Persistence, global stability and attractor size for delay differential equations

Daniel Franco

Universidad Nacional de Educación a Distancia (UNED), Spain

IMDETA, November 2023

What is the talk about?

Asymptotic behaviour of delay equations

1. Difference equations
2. Control theory

Collaborators

Chris Guiver (Edinburgh Napier U., UK)

Hartmut Logemann (U. Bath, UK)

Juan Perán (UNED)

Juan Segura (EADA Business School, Spain)

Difference equations tools

D. Franco, C. Guiver, H. Logemann, J. Perán, *Electron. J. Qual. Theory Differ. Equ.*, 2020.

Delay equation model

$$x'(t) = -\mu\big(x(t) - f(x(t-h))\big), \quad t > 0, \tag{1}$$

with $\mu, h > 0$, $f\colon I \subset \mathbb{R} \to I$, and initial condition $\xi \in \mathcal{C}([-h, 0], I)$.

- Nicholson's blowflies equation (Nature, 1980)

$$f(x) = \frac{1}{\mu} x \mathrm{e}^{-x},$$

- Mackey–Glass equation (Science, 1977)

$$f(x) = \frac{1}{\mu} \frac{ax}{1 + x^b}, \quad a > 0,\ b \geq 1.$$

Main contibutions by: Bellman, Cook, Hale, Krisztin, Mallet-Paret, Nussbaum, Sell, Smith, Walther . . .

Conditions on f

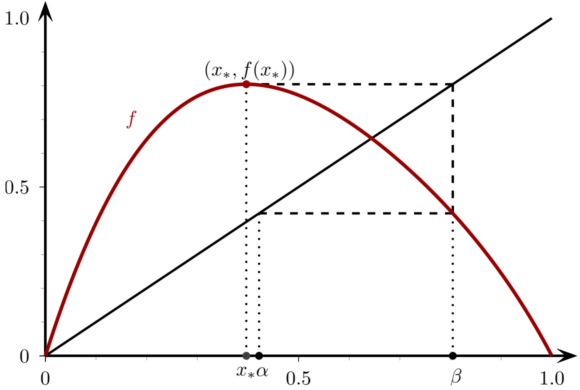
f unimodal (U)

$f\colon (a,b) \subset \mathbb{R} \to (a,b)$ is differentiable, with $-\infty \leq a < b \leq +\infty$; satisfies that there is a unique x_* such that $f'(x) > 0$ if $a \leq x < x_*$, $f'(x_*) = 0$, and $f'(x) < 0$ if $x_* < x < b$; and that there exists $K \in (x_*, b)$ such that $f(K) = K$, $f(x) > x$ for $x \in (a, K)$, and $f(x) < x$ for $x \in (K, b)$.

Condition (L)

Condition **(U)** holds and $f(f(x_*)) > x_*$.

Conditions on f



Denote $\beta := f(x_*)$ and $\alpha := f(\beta)$.

Size of the global attractor

Lemma

*If **(L)** holds, then for any solution $x(t;\xi)$ of (1) with $\xi \in \mathcal{C}([-h,0],(a,b))$ there exists t_0 s.t. $x(t;\xi) \in [\alpha,\beta]$ for $t \geq t_0$.*

Problem

The interval $[\alpha,\beta]$ might have a proper subinterval which contains the global attractor of (1). Estimate the sharpest attracting interval when condition **(L)** holds.

G. Röst, J. Wu, *Proc. R. Soc. Lond. Ser. A*, 2007.

Dichotomy

Theorem

*If **(L)** holds and f satisfies*

(S) $(Sf)(x) < 0$ *on* $[\alpha, \beta]$, *where*

$$(Sf)(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

Then, exactly one of the following holds:

1. $f'(K) \geq -1$ *and the global attractor of* (1) *for all values of the delay is* $\{K\}$.
2. $f'(K) < -1$ *and the sharpest invariant and attracting interval containing the global attractor of* (1) *for all values of the delay is* $[\bar{\alpha}, \bar{\beta}]$, *where* $\{\bar{\alpha}, \bar{\beta}\}$ *is the unique nontrivial 2-cycle (i.e.,* $\bar{\alpha} = f(\bar{\beta})$ *and* $\bar{\beta} = f(\bar{\alpha})$*) of the map* f *in* $[\alpha, \beta]$.

