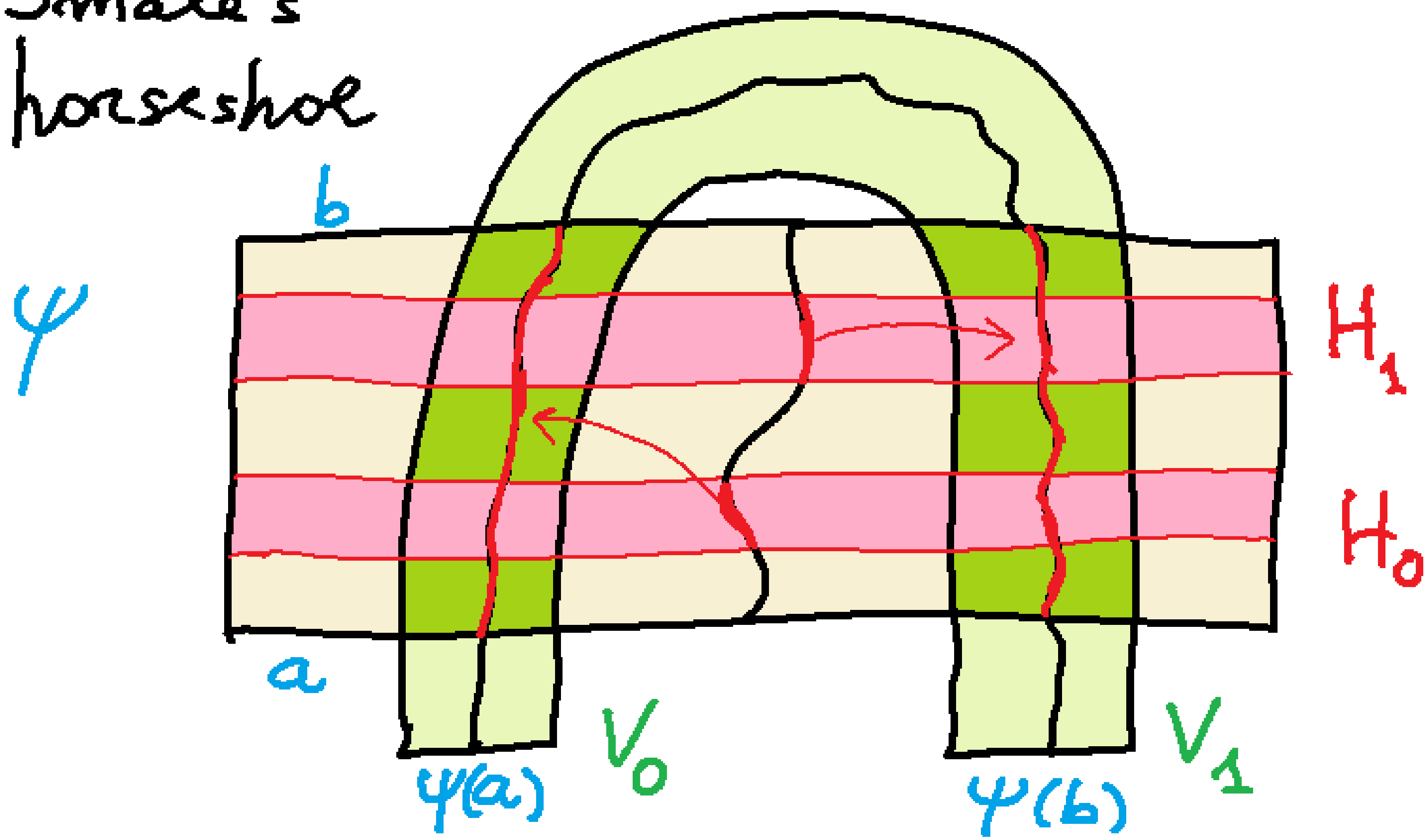
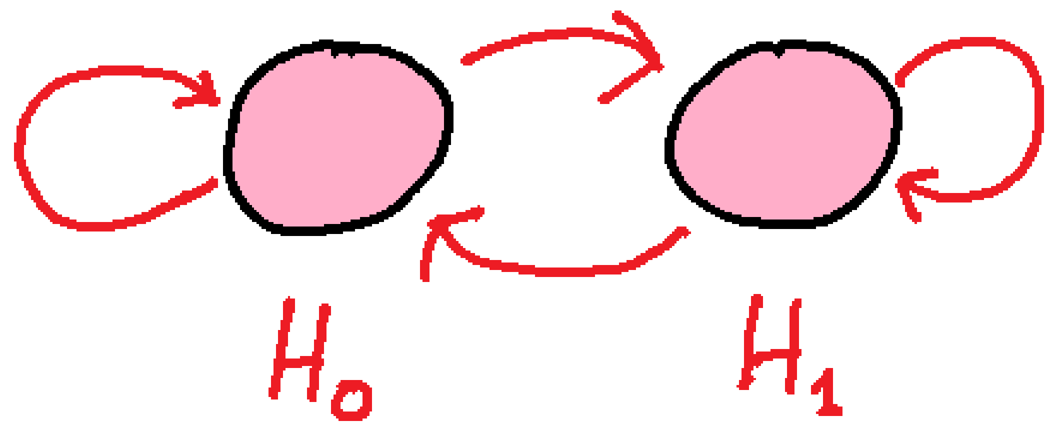


Smale's horseshoe



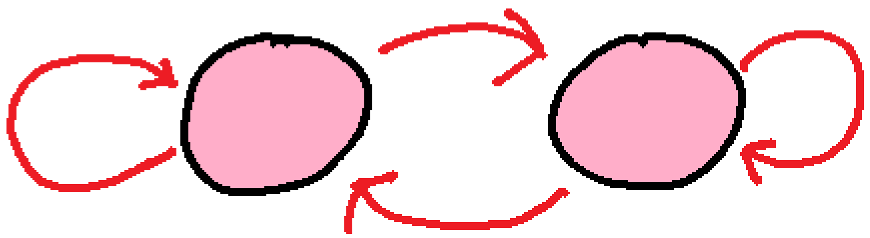
Smale's horseshoe





$\longrightarrow \;\equiv$ "crossing relation"
or
"covering relation"

Burns & Weiss, Mischaikow, Mrozek, Kennedy & Yorke, Srzednicki, Zgliczynski, Wójcik, Gidea, ...



