Quasilinear parabolic systems describing taxis and chemical signaling in predator-prey models.

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Seminar at Łódź University of Technology , 12/10/2022

Outline:

- Keller-Segel model of chemotaxis
- A typical prey-predator model
- Chemical signalling
- Direct/indirect taxis & pursuit-evasion models
- Linearisation at the space-homogeneous steady state and space-time patterns formation
- Numerical solutions
- Blow-up versus existence of global-in-time solutions

The talk is based on the joint papers with former ERCIM scholar Purnedu Mishra (at present in Norwegian University of Life Science)

1. Purnedu Mishra, D.W. *Repulsive chemotaxis and predator evasion in predator prey models with diffusion and prey taxis* Math. Models. Methods. Appl. Sciences (M3AS) (2022)
2. Purnendu Mishra, D.W, *Indirect taxis drives spatio-temporal patterns in an extended Schoener's intraguild predator–prey* , Appl. Math. Letters, (2022)
3. Purnendu Mishra, D.W, *Pursuit-evasion dynamics for Bazykin-type predator-prey model with indirect predator taxis*, preprint.

Transport equation

1. V- constant vector field $V : \mathbb{R}^n \to \mathbb{R}^n$
2. $N = N(x, t)$- gęstość , $N : \mathbb{R}^n \times (0\,, \infty) \to \mathbb{R}$

$$N_t = -\nabla \cdot NV\,, \quad N(x,0) = N_0(x)$$

3. The solution :

$$N(x, t) = N_0(x - Vt)$$

The Keller-Segel model of chemotaxis

Patlak (1953) - Keller-Segel model (1972)

$W = W(x,t)$ density of some chemical released by the members of with density $N(x,t)$, $x \in \Omega \subset \mathbb{R}^n$ with smooth boundary

χ-chemotactic sensitivity parameter

$$\begin{cases} N_t = D_N \Delta N + / - \nabla \cdot (\chi N \nabla W) \\ W_t = D_W \Delta W + \gamma N - \mu W \\ \quad \text{with homogeneous Neumanna boundary condition} \\ \quad \langle \nabla N \,, \nu \rangle = \langle \nabla W \,, \nu \rangle = 0, \quad \text{on} \quad \partial\Omega, \ t > 0\,. \end{cases}$$

- $(-)$ chemoattractant $(+)$ chemorepellent
- Early stages of the fruit body formation in slime mold *Dictostelium Discoideum*)
- For $n = 1$ -global in time classical solution (Nagai, 1995)
- For $n = 2$ - global solution for $\int_\Omega N_0 dx$ small enough, otherwise $T_{max} < \infty$.(Nagai, Senba, Yosida; 1997, Biler, 1998)

A typical prey-predator model (O.D.E. case)

$N(t)$—prey density,

$P(t)$—predator density

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - F(N)P := R_N(N,P)$$
$$\frac{dP}{dt} = bF(N)P - \delta P := R_P(N,P)\,,$$

$F = F(N)$-functional response e.g. -amount of food (prey) consumed per predator per unite of time, Holling's type II function:

$$F = F_H(N) = \frac{aN}{1 + T_h aN} \quad a\,, b > 0\,,$$

1. The Rosenzweig-MacArthur prey-predator model (1963)

   r - growth rate, δ - death rate, a-attack rate, T_h- handling time.

   For $K = \infty$ and $T_h = 0$ we get the Lotka-Volterra model.

   b— efficiency of conversion of food into offspring
2. For some set of parameters there is a unique globally stable steady state which may lose stability and limit cycle emerges via the Hopf bifurcation

Chemical signalling

- Many chemicals (e.g. pheromones, kairomones) released by animals are used as means of inter and intraspecific communication - (chemical signaling) and sense of smell is a primary means by which prey animals detect predators or prey and trigger suitable behavioral responses .
- The chemical signal my be released by predator/prey itself (odor of predator or prey) or it may be released due to damage of prey captured
(e.g. blood in aquatic ecosystems).
- Let W be a chemical released by prey or predator then the corresponding equation reads

$$W_t = d_3 \Delta W + g(N, P) - \mu W$$

where $g = g(N, P)$ is the rate of chemical signal production and μ is the degradation rate

$$g(N, P) = \gamma P \quad \text{or} \quad g(N, P) = \gamma N \quad \text{or } g(N, P) = \beta F(N) P \,,$$

