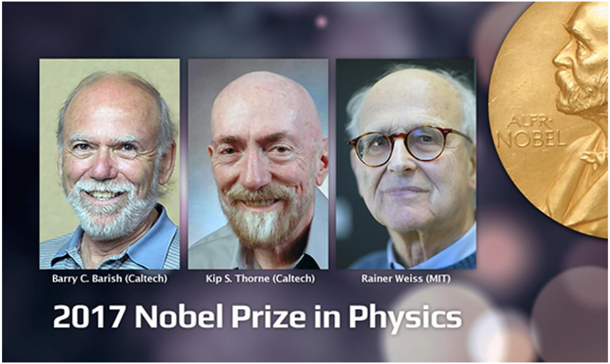
The nonlinear stability of black holes: an overview

Elena Giorgi[1]

[1] Columbia University

IMDETA Lecture, June 2026

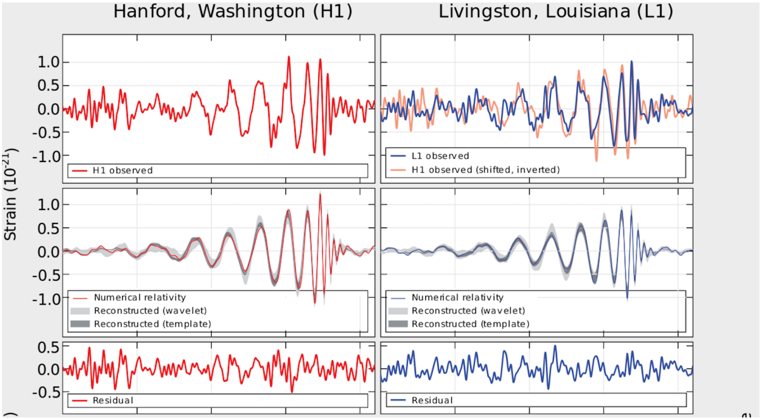
Why studying black holes?



"for decisive contributions to the LIGO detector and the observation of gravitational waves"

Gravitational waves finally captured

On 14 September 2015, the universe's gravitational waves were observed for the very first time. The waves, which were predicted by Albert Einstein a hundred years ago, came from a collision between two black holes. It took 1.3 billion years for the waves to arrive at the LIGO detector in the USA.


Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual

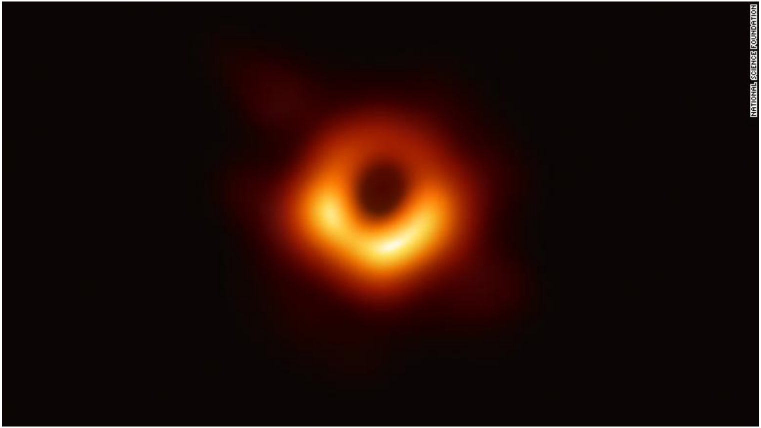
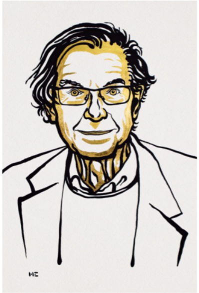


Figure: Event Horizon Telescope, April 2019



Roger Penrose

Prize share: 1/2

Reinhard Genzel

Prize share: 1/4

Andrea Ghez

Prize share: 1/4

"for the discovery that black hole formation is a robust prediction of the general theory of relativity"

"for the discovery of a supermassive compact object at the centre of our galaxy"

General Relativity and the Einstein equation

According to Einstein's theory of General Relativity, a **spacetime** is a 4-dimensional manifold $\mathcal{M}$ equipped with a Lorentzian metric g satisfying the **Einstein field equations**

$$\mathrm{Ric}(g) - \frac{1}{2}\,\mathrm{R}(g)\; g = \mathrm{T}_{\mathrm{matter}}\,.$$

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$$g_0 = -dt^2 + dx^2 + dy^2 + dz^2.$$

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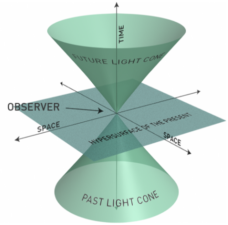
A tangent vector v is either

1. timelike, $g(v,v) < 0$
2. null, $g(v,v) = 0$
3. spacelike, $g(v,v) > 0$.