$\longrightarrow$ $\equiv$ "crossing relation" or "covering relation"

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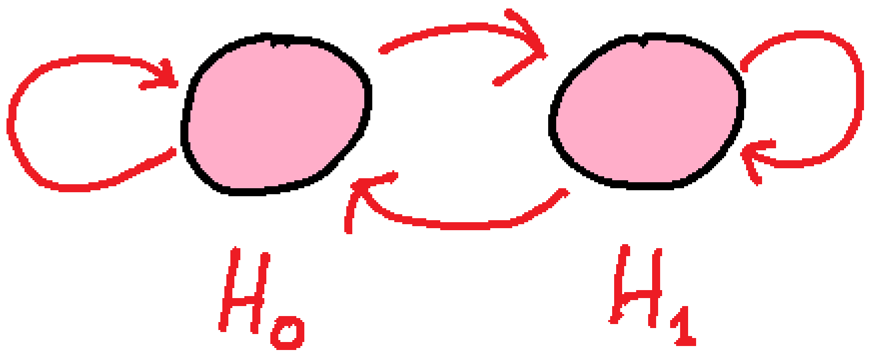
Given any sequence of (two) symbols

0001000110 ----

there is a point which follows, by the map φ, the same itinerary

$H_0 H_0 H_0 H_1 H_0 H_0 H_0 H_1 H_1$



$\rightarrow$ $\equiv$ "crossing relation" or "covering relation"

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Given any sequence of (two) symbols

0001000110 ----

there is a point which follows, by the map ψ, the same itinerary

$H_0 H_0 H_0 H_1 H_0 H_0 H_0 H_1 H_1$

and if the sequence is periodic .... there is a periodic point.

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4. Applications to Duffing type equations

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The goal is to apply the theory of Linked Twist Maps to a periodically perturbed planar system

$$(S_{\text{pert}}) \qquad \begin{cases} x' = y \\ y' = -g(x) + p(t), \end{cases}$$

4. Applications to Duffing type equations

The goal is to apply the theory of Linked Twist Maps to a periodically perturbed planar system

$$(S_{\text{pert}}) \qquad \begin{cases} x' = y \\ y' = -g(x) + p(t), \end{cases}$$

using some information on the associated autonomous unperturbed system

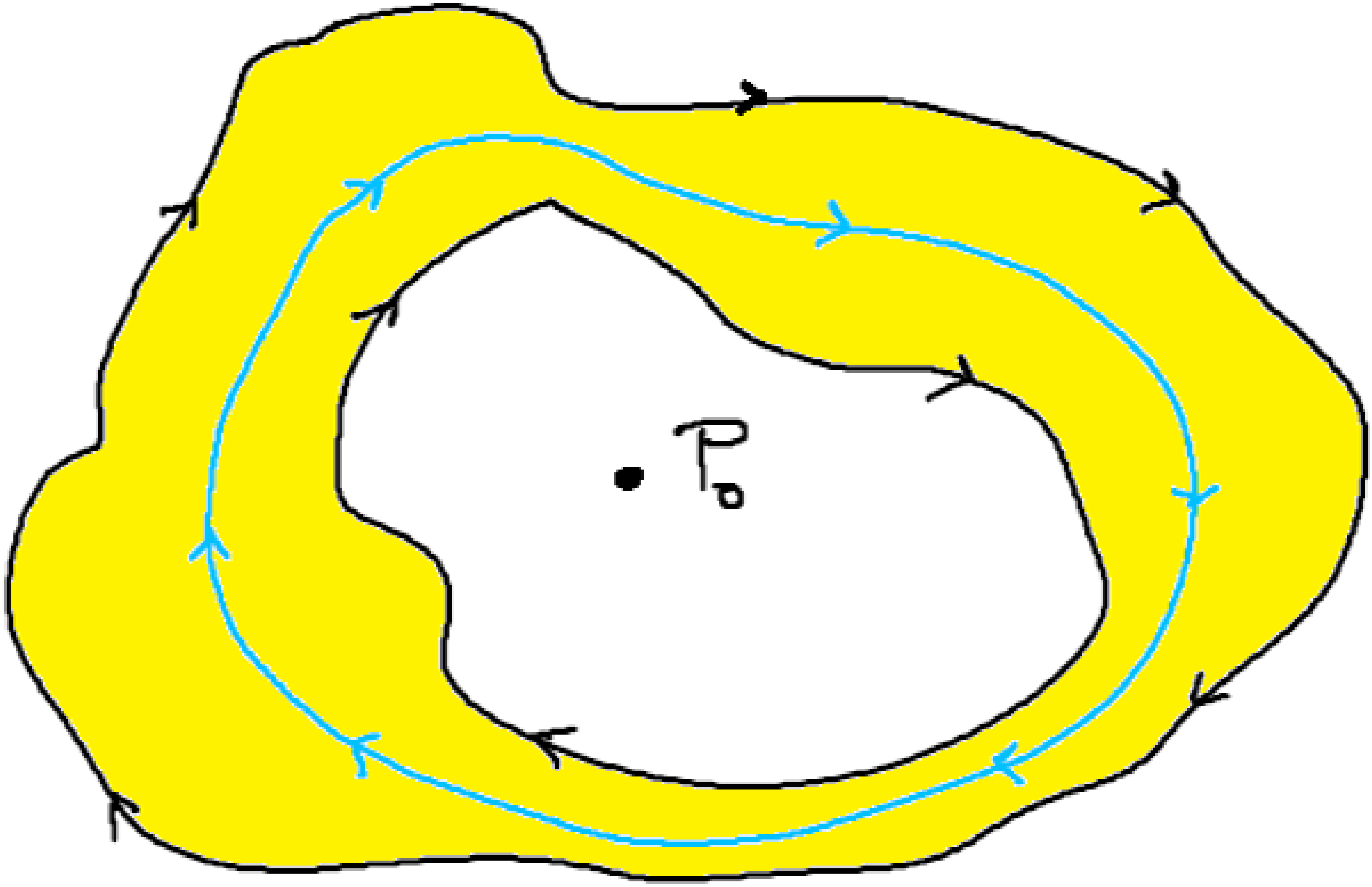
$$(S_{\text{aut}}) \qquad \begin{cases} x' = y \\ y' = -g(x). \end{cases}$$

A recent contribution for an equation with a singularity at the origin is given in