Terminology: Direct/indirect prey taxis and/or predator taxis

- direct prey-taxis is a directed movement of predator toward the gradient of prey density,
- **indirect prey-taxis** is a directed movement of predator toward the density gradient of a chemical released by prey,
- direct repulsive predator taxis is the directed movement of prey in the opposite direction to the gradient of predator density.
- **indirect repulsive predator taxis** is a directed movement of prey in the opposite direction to the density gradient of a chemical released by predator.
- **pursuit- evasion** model includes both direct/indirect prey taxis (**pursuit**) and repulsive direct/indirect predator taxis (**evasion**).

- In the context of predator-prey models the term indirect taxis was first used for a simplistic model in J.I. Tello, D.W. (M3AS, 2016).
- Similar idea was also used in in a different context in K. Fujie, T.Senba, (JDE, 2017)
- Tao, M. Winkler (J.Eur.Math. Soc., 2017)

The prey-predator model with prey taxis (direct)

$$P_t = d_P \Delta P - \xi \nabla \cdot P \nabla N + bF(N)P - \delta P \,,$$

$$N_t = d_N \Delta N + rN \left(1 - \frac{N}{K}\right) - F(N)P \,.$$

with homogeneous Neumann boundary conditions (no-flux) and initial conditions

on smooth boundary $\partial\Omega$, $\Omega \subset \mathbb{R}^n$ and initial conditions. ($\xi > 0$)

- P.Kareiva, G.T. Odel (Am. Naturalist 1987),
- Prey-taxis was found to stabilize prey-predator interactions, no pattern formation is possible if ($\xi > 0$!)-J.M. Lee, T. Hillen and M.A. Lewis (J. Biol. Dyn., 2009)

Global-in-time existence of solutions:

- $n \geqslant 1$ (with volume filling effect) B. Ainseba, at.al.(NARWA, 2008), Y. Tao (NARWA, 2010)
- $n \geqslant 1$ (classical sol., for small ξ with $F(N)$ bounded) - S. Wu, J.Shi, B.Wu (JDE 2016); D.Li (DCDS 2021)
- $n \leqslant 2$ (classical sol.)- H.-Y Jin, Z.Wang (JDE, 2017), T. Xiang (Nonlin Anal, 2018), D. Li (DCDS, 2021)
- $n \leqslant 5$ (**weak solutions**) M. Winkler (JDE, 2017)

Pursuit-evasion predator-prey model with direct taxis

$$\begin{cases} P_t = d_P \Delta P - \xi \nabla \cdot P \, \nabla N + R_P(P, N)\,, \\ N_t = d_N \Delta N + \chi \nabla \cdot N \, \nabla P + R_N(P, N)\,, \\ \quad \text{with homogeneous Neumann boundary conditions (no flux)} \end{cases}$$

- **The main part of the system is not upper triangular** (full cross diffusion system)
- Formal stability/instability analysis, travelling waves)- Y. Tyutyunov, L. Titova, R.Arditi (Math. Mod.. Nat. Phenom., 2007)

  Global-in-time existence of solutions
- $n \leqslant 3$- (class. sol. in a neighbourhood of the constant steady state) M. Fuest (SIAM J. MAth. Anal, 2020)
- $n = 1$ - (no restriction on the size of initial data, approximation by 6-th order operators) Y.Tao, M. Winkler (J.F.A, 2021) , (Nonliner Anal. RWA, 2022)

Pursuit evasion model - indirect taxis for both prey and predator

$$P_t = d_P \Delta P - \chi \nabla \cdot (P \nabla U) + R_P(P,N) - \delta_1 P^2 ,$$

$$N_t = d_N \Delta N + \xi \nabla \cdot (N \nabla W) + R_N(P,N) - r_1 N^2 ,$$

$$W_t = d_W \Delta W + \alpha_w P - \mu_w W ,$$

$$U_t = d_U \Delta U + \alpha_u N - \mu_U U ,$$

with homogeneous Neumann boundary conditions (no-flux)