General Relativity predicts the existence of **black hole spacetimes**.

These are spacetimes whose metric approaches Minkowski "at infinity" (*asymptotically flat*) and that contain a region $\mathcal{BH}$ such that observers and signals originating there cannot reach infinity.

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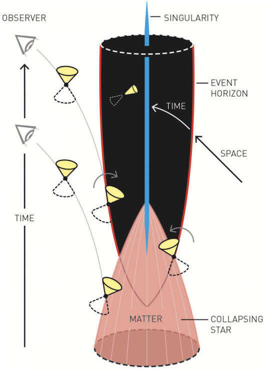
The topological boundary of the black hole region is a null hypersurface called the **event horizon** $\mathcal{H}^+$.



Penrose Diagrams

Penrose diagrams are $(1 + 1)$-dimensional representation of spacetimes.

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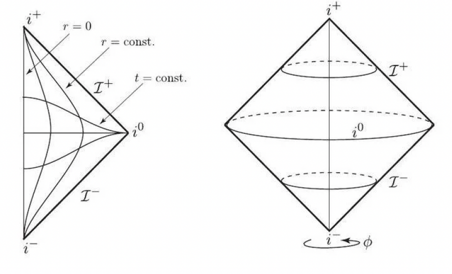
Minkowski:



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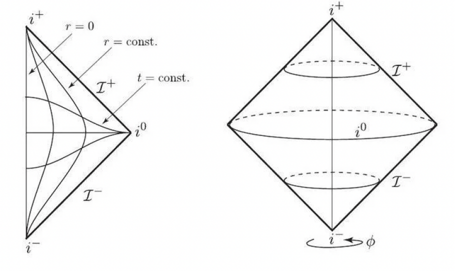
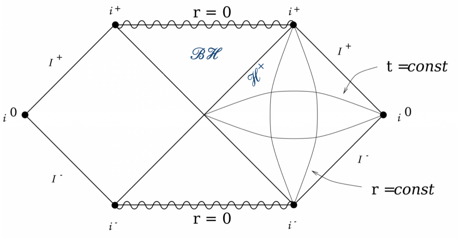


Black hole:



Schwarzschild g_M (1916), for $M \in \mathbb{R}$:

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

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and **Kerr** $g_{M,a}$ (1963), for $|a| \leq M$:

$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2 + a^2)d\phi\right)^2$$

where $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$,

are solutions to the Einstein vacuum equation

$$\mathrm{Ric}(g) = 0.$$

The Einstein equation can be formulated as a quasilinear **hyperbolic system of PDEs**, which in wave coordinates $\Box_g x^\mu = 0$ takes the form

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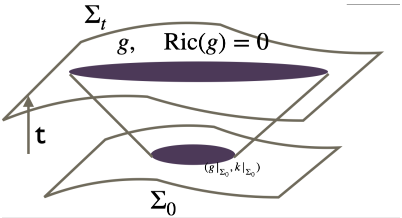
Theorem (Choquet-Bruhat 1952, Choquet-Bruhat-Geroch 1969)

Initial data set for the Einstein equation give rise to a local-in-time smooth solution and such solution is geometrically unique.



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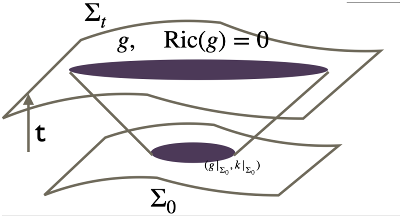

The issue of **stability** of black holes starts with the local well-posedness guaranteed by Choquet-Bruhat's theorem and asks:

what is the long time behavior of perturbed solutions to the Einstein equation?