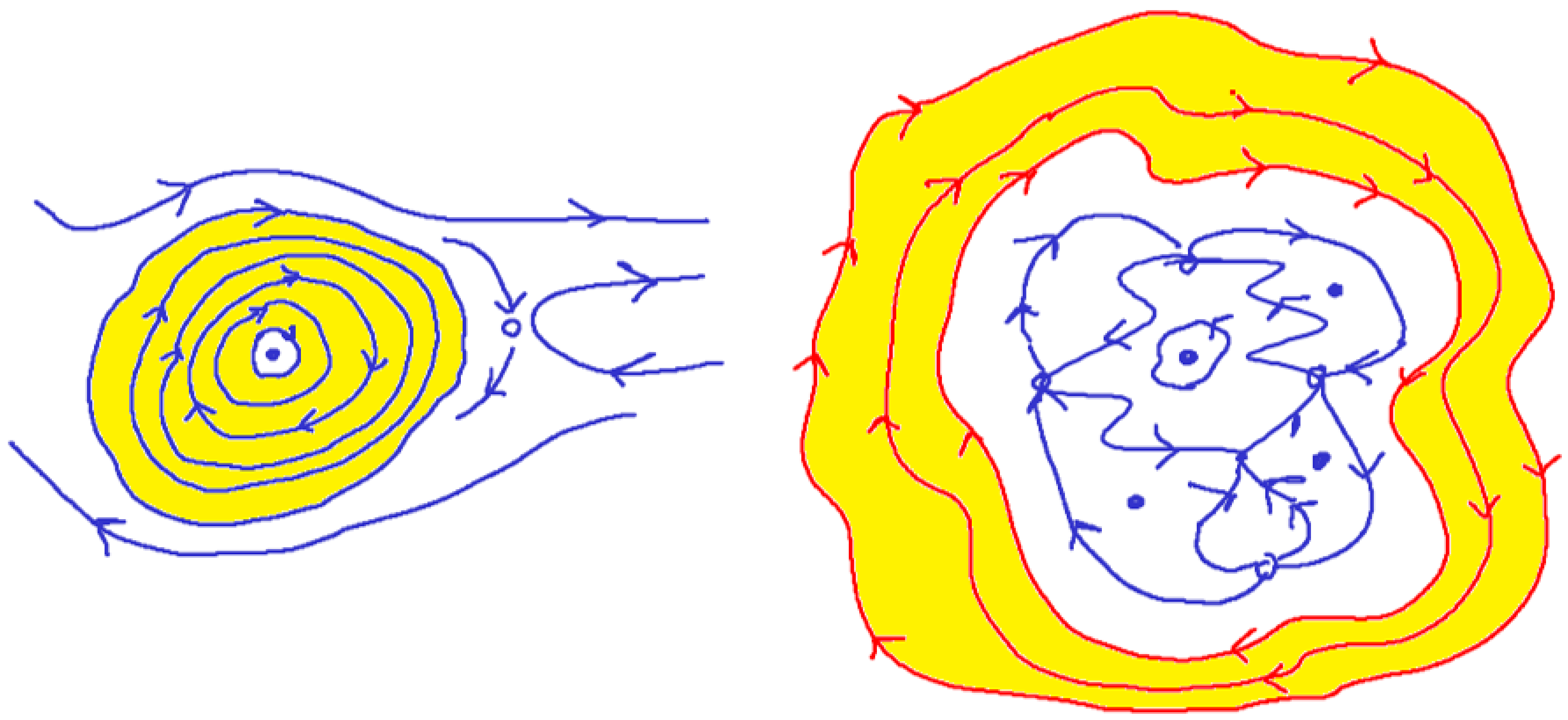
[**L. Burra, F.Z.**, Monotonicity of the period function and chaotic dynamics in a class of singular ODEs, *J. Math. Anal. Appl.* (2022), Paper No. 125814, 17 pp.]

The required geometry is that of an annulus filled by periodic orbits of system (S_{aut}) and the twist condition will be obtained by proving that there is some gap between the periods of the orbits at the boundary of the annulus.

The required geometry is that of an annulus filled by periodic orbits of system (S_{aut}) and the twist condition will be obtained by proving that there is some gap between the periods of the orbits at the boundary of the annulus.
In order to prove such a twist condition, a convenient approach is that of proving the monotonicity of the period map.


P0

An annulus filled by periodic orbits around an equilibrium point . . .



. . . or annulus filled by periodic orbits near infinity.

Monotone period-map

Monotone period-map

The search of sufficient conditions on the vector field guaranteeing that the period map is monotone, is still today a topic of great interest, as one can see from the large number of articles in this area.

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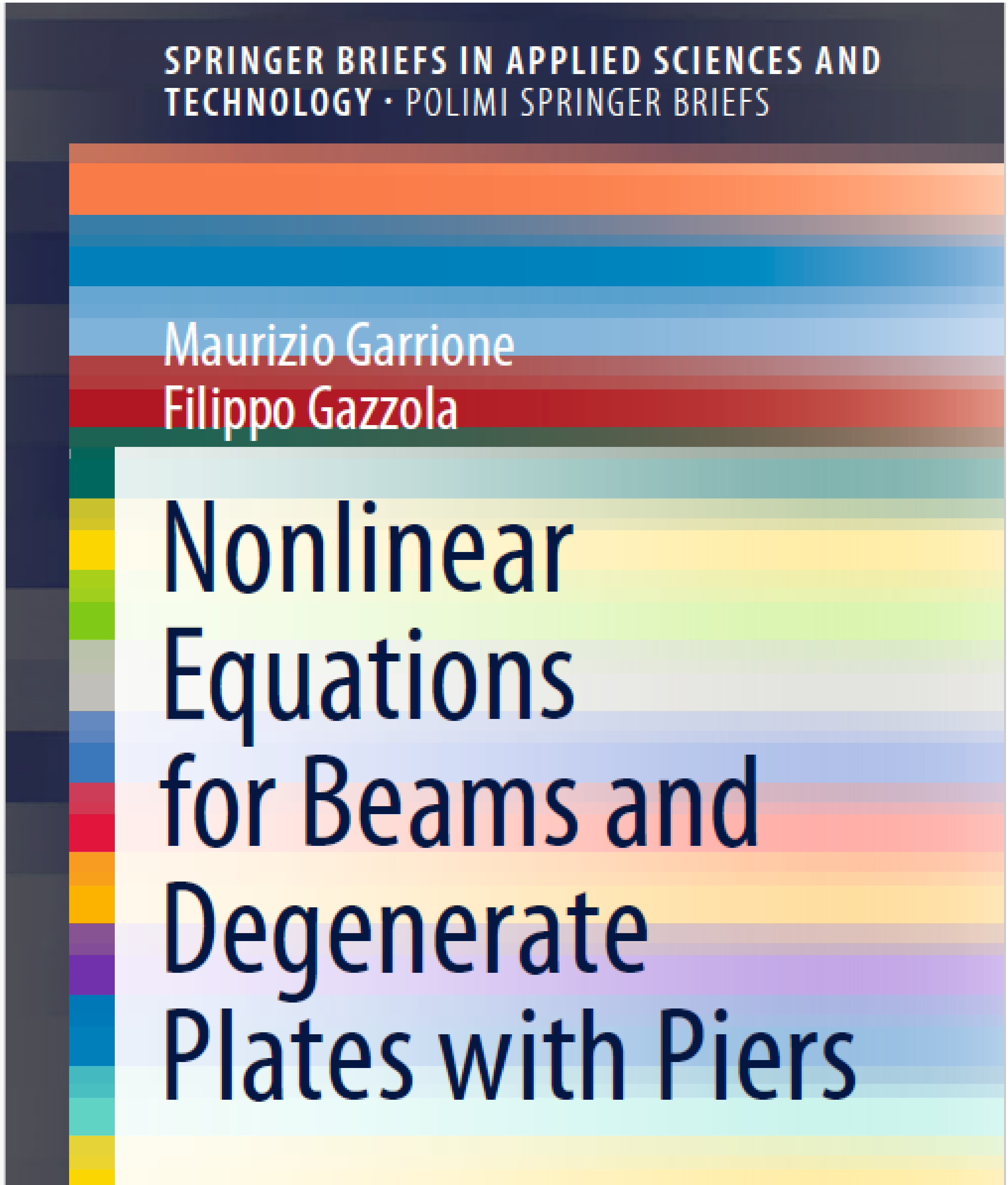
5. An application to a suspension bridge model