- **The main part of the system is upper triangular**
  Global-in-time existence of solutions:
- $n \leqslant 3$ (with χ and ξ small enough or δ_1 , r_1 big enough) - S. Wu (JMAA, 2022)

Pursuit -evasion prey-predator model with indirect repulsive predator taxis and prey taxis

$$P_t = D_P \Delta P - \nabla \cdot (\xi P \nabla N) - \delta P + bF(N)P\,,$$

$$N_t = D_N \Delta N + \nabla \cdot (\chi N \nabla W) + rN\left(1 - \frac{N}{K}\right) - F(N)P\,,$$

$$W_t = D_W \Delta W + \gamma P - \mu W\,.$$

- Model B : ($\chi > 0$ $\xi = 0$) **indirect repulsive predator taxis**
- Model A :($\chi > 0$ $\xi > 0$) **pursuit-evasion model**
- Basic $L^1(\Omega)$ estimate :

$$\frac{d}{dt}\left(\int_\Omega P(x,t)dx + b\int_\Omega N(x,t)dx\right) + C_1\left(\int_\Omega P(x,t)dx + b\int_\Omega N(x,t)dx\right) \leqslant C_2$$

where C_1 and C_2 are positive constants.

Global existence in time

Theorem

Suppose that $P_0\,,N_0\,,W_0 \in W^{1,r}(\Omega)$, $r > n$ *are non-negative functions.*
For Model A and Model B there exists the unique local non-negative classical solution (N, P, W) *satisfying boundary and initial defined on* $\bar{\Omega} \times [0\,, T_{max})$ *such that*

$$(N, P, W) \in (C([0\,, T_{max}) : W^{1,r}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0\,, T_{max})))^3\,.$$

- *Moreover,* $T_{max} = \infty$ *and the solution is uniformly* L^∞*- bounded in the case of*
- *Model B* $(\chi > 0, \xi = 0)$ *for all* $n \geqslant 1$
- *Model A* $(\chi > 0, \xi > 0)$ *in the case of* $n = 1$.

- P. Mishra, D.W. (Math. Mod. & Methods in Appl. Sc. (M3AS), 2022)

Linear stability analysis for Model B and Hopf bifurcation

- The coexistence steady state to Model B is of form

$$\bar{E} = (\bar{N}, \bar{P}, \bar{W}) \quad \text{where} \quad \bar{W} = \frac{\mu}{\gamma}\bar{P}\,.$$

- A complex number belongs to the spectrum of the linearization of Model B at $\bar{E}$ iff it is an eigenvalue of the following stability matrix :

$$M_j = \begin{pmatrix} -D_1 h_j + a_{11} & a_{12} & -\chi\bar{N}h_j \\ a_{21} & -D_2 h_j + a_{22} & 0 \\ 0 & a_{32} & -D_3 h_j + a_{33} \end{pmatrix}.$$

where $\{h_j\}_{j=0}^{\infty}$ denotes the eigenvalues of the Laplace operator $-\Delta$ with homogeneous Neumann boundary condition and $[a_{i,j}]$ is the Jacobian matrix for O.D.E. case.

$$a_{11} < 0\,, \quad a_{12} < 0\,, \quad a_{21} > 0\,, \quad a_{22} \leqslant 0 \quad a_{32} > 0 \quad a_{33} < 0\,.$$

- For any $\chi > 0$ considered as a **bifurcation parameter**: $\det M_j < 0$ and $\mathrm{tr} M_j < 0$.

Linear stability analysis for Model B and Hopf bifurcation

The dispersal equation of stability matrix M_j is following

$$\lambda^3 + \rho_j^{(1)}\lambda^2 + \rho_j^{(2)}\lambda + \rho_j^{(3)}(\chi) = 0$$