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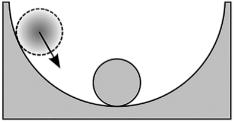

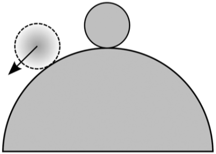

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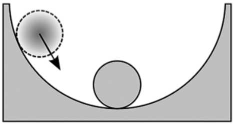

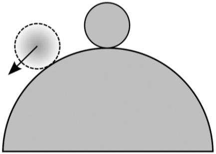

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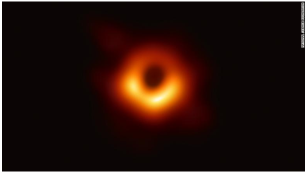

Is it Kerr?

It is an empirical observation that the astrophysical black holes observed in space are stable, but the Kerr family can truthfully be used to represent real black holes only if it is stable.

Translating in PDE language

Let the Einstein equation be represented by the non-linear operator

$$\mathcal{P}[\phi] = 0, \tag{1}$$

and let ϕ_λ a family of stationary solutions to (1), i.e. $\mathcal{P}[\phi_\lambda] = 0$.

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There are various levels of increasing difficulty for the stability problem:

1. Consider the **linearized** equation

$$(d\mathcal{P})|_{\phi_\lambda}(\psi) = 0 \tag{2}$$

and prove that

1a) separated solutions of (2), of the form $\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta)$, do not exponentially grow in time: **mode stability**

1b) all solutions of (2) decay in time: **linear stability**

2. all solutions of the full non-linear equation (1) decay in time: **non-linear stability**.

- **Mode stability**: There are no exponentially growing modes for separated solutions $\psi(t,r,\theta,\phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta)$ of the Einstein equation in Kerr [Whiting 1989, Shlapentokh-Rothman 2015].

- **Scalar wave equation on black hole backgrounds**: General solutions to

$$\Box_g \psi = 0$$

arising from regular initial data remain bounded and decay in time in Schwarzschild [Kay-Wald 1987, Blue-Soffer 2003, Dafermos-Rodnianski 2008], and in Kerr [Dafermos-Rodnianski 2008, Tataru-Tohaneanu 2008, Andersson-Blue 2009, Dafermos-Rodnianski-Shlapentokh-Rothman 2014]

- **Non-linear stability of Minkowski space**: Solutions to the non-linear Einstein vacuum equation which are small perturbations of Minkowski give rise to a complete spacetime which converges to Minkowski space. [Christodoulou-Klainerman 1993, Lindblad-Rodnianski 2004, Bieri 2009]

Scalar wave equation on black holes

The aim is to obtain **energy estimates** for general solutions to

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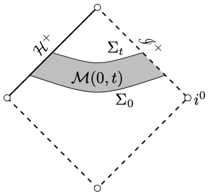
- *energy boundedness*:

$$E[\psi](t) \leq C E[\psi](0), \qquad E[\psi](t) = \int_{\Sigma_t} |\partial\psi|^2$$

- *integrated local energy decay (Morawetz) estimate*:

$$\mathrm{Mor}[\psi](0,t) \leq C E[\psi](0), \qquad \mathrm{Mor}[\psi](0,t) = \int_{\mathcal{M}(0,t)} |\partial\psi|^2$$



In Minkowski space $\mathbb{R}^{1+3}$, solutions to $\Box_{g_0}\psi = 0$ conserve energy:

$$0 = \Box_{g_0}\psi \cdot \partial_t\psi = \big(-\partial_t^2\psi + \triangle\psi \big) \cdot \partial_t\psi = -\frac{1}{2}\partial_t\big(|\partial_t\psi|^2 + |\nabla\psi|^2\big).$$

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In general, we apply the divergence theorem to certain energy currents, constructed from the *energy-momentum tensor*:

$$\mathcal{Q}[\psi]_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2} g_{\mu\nu}\partial_\lambda\psi\partial^\lambda\psi.$$

The divergence of the energy-momentum tensor satisfies

$$D^\mu \mathcal{Q}[\psi]_{\mu\nu} = \partial_\nu\psi \cdot \Box_g\psi.$$

If $\Box_g\psi = 0$, then $D^\mu \mathcal{Q}[\psi]_{\mu\nu} = 0$.