5. An application to a suspension bridge model

$$\begin{cases} u_{tt} + \delta_u u_t + u_{xxxx} + \sigma\Big(\displaystyle\int_I (u^2+\theta^2)\Big)u + 2\sigma\Big(\displaystyle\int_I u\theta\Big)\theta + f(u+\theta) + f(u-\theta) = p_u(x,t) \\ \theta_{tt} + \delta_\theta \theta_t - \theta_{xx} + 2\sigma\Big(\displaystyle\int_I u\theta\Big)u + \sigma\Big(\displaystyle\int_I (u^2+\theta^2)\Big)\theta + f(u+\theta) - f(u-\theta) = p_\theta(x,t), \end{cases}$$


SPRINGER BRIEFS IN APPLIED SCIENCES AND TECHNOLOGY · POLIMI SPRINGER BRIEFS
Maurizio Garrione
Filippo Gazzola
Nonlinear Equations for Beams and Degenerate Plates with Piers

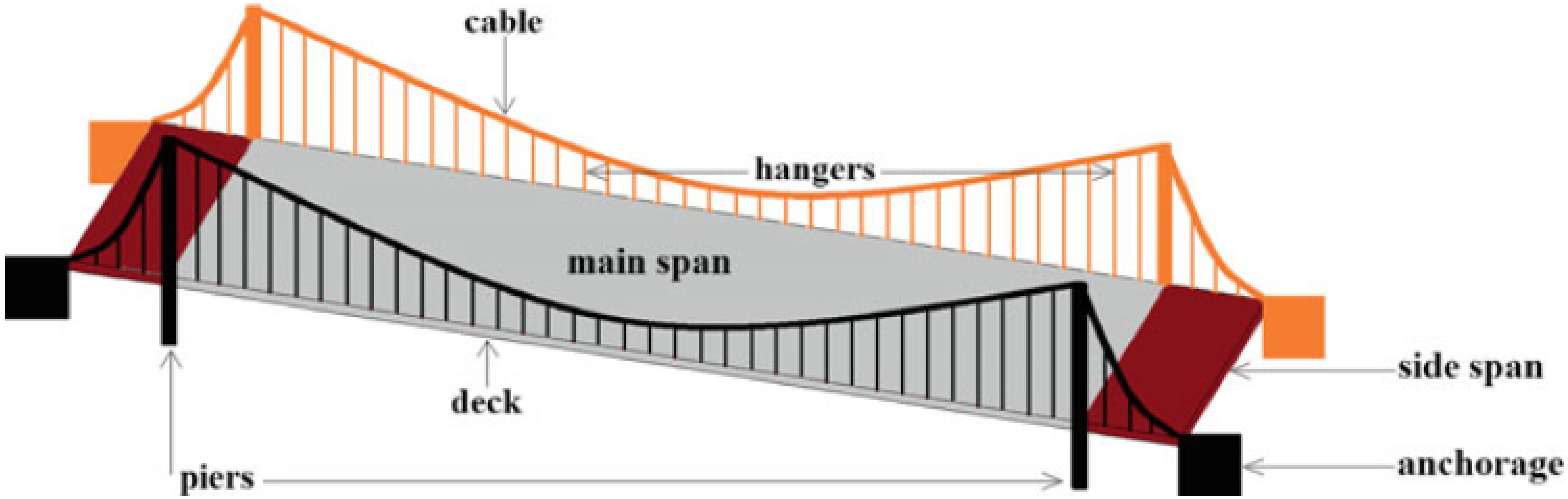


Fig. 1 Sketch of a suspension bridge

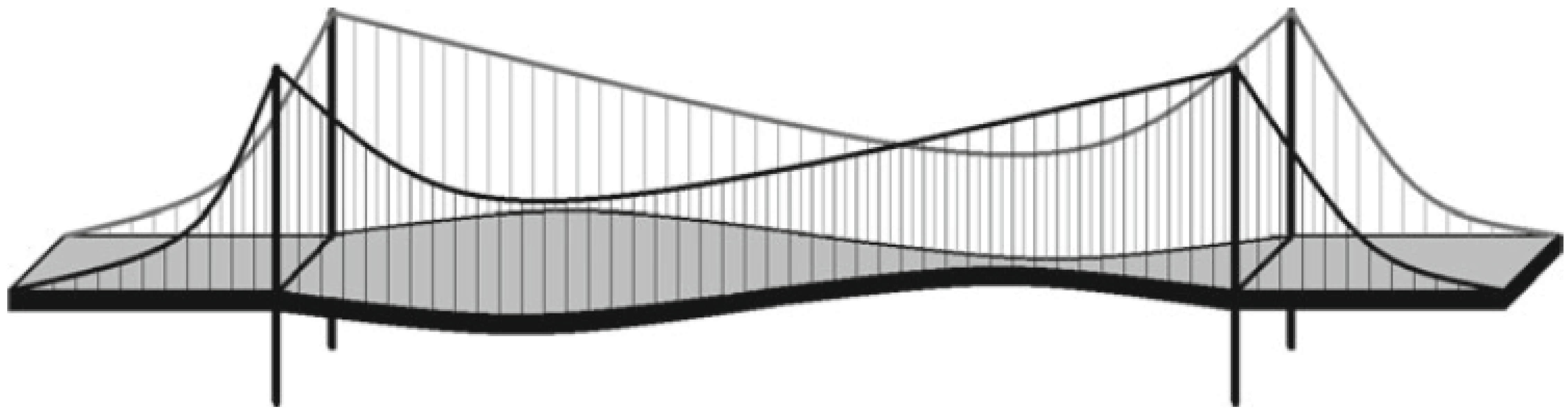


Fig. 1.2 Qualitative behavior of the oscillations at the TNB the day of the collapse