E. Liz, G. Röst, *Discrete Contin. Dyn. Syst.* 2009.

$$x'(t) = -\mu\big(x(t) - f(x(t-h))\big)$$

Related difference equation

$$x_n = f(x_{n-1}), \quad x_0 \in I. \tag{2}$$

Lemma

If there exists an interval $I_0 \subset I$ such that

$$\inf I_0 \leq \liminf_{n\to+\infty} f^{(n)}(x) \leq \limsup_{n\to+\infty} f^{(n)}(x) \leq \sup I_0 \quad \forall\, x \in I\,,$$

then the solutions of (1) *satisfy*

$$\inf I_0 \leq \liminf_{t\to+\infty} x(t,\xi) \leq \limsup_{t\to+\infty} x(t,\xi) \leq \sup I_0$$

$\forall\, h > 0,\ \forall\, \xi \in \mathcal{C}([-h, 0], I)\,.$

A. F. Ivanov, A. N. Sharkovsky, Dynam. Report. Expositions Dynam. Systems, 1992.

T. Yi, X. Zhou, Proc. Roy. Soc. A, 2010.

2-cycles and global stability

Theorem

The following statements are equivalent:

- K is a global attractor for (2).
- $f^{(2)}(x) \neq x$

W. A. Coppel, *The solution of equations by iteration.* Proc. Cambridge Phil. Soc., 1955.

Theorem

For S-unimodal maps there exists a global attracting 2-cycle for the difference equation (2).

D. Singer, SIAM J. Appl. Math. 1978.

Rewrite $x_n = f(x_{n-1})$ as

$$y_n = y_{n-1} + g(y_{n-1}), \qquad y_0 \in \operatorname{dom} g, \tag{3}$$

where $g \in \mathcal{C}^1(\alpha, \beta)$, $\alpha < \operatorname{id} + g < \beta$, $g' < 0$, $g(\beta) < 0 < g(\alpha)$.

Definition

Define $\sigma_g \colon (-b_g, b_g) \to (0, +\infty)$ by

$$\sigma_g(u) = \begin{cases} \frac{g^{-1}(-u) - g^{-1}(u)}{u}, & u \neq 0, \\ \frac{-2}{g'(y_g)}, & u = 0, \end{cases}$$

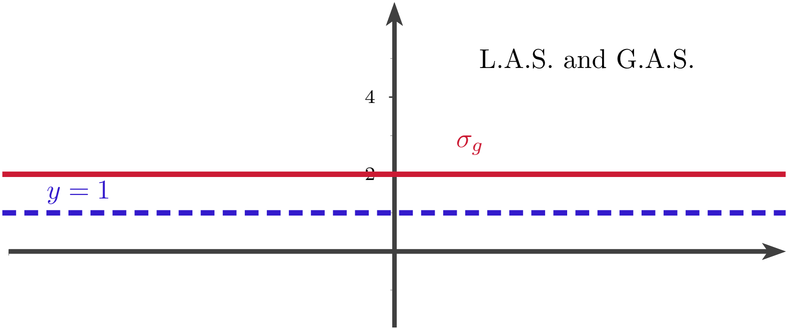
where $b_g := \min\{-\inf g, \sup g\}$.

Property

2-cycles correspond with solutions of $\sigma_g(u) = 1$.

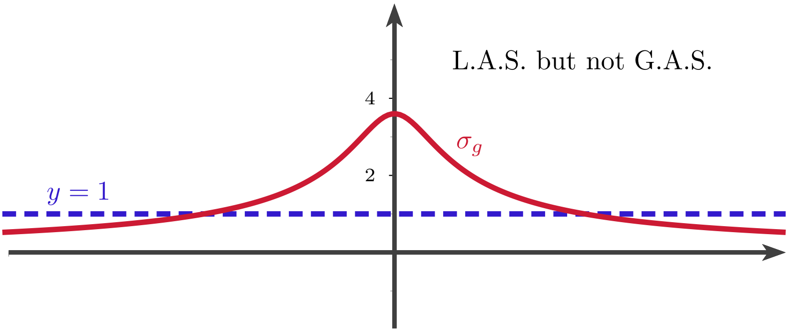
Theorem

1. $\sigma_g(0) > 1 \implies$ *K is L.A.S.*
2. $\sigma_g(0) < 1 \implies$ *K is unstable.*
3. *K is G.A.S.* $\iff 1 < \sigma_g(u)$ *for all* $u \neq 0$.