For a vectorfield X, one can construct the current

$$\mathcal{P}_\mu^{(X)} = \mathcal{Q}[\psi]_{\mu\nu} X^\nu.$$

The divergence of the current satisfies

$$D^\mu \mathcal{P}_\mu^{(X)} = X(\psi) \cdot \Box_g \psi + \mathcal{Q}[\psi] \cdot \mathcal{L}_X g.$$

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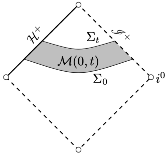
By the divergence theorem,



$$\int_{\Sigma_t} \mathcal{P}^{(X)}_{\mu} n^{\mu}_{\Sigma_t} + \int_{\mathcal{M}(0,t)} D^{\mu}\mathcal{P}^{(X)}_{\mu} = \int_{\Sigma_0} \mathcal{P}^{(X)}_{\mu} n^{\mu}_{\Sigma_0},$$

Goal: combine vectorfields X such that

$$\int_{\Sigma_t} \mathcal{P}^{(X)}_{\mu} n^{\mu}_{\Sigma_t} \qquad \text{is positive definite, such as for } X = \partial_t$$

$$\int_{\mathcal{M}(0,t)} D^{\mu}\mathcal{P}^{(X)}_{\mu} \qquad \text{is positive definite, such as modifications of } X = \partial_r$$

In Kerr one encounters additional difficulties:

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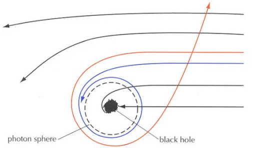
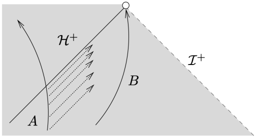
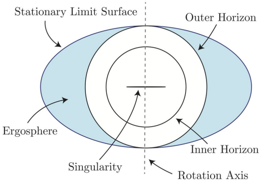
- **Existence of trapped null geodesics.** Remarkably, those trapped null geodesics are unstable.



- **Trapping properties of the event horizon.**
  The *red-shift effect* associated to the event horizon overcomes it[Dafermos-Rodnianski 2005].



- **Superradiance.**
  The stationary Killing field ∂_t is timelike only outside of the *ergoregion*. As a consequence, the associated conserved energy $\int_{\Sigma_t} \mathcal{P}^{(\partial_t)}_{\mu} n^{\mu}_{\Sigma_t}$ fails to be positive definite.



Nevertheless, solutions to the wave equation are stable in Kerr!

Theorem (Dafermos-Rodnianski-Shlapentokh-Rothman (2014))

General solution ψ of $\Box_{g_{M,a}}\psi = 0$ on Kerr for $|a| < M$ arising from bounded initial energy have bounded energy flux. In particular, ψ satisfies uniform pointwise bounds.

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- In Kerr, the trapped null geodesics are not confined on a hypersurface in physical-space: requires a more refined analysis involving both the vectorfield method and Fourier or mode decompositions [Tataru-Tohaneanu 2011, Dafermos-Rodnianski 2011, Dafermos-Rodnianski-Shlapentokh-Rothman 2014].
- For $|a| \ll M$, it is possible to obtain the analysis of the solution in physical-space by extending the vectorfield method to include second order operators. [Andersson-Blue 2009]

Nonlinear stability of Minkowski

Consider the nonlinear scalar wave equation in Minkowski

$$\Box_{g_0}\psi \quad = \quad (\partial_t\psi)^2$$

with smooth compactly supported initial data of size ϵ.

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Perturbations of the trivial solution to $\Box_g g = \mathcal{N}(g, \partial g)$ do not blow up because of the null condition: "bad" nonlinear terms such as $(\partial_t\psi)^2$ are not present in the Einstein equation:

$$\Box\psi = m(d\psi, d\psi), \qquad \text{with} \quad m(\xi,\xi) = 0 \qquad \text{if } g(\xi,\xi) = 0.$$

Nonlinear stability of a black hole family: sample statement

Theorem

Initial data sufficiently close to a member of a *black hole family* *evolve, under the Einstein equations, to a spacetime which:*

- *has a complete future null infinity and a future-complete event horizon,*
- *settles down to a (possibly different) member of the* *same black hole family*.