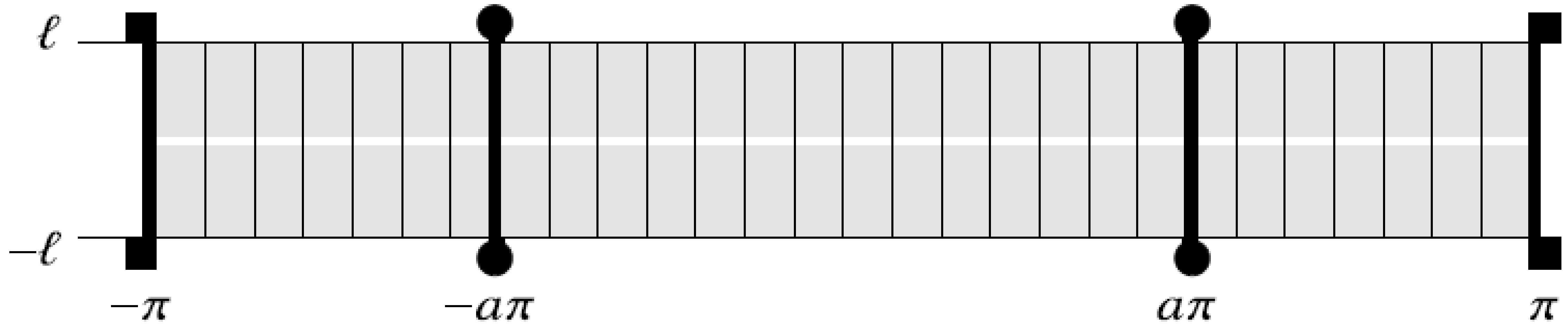


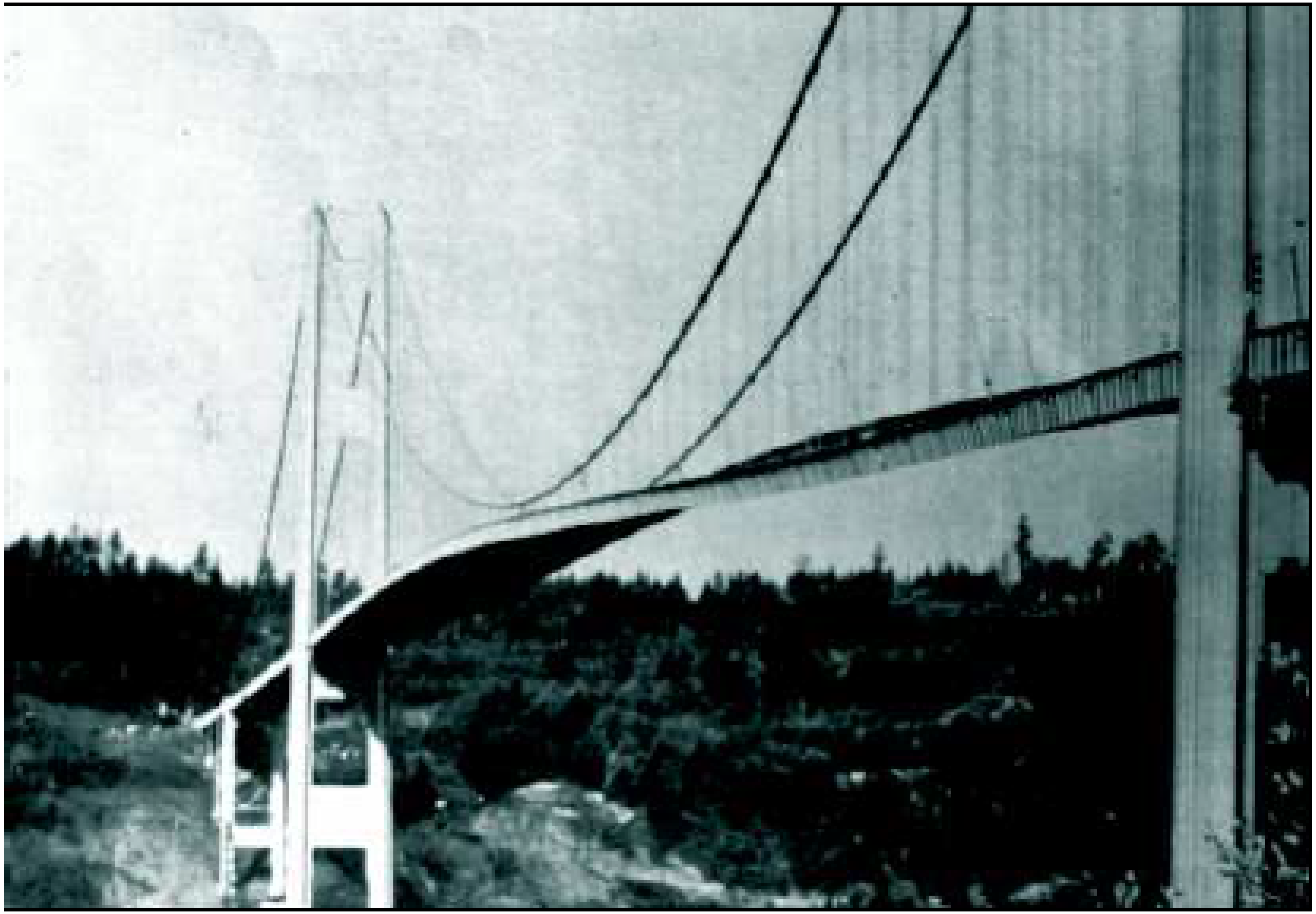
Fig. 1.3 A fish-bone model for a bridge with piers

This energy then leads to the following *nonlocal* system, for $x \in I$ and $t > 0$:

$$\begin{cases} Mu_{tt} + EIu_{xxxx} + 2\gamma\Big(\int_I (u^2+\theta^2)\Big)u + 4\gamma\Big(\int_I u\theta\Big)\theta + f(u+\theta) + f(u-\theta) = 0 \\ \frac{M}{3}\theta_{tt} - \mu\theta_{xx} + 4\gamma\Big(\int_I u\theta\Big)u + 2\gamma\Big(\int_I (u^2+\theta^2)\Big)\theta + f(u+\theta) - f(u-\theta) = 0. \end{cases} \tag{1.8}$$

If $\gamma = 0$ and f is linear, then system (1.8) decouples into two linear equations. The definition of *weak solution* of (1.8) will be given in Sect. 4.2.