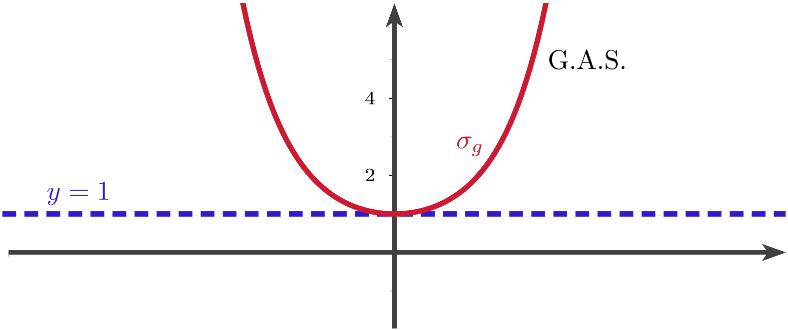
Theorem

1. $\sigma_g(0) > 1 \implies K$ *is L.A.S.*
2. $\sigma_g(0) < 1 \implies K$ *is unstable.*
3. K *is G.A.S.* $\iff 1 < \sigma_g(u)$ *for all* $u \neq 0$.



Theorem

1. $\sigma_g(0) > 1 \implies$ *K is L.A.S.*
2. $\sigma_g(0) < 1 \implies$ *K is unstable.*
3. *K is G.A.S.* $\iff 1 < \sigma_g(u)$ *for all* $u \neq 0$.



Proposition

1. If $\left(g^{-1}\right)'$ is strictly concave, then the difference equation has at most one nontrivial period-2 solution.
2. If $\left(g^{-1}\right)'$ is strictly concave and $g'(y_g) \geq -2$, then y_g is G.A.S.

Assume $g \in \mathcal{C}^3(\operatorname{dom} g)$. Since

$$(g^{-1})'''(u) = \frac{3(g''(y))^2 - g'(y)g'''(y)}{(g'(y))^5} \quad \forall\, u = g(y),\ y \in (a, b)\,,$$

a sufficient condition for the strict concavity of $\left(g^{-1}\right)'$ is

$$3(g'')^2 - g'g''' > 0.$$

Proposition

1. If $\left(g^{-1}\right)'$ is strictly concave, then the difference equation has at most one nontrivial period-2 solution.
2. If $\left(g^{-1}\right)'$ is strictly concave and $g'(y_g) \geq -2$, then y_g is G.A.S.

Assume $g \in \mathcal{C}^3(\mathrm{dom}\, g)$. Since

$$(g^{-1})'''(u) = \frac{3(g''(y))^2 - g'(y)g'''(y)}{(g'(y))^5} \quad \forall\, u = g(y),\ y \in (a, b)\,,$$

a sufficient condition for the strict concavity of $\left(g^{-1}\right)'$ is

$$3(g'')^2 - g'g''' > 0.$$

Theorem

*Assume that **(L)** holds, that f is three times differentiable and satisfies*

$$3(f'')^2 - (f' - 1)f''' > 0\,, \tag{4}$$

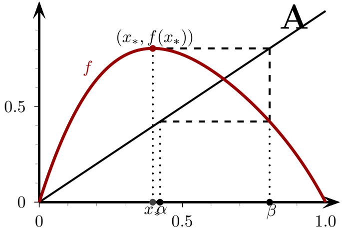
on the interval (α, β)*. Then, exactly one of the following holds:*

1. $f'(K) \geq -1$ *and the global attractor of* (1) *for all values of the delay is* $\{K\}$*.*
2. $f'(K) < -1$ *and the sharpest invariant and attracting interval containing the global attractor of* (1) *for all values of the delay is* $[\bar{\alpha}, \bar{\beta}]$*, where* $\{\bar{\alpha}, \bar{\beta}\}$ *is the unique nontrivial 2-cycle of the map f in* $[\alpha, \beta]$*.*