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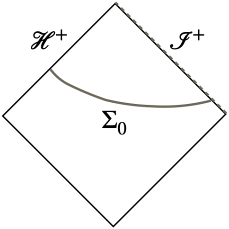

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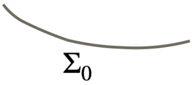

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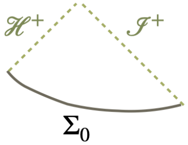

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Proved for:

Schwarzschild for axially symmetric polarized perturbations [Klainerman-Szeftel 2018], and for data which lie on a codimension-3 submanifold of moduli space [Dafermos-Holzegel-Rodnianski-Taylor 2021];

Kerr for $|a| \ll M$ [Klainerman-Szeftel 2019, 2021, Shen 2022, G.-Klainerman-Szeftel 2022]

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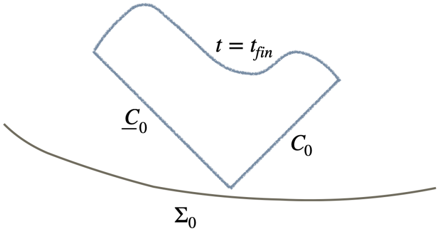

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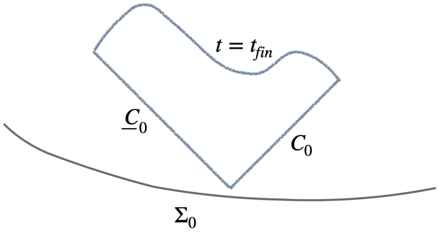

Consider

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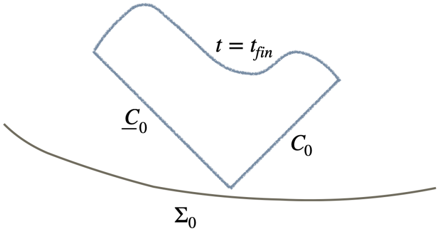

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A subset $\mathcal{B} \subset [0, \infty)$ which is non-empty, open and closed is $[0, \infty)$.

bootstrap [**boot**-strap] SHOW IPA

adjective

3. relying entirely on one's efforts and resources:
   The business was a bootstrap operation for the first ten years.
4. self-generating or self-sustaining:
   a bootstrap process.

Two magic words

bootstrap [**boot**-strap] SHOW IPA

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bootstrap [**boot**-strap] SHOW IPA

adjective

3. relying entirely on one's efforts and resources:
   The business was a bootstrap operation for the first ten years.
4. self-generating or self-sustaining:
   a bootstrap process.

gauge [geyj] SHOW IPA

If g is a solution of the Einstein equation $\mathrm{Ric}(g) = 0$, then the pullback through any diffeomorphism $\phi^*(g)$ is also an equivalent solution.

- **Bootstrap assumptions** measure in a quantitative way how the bootstrap region is close to the perturbed family of black holes. Schematically

$$\sup_{\mathcal{M}_{fin}} |g - g_{M_f, a_f}|, \quad |\Gamma - \Gamma_{M_f, a_f}|, \quad |R - R_{M_f, a_f}| \leq \epsilon.$$

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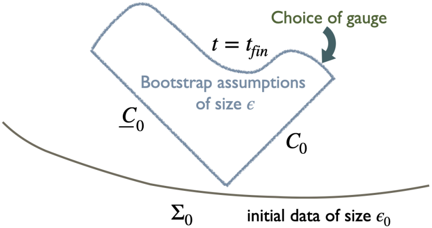

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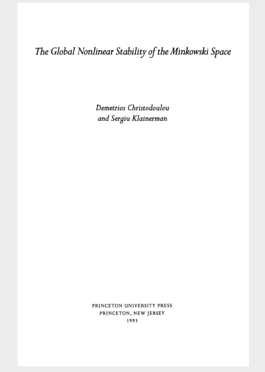
How to study the Einstein equation?

Alternatively to Choquet-Bruhat's approach, we can use that

$$\mathrm{Ric}(g) = 0 \iff D_{[\alpha} \mathrm{Riem}_{\beta\gamma]\delta\epsilon} = 0, \quad \mathrm{div}\ \mathrm{Riem} = 0.$$

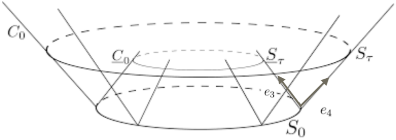
Also, instead of using coordinates, we can use "frames", i.e. vectorfields geometrically defined along the manifold.