In a slightly different setting, involving mixed space-time fourth order derivatives, a linear version of (1.8) with coupling terms was suggested by Pittel–Yakubovich [18], see also [23, p. 458, Chap. VI]. A nonlinear f was considered in (1.8) by Holubová–Matas [16], who were able to prove well-posedness for a forced-damped version of (1.8). Also in [5] the well-posedness of an initial-boundary value problem for (1.8) is proved for a wide class of nonlinear forces f. The fish-bone model described by (1.8), with *nonlinear* f, is able to display a possible transition between vertical and torsional oscillations within the main span: the former are described by u whereas the latter are described by θ.



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001

LARGE-AMPLITUDE PERIODIC OSCILLATIONS IN SUSPENSION BRIDGES: SOME NEW CONNECTIONS WITH NONLINEAR ANALYSIS*

A. C. LAZER† AND P. J. MCKENNA‡

Abstract. This paper surveys an area of nonlinear functional analysis and its applications. The main application is to the existence and multiplicity of periodic solutions of a possible mathematical models of nonlinearly supported bending beams, and their possible application to nonlinear behavior as observed in large-amplitude flexings in suspension bridges. A second area, periodic flexings in a floating beam, also nonlinearly supported, is covered at the end of the paper.

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https://doi.org/10.1186/s13661-023-01801-7

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Existence of exponential attractors for the coupled system of suspension bridge equations

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Abstract

In this paper, we investigate the asymptotic behavior of the coupled system of suspension bridge equations. Under suitable assumptions, we obtain the existence of exponential attractors by using the decomposing technique of operator semigroup.

Keywords: Coupled suspension bridge equations; Decomposition of operator; Exponential attractors

1 Introduction

In the paper, we consider the following system, which describes the vibrating beam equation coupled with a vibrating string equation:

$$\begin{cases} u_{tt} + \alpha u_{xxxx} + \delta_1 u_t + k(u-v)^+ + f_B(u) = h_B, & \text{in } (0,L) \times \mathbb{R}^+, \\ v_{tt} - \beta v_{xx} + \delta_2 v_t - k(u-v)^+ + f_S(v) = h_S, & \text{in } (0,L) \times \mathbb{R}^+ \end{cases} \tag{1}$$

ELSEVIER

Journal of Computational and Applied Mathematics 52 (1994) 113–140

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Periodic oscillations for a nonlinear suspension bridge model

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[b] *Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovak Republic*
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Received 20 April 1992

Abstract

We look for time-periodic solutions of the suspension bridge equation. Lazer and McKenna showed that for a certain configuration of the parameters, one may expect the existence of large-amplitude periodic solutions having the same period as the forcing term. We prove the existence of large-scale subharmonic solutions.

Keywords: Periodic solutions; Poincaré–Birkhoff; Suspension bridges

The model we want to study is analogous to Lazer and McKenna's. Let $v(t, x)$ be the downward displacement of the bridge at the point x and time t, and denote, for any real number α, by α^+ its positive part (i.e., α^+ is equal to α when α is positive, and to 0 when α is negative). We consider the partial differential equation

$$m\frac{\partial^2 v}{\partial t^2} + d\frac{\partial v}{\partial t} + EI\frac{\partial^4 v}{\partial x^4} + \kappa[v+h]^+ = mg + F(t, x),$$

$$u''(t) + \delta u'(t) + \lambda u(t) + g\left[\left(h^{-1}u(t) + 1\right)^{+} - 1\right] = e(t). \tag{1.1}$$

We will look for time-periodic solutions of (1.1). Lazer and McKenna already showed that for a certain configuration of the parameters one may expect the existence of large-amplitude periodic solutions having the same period of the forcing term $e(t)$. Our attention will instead be directed in proving the existence of large-amplitude subharmonic solutions, i.e., periodic solutions having as period an integer multiple of the forcing's period.

We will prove that, if $e(t)$ is a periodic function with mean value zero, $\delta = 0$ and λ is small enough, the above equation has large-amplitude subharmonic solutions. The appearance of this type of solutions is not related to the period nor to the amplitude of the forcing $e(t)$, and in this regard they seem to well simulate the behaviour of oscillating bridges. By a numerical simulation we will show that, for a long and flexible bridge, the coefficient λ is in fact sufficiently small, and subharmonic solutions can be seen. Notice that subharmonic solutions for an equation like (1.1) had already been observed numerically in [31] by a different approach, and a theoretical explanation was asked for this phenomenon.

$$\begin{cases} u_{tt} + \alpha u_{xxxx} + \delta_1 u_t + k(u-v)^+ + f_B(u) = h_B, & \text{in } (0,L) \times \mathbb{R}^+, \\ v_{tt} - \beta v_{xx} + \delta_2 v_t - k(u-v)^+ + f_S(v) = h_S, & \text{in } (0,L) \times \mathbb{R}^+ \end{cases}$$

$$\begin{cases} u_{tt} + \delta_u u_t + u_{xxxx} + \sigma\Big(\int_I (u^2+\theta^2)\Big)u + 2\sigma\Big(\int_I u\theta\Big)\theta + f(u+\theta) + f(u-\theta) = p_u(x,t) \\ \theta_{tt} + \delta_\theta \theta_t - \theta_{xx} + 2\sigma\Big(\int_I u\theta\Big)u + \sigma\Big(\int_I (u^2+\theta^2)\Big)\theta + f(u+\theta) - f(u-\theta) = p_\theta(x,t), \end{cases}$$

One of the possible issues related to system (1) is then to study the energy transfers between different Fourier components. Indeed, it is well understood that in the famous Tacoma Narrows Bridge collapse in 1940 a crucial role was played by a sudden switch from a longitudinal to a torsional oscillation. This corresponds to some *instability* of the structure which, from an analytic point of view, can be effectively highlighted starting from the analysis of bi-modal solutions, having the form

$$(u(x,t),\theta(x,t)) = \big(w(t)e_\lambda(x), z(t)\eta_\kappa(x)\big), \tag{4}$$

for fixed eigenvalues λ and κ of (2) and (3), respectively. In this way, (1) is reduced to the 2×2 ODE system