where

$$\begin{aligned}
\rho_j^{(1)} &= -\mathrm{tr} M_j = -(a_{11} + a_{22} + a_{33}) + (D_1 + D_2 + D_3)h_j\,, \\
&:= \alpha_0 + \alpha_1 h_j\,, \\
\rho_j^{(2)} &= a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} + a_{22}a_{33} \\
&\quad + h_j(-a_{22}D_1 - a_{33}D_1 - a_{11}D_2 - a_{22}D_3 - a_{11}D_3 - a_{33}D_2) \\
&\quad + h_j^2(D_1D_2 + D_1D_3 + D_2D_3) \\
&\quad := \beta_0 + \beta_1 h_j + \beta_2 h_j^2, \\
\rho_j^{(3)}(\chi) &= -\det M_j = -a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} \\
&\quad + h_j(a_{22}a_{33}D_1 + a_{11}a_{22}D_3 - a_{12}a_{21}D_3 + a_{11}a_{33}d_2) \\
&\quad + h_j^2(-a_{22}D_1D_3 - a_{33}D_1D_2 - a_{11}D_2D_3) + D_1D_2D_3h_j^3 + \chi a_{21}a_{32}\bar{N}h_j, \\
&\quad\; = (\gamma_0 + \gamma_1 h_j + \gamma_2 h_j^2 + \gamma_3 h_j^3) + \chi(\gamma_4 h_j) := \rho_j^{(3,1)} + \chi\rho_j^{(3,2)} > 0
\end{aligned}$$

where we have denoted $\rho_j^{(3)}(\chi) = \rho_j^{(3,1)} + \chi\rho_j^{(3,2)}$. It can be checked that all coefficients α_j , β_j , γ_j are positive.

Linear stability analysis for Model B and Hopf bifurcation

1. $\bar{E}$ is linearly stable if and only if for each $j \geqslant 0$ matrices M_j have eigenvalues with negative real parts which according to the Routh-Hurtwitz stability criterion is equivalent to the conditions

$$\rho_j^{(1)} > 0, \ \rho_j^{(3)} > 0,$$
$$\text{and} \quad Q_j := \rho_j^{(1)}\rho_j^{(2)} - \rho_j^{(3)}(\chi) = \rho_j^{(1)}\rho_j^{(2)} - \rho_j^{(3,1)} - \chi\rho_j^{(3,2)} > 0 \qquad \text{for all } j \geqslant 0 .$$

2. There exists $\chi^H > 0$ such that

$$\chi^H = \min_{j \in \mathbb{N}_+} \tilde{\Psi}(h_j) := \left\{ \frac{\rho_j^{(1)}\rho_j^{(2)} - \rho_j^{(3,1)}}{\rho_j^{(3,2)}} \right\} \tag{1}$$

and the steady state $\bar{E}$ is stable if $\chi < \chi^H$.

3. If

$$\tilde{\Psi}(h_j) \neq \tilde{\Psi}(h_k) \quad \text{for} \quad j \neq k$$

then the minimum is attained for a singe $j = j_0$.

4. Since $\mathrm{tr} M_{j_0} < 0$ and $\det M_{j_0} < 0$ there is one real negative eigenvalue and a pair of conjugate eigenvalues which cross imaginary axis for $\chi = \chi^H$ with the transversality condition being satisfied.

Theorem

There exist $\chi^H > 0$ such that steady state $\bar{E}$ in model B is locally asymptotically stable if $\chi < \chi^H$. Morrover, at χ^H a solution periodic in space and time emerges according to the Hopf bifurcation mechanism.

5. based on result of Amann, 1991

Digression: Shoener's model of intra-gild prey-predator interactions

- predator and prey exploit common resources

$$\begin{cases} P_t = d_1 \Delta P - \xi \nabla \cdot (P \nabla U) + \dfrac{b_1 RcP}{\gamma + cP + eN} - \delta P + d_1 NP\,, \\ N_t = d_2 \Delta N + \dfrac{b_2 ReN}{\gamma + cP + eN} - \delta' N - d_2 NP\,, \\ U_t = d_U \Delta U + \alpha N - \beta U\,, \end{cases}$$

  with homogeneous Neumann boundary condition
- Kinetic O.D.E model has no periodic orbits (Bendixona-Dulac ctriterion)
- For the Schoener model with indirect prey taxis for ξ big enough there appears Hopf bifurcation (periodic in space and time).
- O.D.E. – E. Ruggieri, S.J. Schreiber, Math. Biosc. Eng. 2005.
- P.D.E. – P. Mishra, D. W., Applied Mathematics Letters 2022.