The Global Nonlinear Stability of the Minkowski Space

Demetrios Christodoulou
and Sergiu Klainerman

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY
1993

(M, g) can be foliated by spacelike 2-spheres $(S, \not{g})$, and to each point of M, we can associate a null frame $\{e_3, e_4, e_a\}$, with $\{e_3, e_4\}$ null vectors and $\{e_1, e_2\}$ being orthonormal tangent vectors to $(S, \not{g})$.



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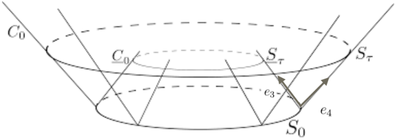

Project all geometric quantities along the null frame $\{e_3, e_4\}$, and obtain tensors on the spheres S, such as

- Riemann curvature

$$\alpha_{ab} : \quad = R(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} := R(e_a, e_3, e_b, e_3), \qquad \ldots$$

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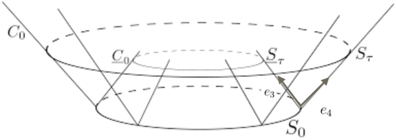

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Then project the Einstein equation $D_{[\alpha} \, \mathrm{Riem}_{\beta\gamma]\delta\epsilon} = 0, \quad \mathrm{div} \;\, \mathrm{Riem} = 0$ to the spheres S, and obtain **many** tensorial equations on the sphere.

The Kerr family admits a special null frame, called *principal null frame*, which diagonalizes the curvature. This means that for a particular choice of $\{e_3^{PN}, e_4^{PN}\}$, then

$$R_{a4b4} = R_{a3b3} = R_{a343} = R_{a434} = 0, \qquad R_{abcd}, R_{3434} \neq 0$$

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In fact, they simplify dramatically and become tractable. More precisely, the symmetric 2-tensors on the spheres

$$\alpha_{ab} = R_{a4b4}, \qquad \underline{\alpha}_{ab} = R_{a3b3}$$

satisfy a **second order PDE** which is wave-like and decouples from all the other components in the linearization.

The Teukolsky equation

The curvature components

$$\alpha_{ab} \quad = R_{a4b4}, \qquad \underline{\alpha}_{ab} = R_{a3b3}$$

- vanish in Kerr in the *principal null frame*,
- satisfy a wave equation which is decoupled by all the other quantities in linear theory[Teukolsky 1972] , i.e.

$$\boxed{\mathcal{T}(\alpha) := \Box_g \alpha + c_1(r,\theta)\nabla_{\partial_t}\alpha + c_2(r,\theta)\nabla_{\partial_\phi}\alpha + V(r,\theta)\alpha = 0}$$

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Strategy to improve the bootstrap assumptions of all curvature and metric components of $\mathcal{M}_{fin}$:

1. first improve the norms for "almost" gauge invariant quantities,
2. then improve them for the remaining quantities.

Due to the first order terms, one cannot obtain energy boundedness for the Teukosky equation directly. Instead, one would like to pass

$$\begin{array}{rcl} \text{Teukolsky equation} & \to & \text{Regge-Wheeler eq} \\ \Box_g \alpha - V\alpha = c_1\partial_r \alpha + c_2 \partial_\phi \alpha + c_3 \partial_t \alpha & \to & \Box_g \psi - V\psi = 0 \end{array}$$

with energy-momentum tensor

$$\mathcal{Q}[\psi]_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \big(\partial_\lambda \psi \partial^\lambda \psi + V|\psi|^2 \big).$$

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There is a transformation [Chandrasekhar '80s] that does exactly that! It consists in taking two null derivatives [Dafermos-Holzegel-Rodnianski 2016]

$$\psi = f_1(r) \nabla_3 \big(f_2(r) \nabla_3 \alpha \big). \tag{3}$$

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Applying results from the wave equation, one can obtain boundedness for ψ and subsequently for α from the differential relation (3).