$$\begin{cases} \ddot{w}(t) + \delta_u \dot{w}(t) + \lambda^4 w(t) + \sigma(1+2A_{\lambda,\kappa}^2)z(t)^2 w(t) + \sigma w(t)^3 + \Gamma_{\lambda,\kappa}(w(t),z(t)) = \displaystyle\int_I p_u(x,t)e_\lambda(x)\,dx \\ \ddot{z}(t) + \delta_\theta \dot{z}(t) + \kappa^2 z(t) + \sigma(1+2A_{\lambda,\kappa}^2)w(t)^2 z(t) + \sigma z(t)^3 + \Xi_{\lambda,\kappa}(w(t),z(t)) = \displaystyle\int_I p_\theta(x,t)\eta_\kappa(x)\,dx, \end{cases} \tag{5}$$

where

$$A_{\lambda,\kappa} = \int_I e_\lambda(x)\eta_\kappa(x)\,dx$$

and

$$\Gamma_{\lambda,\kappa}(w,z) := \int_I [f(we_\lambda(x)+z\eta_\kappa(x)) + f(we_\lambda(x)-z\eta_\kappa(x))]e_\lambda(x)\,dx, \tag{6}$$

$$\Xi_{\lambda,\kappa}(w,z) := \int_I [f(we_\lambda(x)+z\eta_\kappa(x)) - f(we_\lambda(x)-z\eta_\kappa(x))]\eta_\kappa(x)\,dx.$$

2 Uni-modal solutions and their period maps

In this section, we consider the solutions of system (5) which satisfy $z \equiv 0$, corresponding via (4) to purely longitudinal uni-modal solutions of (1). In this case, w satisfies the Duffing equation (8), which henceforth we write as

$$\ddot{w} + \delta_u \dot{w} + \lambda^4 w + \sigma w^3 + \Gamma_{f,\lambda}(w) = \int_I p_u(x,t)\, e_\lambda(x), \tag{9}$$

where

$$\Gamma_{f,\lambda}(w) = 2 \int_I f(we_\lambda(x)) e_\lambda(x)\, dx. \tag{10}$$

In particular, in case $p_u \equiv 0$, we are dealing with the homogeneous damped equation

$$\ddot{w} + \delta_u \dot{w} + \lambda^4 w + \sigma w^3 + \Gamma_{f,\lambda}(w) = 0. \tag{11}$$

In order to prove the onset of chaotic dynamics for both large and small solutions of (9), in presence of a suitable stepwise forcing term p_u, it will be crucial to analyze the properties of the time map for the associated unforced and undamped equation

$$\ddot{w} + \lambda^4 w + \sigma w^3 + \Gamma_{f,\lambda}(w) = 0, \tag{12}$$

with $\Gamma_{f,\lambda}$ the transform of a function f satisfying (F). As suggested by Remark 6, such features cannot be fully deduced from the corresponding ones for the equation

$$\ddot{w} + \lambda^4 w + \sigma w^3 + 2f(w) = 0; \tag{13}$$

this point indeed deserves some further comments, which we here provide. Let us first recall that (12)

Proposition 8. *Let e_λ be an odd eigenfunction of problem* (2). *If*

$$f''(0)^2 > \frac{3}{5}\left(\frac{\lambda^4}{2} + f'(0)\right)(f'''(0) + 3\sigma), \quad \textit{with} \;\; f'''(0)\int_I e_\lambda^4(x)\,dx + 3\sigma > 0, \tag{14}$$

then the time map associated with (13) *is locally increasing near the origin, while the time map associated with* (12) *is locally decreasing near the origin.*

6. Chaotic dynamics of longitudinal motions

6. Chaotic dynamics of longitudinal motions

We suppose that p_u is a stepwise function of time, acting on a single longitudinal mode, and we analyze the behavior of the corresponding purely longitudinal $(\theta(x,t) \equiv 0)$ uni-modal solution of

$$\begin{cases} u_{tt} + \delta_u u_t + u_{xxxx} + \sigma\Big(\int_I (u^2+\theta^2)\Big)u + 2\sigma\Big(\int_I u\theta\Big)\theta + f(u+\theta) + f(u-\theta) = p \\ \theta_{tt} + \delta_\theta \theta_t - \theta_{xx} + 2\sigma\Big(\int_I u\theta\Big)u + \sigma\Big(\int_I (u^2+\theta^2)\Big)\theta + f(u+\theta) - f(u-\theta) = p_\theta(x \end{cases} \tag{9}$$

according to the properties of p_u.

This leads us to the study of the periodically perturbed Duffing equation

$$w'' + g(w) = p(t). \tag{10}$$

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$$w'' + g(w) = p(t). \tag{10}$$

As for the forcing term p **in** (10)**, we assume that it is given by**

$$p(t) = p_{1,2}(t) = \begin{cases} p_1, \ t \in [0, T_1) \\ p_2, \ t \in [T_1, T], \end{cases} \tag{11}$$

for $p_1, p_2 \in (g(s_0), +\infty)$**,** $p_1 \neq p_2$ **and** $T = T_1 + T_2$**, giving rise to the two corresponding equilibrium points** $(w_{p_1}, 0)$**,** $(w_{p_2}, 0)$.

The first order planar system equivalent to (10) turns then out to be the switched system

$$\begin{cases} \dot{w} = y \\ \dot{y} = -g(w) + p_{1,2}(t), \end{cases} \tag{12}$$

namely the dynamics is that of two autonomous planar systems with global centers at $(w_{p_1}, 0)$ and $(w_{p_2}, 0)$, respectively, which switch back and forth in a periodic fashion along time intervals of lengths T_1, T_2, respectively.