Numerical solutions of models A and B

- Non-dimensional Rosenzweig-MacArthur model in the frame of model A

$$\begin{cases} N_t = \Delta N + \nabla \cdot (\chi N \nabla W) + rN\Big(1 - N\Big) - \dfrac{aNP}{(1+\beta N)}\,, \\ P_t = d_p \Delta P - \nabla \cdot (\xi P \nabla N) - \delta P + \dfrac{cNP}{(1+\beta N)}\,, \\ W_t = d_w \Delta W + \gamma P - \mu W\,, \end{cases}$$

with non-negative initial and no-flux boundary condition.

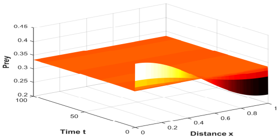
- 1D simulations with the help of MATLAB PDEPE tool ($\Delta t = 0.01, \Delta x = 0.1$)
- 2D simulations with the help of FreeFem++ solver ($\Delta t = 0.01, \Delta x = \Delta y = 0.1$)
- Values of model parameters are assumed to be

$$\begin{cases} r = 0.25,\ \ \beta = 2,\ \ c = 0.85,\ \ a = 0.95,\ \ \delta = 0.17\,, \\ \mu = 0.5,\ \ \gamma = 10,\ \ d_p = 0.01,\ \ d_w = 0.01. \end{cases} \tag{2}$$

- Unique coexistence steady state $E = (0.3333; 0.2924; 5.8490)$ and $\chi^H = 6.889$.
- Initial data : perturbation of the steady state e.g.

$$N(x,0) = \bar{N} + 0.1\cos\left(\frac{j\pi x}{L}\right),\ \ P(x,0) = \bar{P} + 0.1\cos\left(\frac{j\pi x}{L}\right),\ \ W(x,0) = \bar{W} + \cos\left(\frac{j\pi x}{L}\right) \tag{3}$$

Model B; convergence to the steady state (1D simulations)



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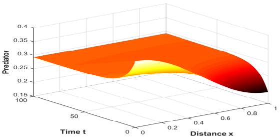

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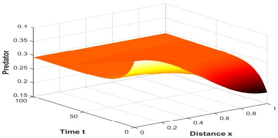

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Figure 1: Model B: Perturbation in model B approaches the constant steady state $\bar{E}$ for $\chi < \chi^H$ with $j = 1$

Initial data $N(x,0) = \bar{N}, P(x,0) = \bar{P} + 0.1e^{-(\frac{x-0.5}{0.2})^2},\ W(x,0) = \bar{W}$

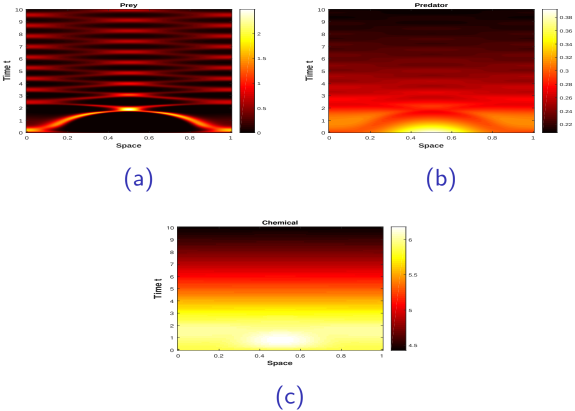
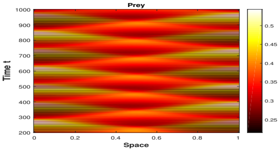


Figure 2: Model B: spatio-temporal patterns for $\chi > \chi^H$.

Model A; periodic solutions (1D simulations)



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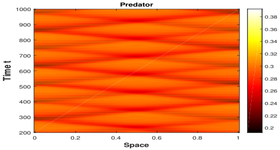

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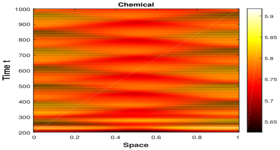
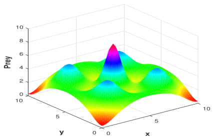


(c)

Figure 3: Model A: space-time patterns in unit domain when $\chi = 5, \ \xi = 0.2$ and symmetrical initial data with $j = 4$.