In nonlinear perturbations of Kerr, the Chandrasekhar transformation gives a **generalized Regge-Wheeler equation** given by:

$$\square_g \psi - V\psi - i\frac{4a\cos\theta}{r^2 + a^2\cos^2\theta}\nabla_{\partial_t}\psi = a \cdot L_\psi[\alpha] + \mathcal{N}[\check{\Gamma}, \check{R}]$$

- The first order term in ∂_t satisfies good divergence properties.
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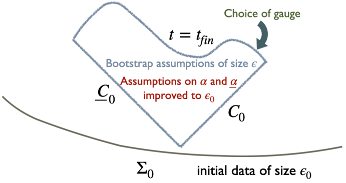
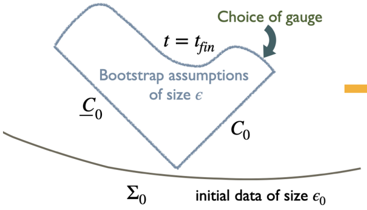
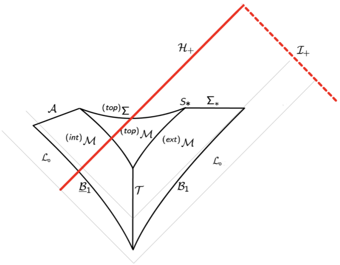
The bootstrap assumptions for α and $\underline{\alpha}$ are improved for $|a| \ll M$, where the interaction between trapping and superradiance is controlled in physical space, but there are promising results to extend it to $|a| < M$ [Teixeira da Costa-Shlapentokh Rothman 2020-2023, Dafermos-Holzegel-Rodnianski-Taylor 2024, Ma-Szeftel 2026]!



The actual picture





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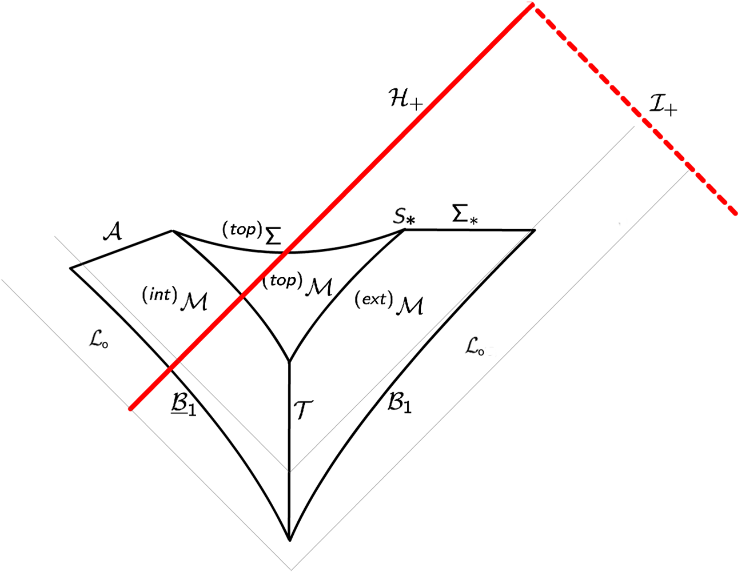
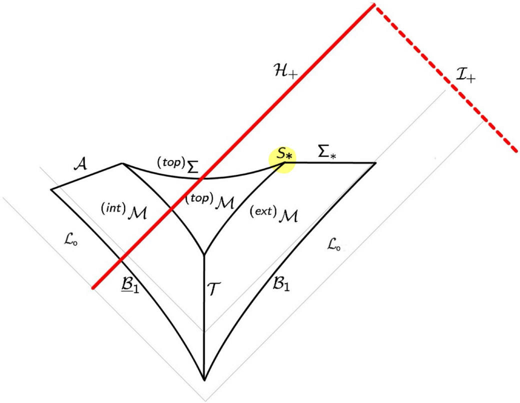


Improvement of the bootstrap assumptions for all the other quantities



- The sphere S_* is an almost-round 2-sphere, unrelated to the initial conditions, on which some geometric quantities vanish. On S_*, we also define the mass M_f and the angular momentum a_f of $\mathcal{M}_{fin}$.