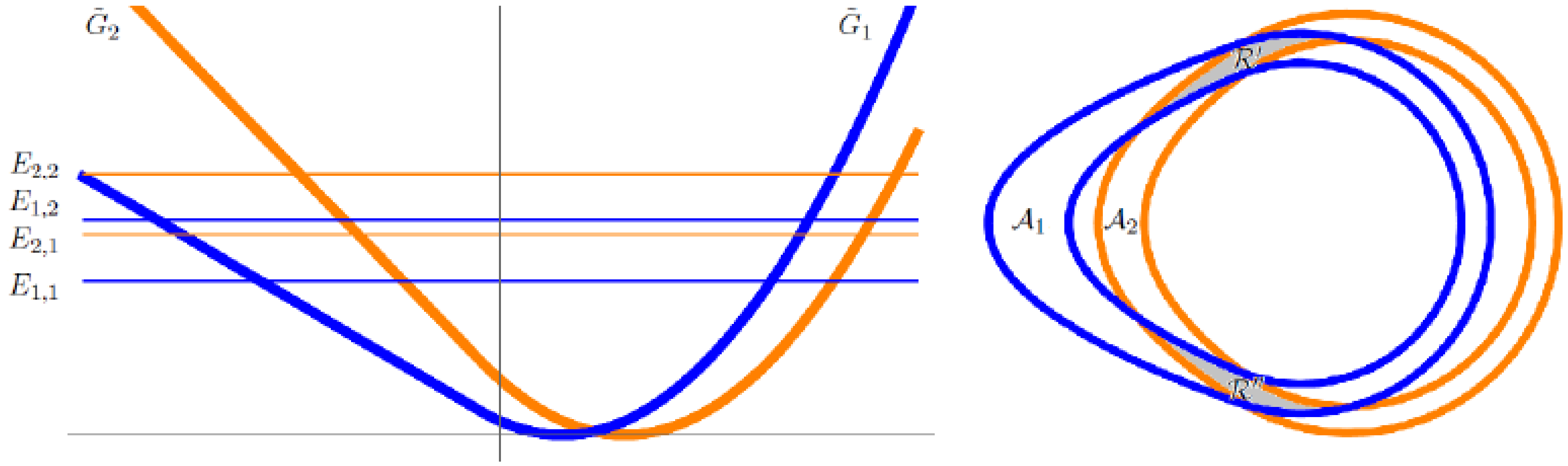


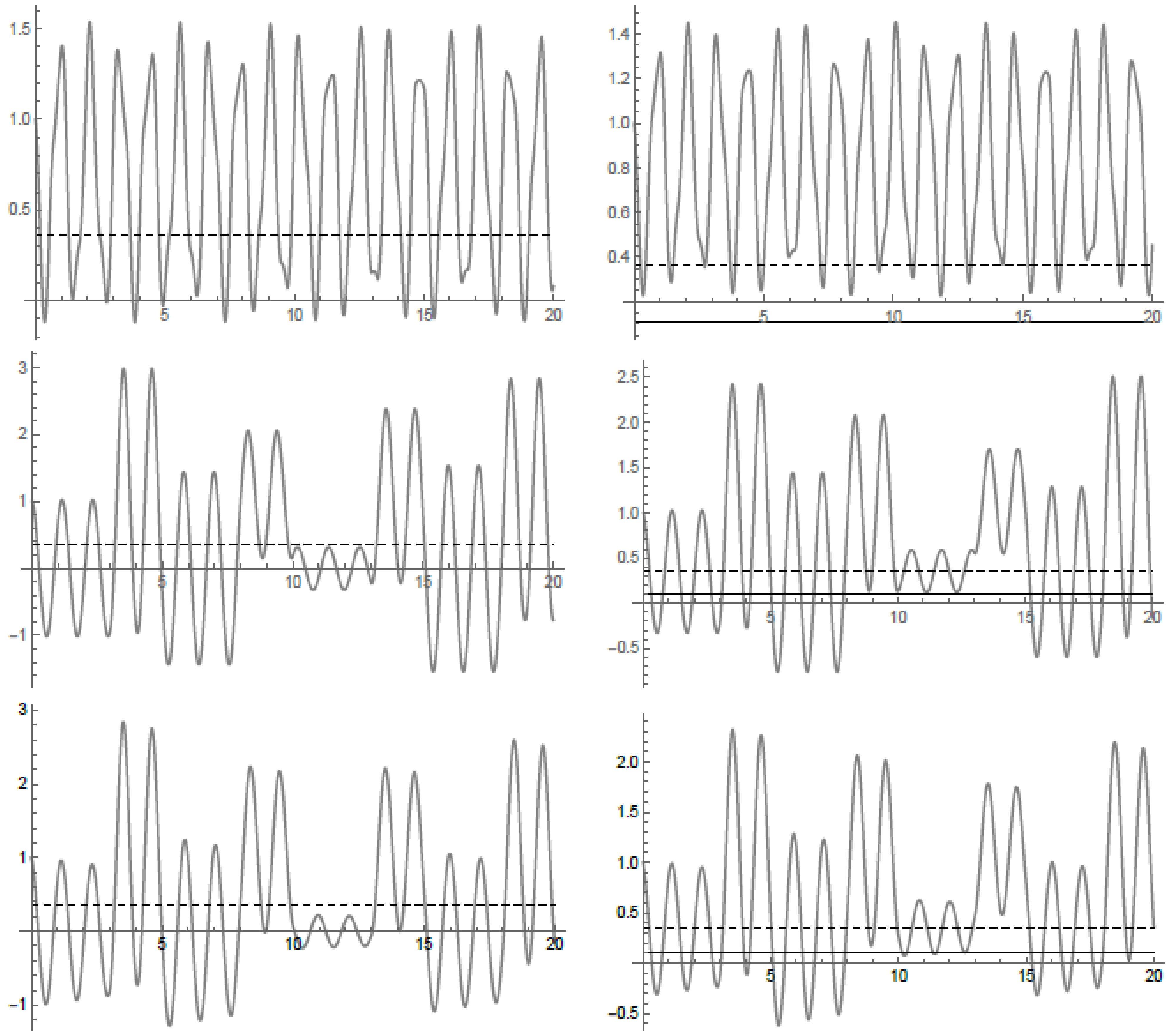
Figure 4: A typical geometry of linked annuli: on the left, the graphs of $\tilde{G}_1$ and $\tilde{G}_2$, on the right the resulting annular regions in the phase plane.

Theorem 2 ***Let*** $\mathcal{A}_1$ ***and*** $\mathcal{A}_2$ ***be two linked annuli as above and suppose that*** $\tau_{i,1} \neq \tau_{i,2}$ ***for*** $i = 1, 2.$ ***Then, given*** $n_1 \geq 1$ ***and*** $n_2 \geq 1$ ***with*** $m = n_1 n_2 \geq 2,$ ***there exist*** T_1^* ***and*** T_2^* ***such that, for each*** $T_1 > T_1^*$ ***and*** $T_2 > T_2^*$***, the Poincaré map*** $\Phi^{[0,T]}$ ***associated with system*** (12) ***induces chaotic dynamics on*** m ***symbols in each of the sets*** $\mathcal{R}'$ ***and*** $\mathcal{R}''$***.***

An analysis of the proof shows that our main result is robust with respect to small perturbations in the terms appearing in the equation. In particular, we can give an application to

$$w'' + \delta w' + h(w, z) = q(t),$$

where, for a suitably small $\varepsilon > 0$, it is assumed that: $0 < \delta < \varepsilon$, $h(w, z) = g(w) + r(w, z)$, where g is a function fulfilling assumption (G) and $\|r\|_{L^\infty(\mathbb{R}^2)} < \varepsilon$, and $q : \mathbb{R} \to \mathbb{R}$ is a T-periodic function (possibly smooth) with $\int_0^T |q(t) - p_{1,2}(t)|\, dt < \varepsilon$. The constant $\varepsilon > 0$ depends on the pair (T_1, T_2) and on the configuration of the linked annuli.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Thank you for your attention!