Gaussian initial data for predator centered in the middle of the square with constant initial data for the prey $N = \bar{N}$ and for the chemical $W = \bar{W}$



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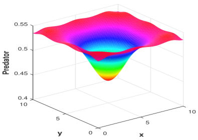

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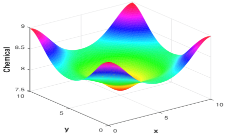

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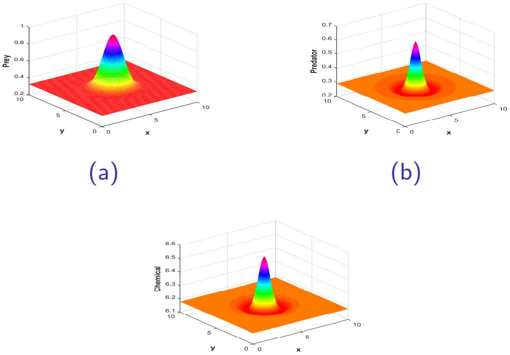
Figure 4: Model B ($\xi = 0$): 2D separation regions for $\chi = 10$ at time step $t = 1500$.

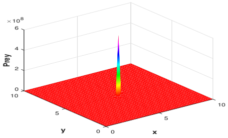
Gaussian initial data for predator and prey centered in the middle of the square with constant initial data $\bar{W}$ for the chemical.



(a) (b)

(c)

Figure 5: Model A: 2D simulation result for model A at time $t = 10$ for $\chi = 0.5, \xi = 10.0$



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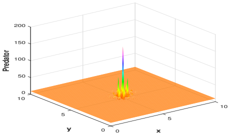

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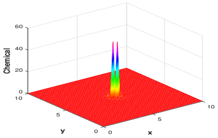

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Figure 6: Model A: numerical indication of blowup at time $t = 134$ for model A for $\chi = 0.5, \xi = 10.0$

How to modify model A to prevent blow-up?—> Model C

- In Model C a minimal modification with respect to model A is made for prevention of blow-up in finite time.
- The kinetic part is as in the classical Bazykin model (1976).
- Density-dependent suppression of velocity in predators is interpreted as the result of interference (kind of regularisation)

$$\begin{cases} P_t = d_P \Delta P - \xi \nabla \cdot P \left(\dfrac{\nabla N}{1 + \sigma P} \right) + bF(N) - \delta P - \delta_1 P^2 \,, \\ N_t = d_N \Delta N + \chi \nabla \cdot N \, \nabla W - F(N)P + rN - r_1 N^2 \,, \\ W_t = d_W \Delta W + \gamma P - \mu W \,, \end{cases}$$

Model C

Theorem

If P_0 , N_0 , $W_0 \in W^{1,r}(\Omega)$, $r > n$ are non-negative functions then there exist global in-time, non-negative classical solution to Model C satisfying boundary and initial condition provided $n \leqslant 3$ and the following restrictions on parameters are satisfied

$$\mathbf{Q}\begin{cases} \delta_1 \geqslant \left(\dfrac{\gamma^2(16+n)}{d_W} + d_W \right) , \\ r_1 \geqslant \left(\dfrac{\chi^2 A_N}{(d_N)^2} + \dfrac{2\chi^2}{d_W} + d_W \right) , \\ \textit{with} \quad A_N = \dfrac{2\left((d_N)^2 + (d_W)^2 + \xi^2 \sigma^{-2} \right)}{d_W} . \end{cases}$$

P.Mishra , D.W., preprint 2022.

Numerical solutions to Model C

- Set of parameters

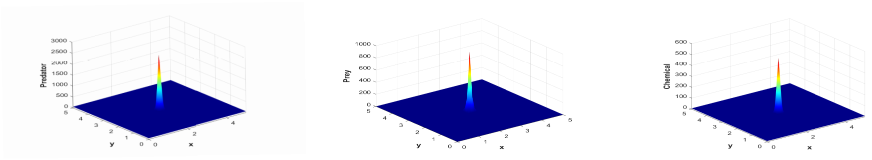
$$r = 2,\ r_1 = 1.8,\ a = 0.7,\ b = 0.9,\ \beta = 2,\ \mu = 0.01,\ \delta = 0.1,\ \delta_1 = 0.15,$$
$$\gamma = 0.015,\ d_n = 1,\ d_p = 0.1,\ d_w = 0.05.$$