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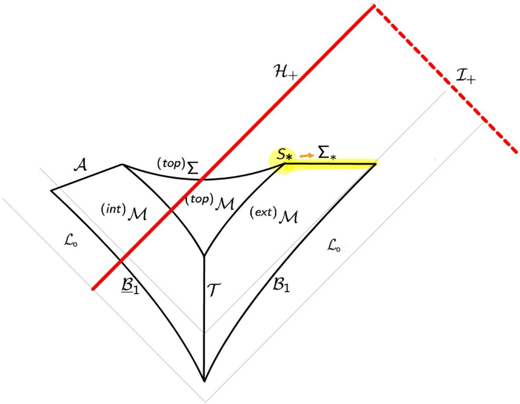
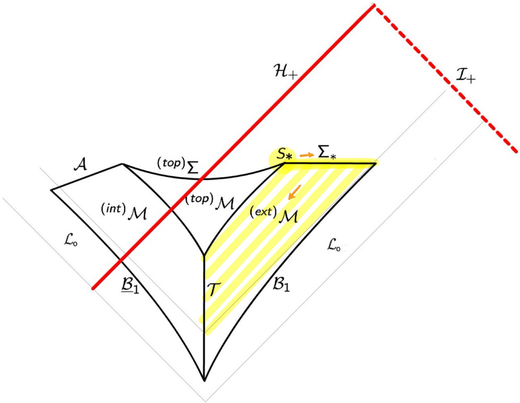


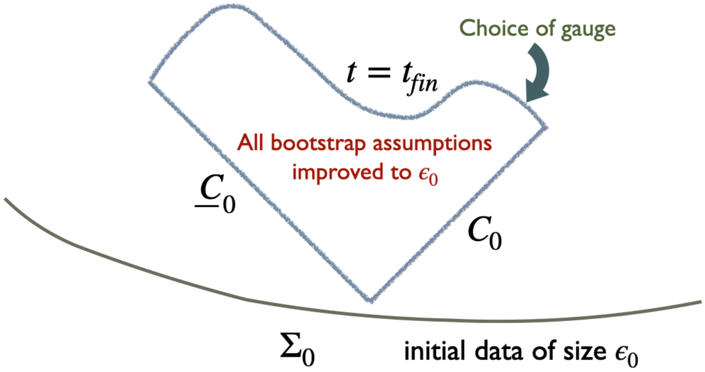
- Σ_* is a spacelike hypersurface, whose sections are spheres on which some geometric quantities are assumed to vanish.

Improvement of the bootstrap assumptions for all the other quantities



- On the asymptotic region and the near horizon region, there exists a hierarchy of renormalized quantities satisfying transport estimates with integrable right hand side.

Conclusion of the proof



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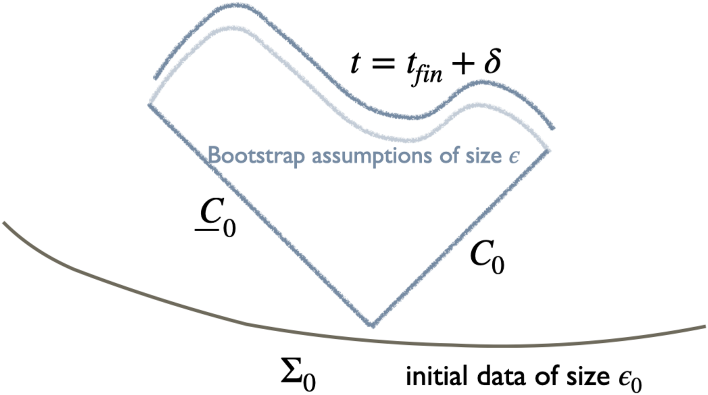

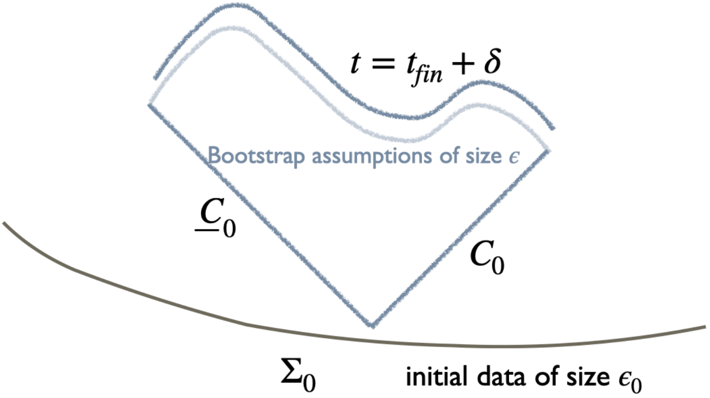

Then $\mathcal{B} = [0, \infty)$, the solution is global, satisfy the estimates obtained and asymptotically converges to g_{M_∞, a_∞}.

Thank you for your attention!