- For this choice of parameters values the restriction **Q** holds if and only if $\sigma > \sigma_c := 19.7$
- For $\sigma < \sigma_c$, num. sol. to Model C exhibits finite-time blow-up of solution and for
- For $\sigma > \sigma_c$ there is prevention of blow-up (global solutions) .
- Initial data: perturbation of the constant steady state $E^\star = (P^\star, N^\star, W^\star) = (0.741, 1.016, 0.74)$

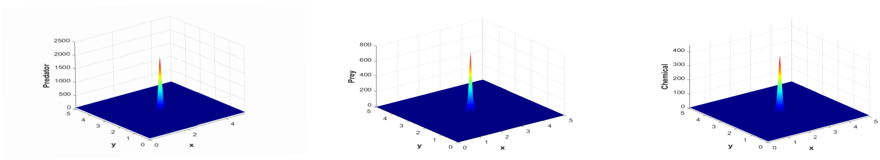
$$P_0(x,y) = P^\star + 500e^{-100((x-2.5)^2+(y-2.5)^2)},$$
$$N_0(x,y) = N^\star + 800e^{-100((x-2.5)^2+(y-2.5)^2)},$$
$$W_0(x,y) = W^\star + 100e^{-100((x-2.5)^2+(y-2.5)^2)}$$

where $(x,y) \in \Omega = (0\,,5) \times (0\,,5)$

Figure 1

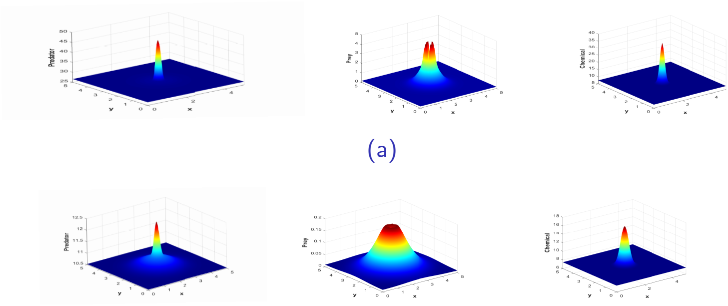


(a)



(b)

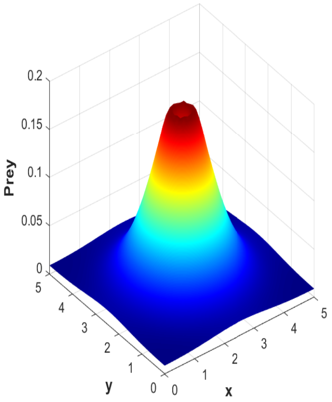
Figure 7: (a) Approximated blowup solution at time $t = 1.5 \times 10^{-4}$ for $\sigma = 0.0$ (b) Approximated blowup solution at time $t = 2.3 \times 10^{-4}$ for $\sigma = 5.0$ subject to initial conditions. It was assumed $\chi = 0.1$, $\xi = 30$.

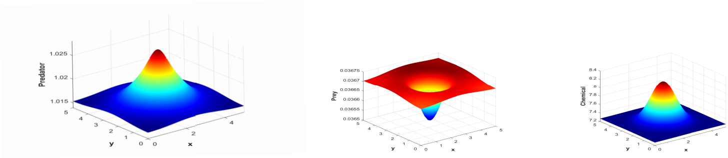


(a)

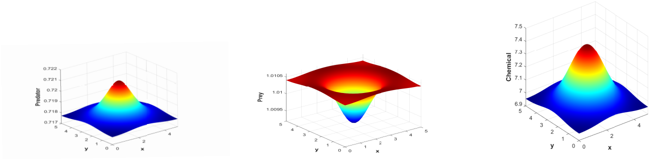
(b)

Figure 8: Snapshots for $\sigma = 25$ at different time steps. (a) $t = 13$, (b) $t = 50$


Prey
0.2
0.15
0.1
0.05
0
y
x
0
1
2
3
4
5



(a)



(b)

Figure 9: Snapshots for $\sigma = 25$ at different time steps. (a) $t = 100$, (b) $t = 500$. All other parameter values and initial condition is same as in figure (7).

Sketch of proof of the global existence for Model C

1. There is a local smooth solution defined on $[\tau\ , T_{max})$ satisfying $L^1(\Omega)$-bound.
2. We begin with the N-equation

$$N_t = d_N \Delta N + \chi \nabla \cdot N \, \nabla W - F(N)P + rN - r_1 N^2$$

3. Using the Gagliardo-Nirenberg inequality and $L^1(\Omega)$-bound one proves that for $n \leqslant 3$

$$\sup_{t\in[\tau\ , T_{max})} \|N(\cdot,t)\|_k \leqslant C_N(k)$$

provided

$$\sup_{t\in[\tau\ , T_{max})} \|\nabla W(\cdot,t)\|_4 \leqslant C'_W \, .$$

Then

$$\sup_{t\in[\tau\ , T_{max}} \|N(\cdot,t)\nabla W(\cdot,t)\|_{4-\varepsilon} \leqslant C''_W$$

4. Using properties of the heat semigroup we infer that

$$\sup_{t\in[\tau\ , T_{max})} \|N(\cdot,t)\|_\infty \leqslant C_N \, .$$

5. Using $L^p - L^q$ estimates for analytic semigroups $(n \leqslant 3)$ we get

$$\sup_{t\in[\tau\ , T_{max})} \|\nabla N(\cdot,t)\|_p \leqslant C'_N \quad for \quad p < 4$$

Next it is easy to deduce by parabolic regularity that

$$\sup_{t\in[\tau\ , T_{max})} \|\nabla P(\cdot,t)\|_\infty \leqslant C_P \, , \qquad \sup_{t\in[\tau\ , T_{max})} \|\nabla W(\cdot,t)\|_\infty \leqslant C_W$$

Sketch of proof of the global existence for Model C

- The most complicated part of the proof is to find estimate on $\|\nabla W(\cdot,t)\|_4$. To this end we derive differential inequality

$$y'(t) + y(t) \leqslant Const. \quad \text{for} \quad t \in [\tau \,, T_{max})$$

where for suitable constants A_1 and A_2

$$y(t) = \int_\Omega |\nabla W|^4 + \int_\Omega P|\nabla W|^2 + \int_\Omega N|\nabla W|^2 + A_1 \int_\Omega N^2 + A_2 \int_\Omega P^2 \,.$$

- We use Bochner's type inequality : For $W \in C^2(\bar{\Omega})$ there holds

$$2\nabla W \nabla \Delta W = \Delta|\nabla W|^2 - 2|D^2 W|^2$$

and

- Mizoguchi-Souplet inequality : for $u \in C^2(\bar{\Omega})$ satisfying $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and Ω there holds the following pointwise inequality

$$\frac{\partial |\nabla u|^2}{\partial \nu} \leqslant K|\nabla u|^2 \quad \text{on} \quad \partial\Omega$$

where K depends on the curvature of $\partial\Omega$.

Lemma

Let (P, N, W) be a solution to Model C. Then there exists a constant $C > 0$ such that for $t \in (0, T_{max})$.

$$\frac{d}{dt}\int_\Omega |\nabla W|^4 + d_W \int_\Omega \left|\nabla(|\nabla W|^2)\right|^2 + 4\mu \int_\Omega |\nabla W|^4$$
$$\leqslant \gamma^2 \left(\frac{16+n}{d_W}\right) \int_\Omega |\nabla W|^2 P^2 + C\,.$$

Corollaries and open questions

- Chemical signalling may destabilize a space-homogeneous steady state in a prey -predator model and gives rise to space-time dependent pattern formation.
- When an O.D.E. model is extended to a P.D.E model with taxis terms some mechanism of blow-up prevention might be necessary to be built in the model.
- Non of the two taxis mechanisms studied in Model C alone can lead to the blow-p for $n = 2$. Their cumulative effect (for $\sigma = 0$) leading to blow-up demands farther investigation.
- Are there any weak solutions for for Model C when $\sigma = 0$, weak enough to grasp the singular solutions?

Thank you.