Example

Consider equation (1) with $f\colon (0,1) \to (0,1)$ given by

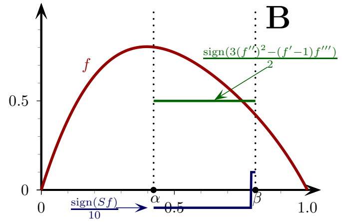
$$f(x) = \frac{19}{20}x(1-x)(5-4x+2x^3).$$



Example

Consider equation (1) with $f\colon (0,1) \to (0,1)$ given by

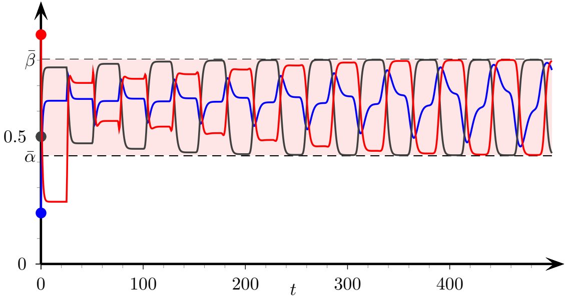
$$f(x) = \frac{19}{20}x(1-x)(5-4x+2x^3).$$



Example

Consider equation (1) with $f\colon (0,1) \to (0,1)$ given by

$$f(x) = \frac{19}{20} x(1-x)(5-4x+2x^3).$$



$$x'(t) = -\mu\big(x(t) - f(x(t-h))\big) \tag{1}$$

$$\Uparrow$$

$$x_n = f(x_{n-1}) \tag{2}$$

$$\Updownarrow$$

$$y_n = y_{n-1} + g(y_{n-1}) \tag{3}$$

Topological conjugacy

- $g = f - \mathrm{id}$ is the natural choice to rewrite (2) in the form (3).
- But any topologically conjugate equation of (2) belonging to model (3) will give a condition on f.
- If f is positive and $x \mapsto f(x)/x$ is decreasing, we have

Theorem

Assume that **(L)** holds, that $d(x) := f(x)/x$ is three times differentiable with $d' < 0$, and that

$$3(g'')^2 - g'g''' > 0\,,$$

on the interval $(\ln \alpha, \ln \beta)$, where $g := \ln \circ d \circ \exp$. Then, the dichotomy holds.

Topological conjugacy

- $g = f - \mathrm{id}$ is the natural choice to rewrite (2) in the form (3).
- But any topologically conjugate equation of (2) belonging to model (3) will give a condition on f.
- If f is positive and $x \mapsto f(x)/x$ is decreasing, we have

Theorem

Assume that **(L)** holds, that $d(x) := f(x)/x$ is three times differentiable with $d' < 0$, and that

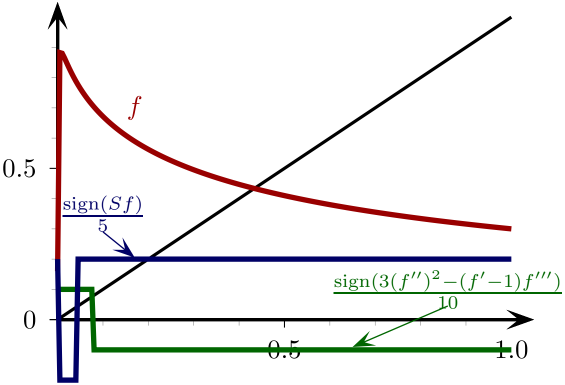
$$3(g'')^2 - g'g''' > 0\,,$$

on the interval $(\ln\alpha, \ln\beta)$, where $g := \ln \circ d \circ \exp$. Then, the dichotomy holds.

Example

Consider equation (1) with $f\colon (0,1) \to (0,1)$ given by

$$f(x) = \frac{3}{10}x\left(1 - \frac{1}{10}\ln(x)\right)^{15}.$$



- Nicholson's blowflies equation

$$f(x) = \frac{1}{\mu} x e^{-x},$$

- Mackey–Glass equation

$$f(x) = \frac{1}{\mu} \frac{ax}{1 + x^b}, \quad a > 0,\ b \geq 1.$$

Nicholson's blowflies equation

In this case, $g(x) = \ln(1/\mu) - e^x$ and $3(g'')^2 - g'g''' = 2e^{2x} > 0$.

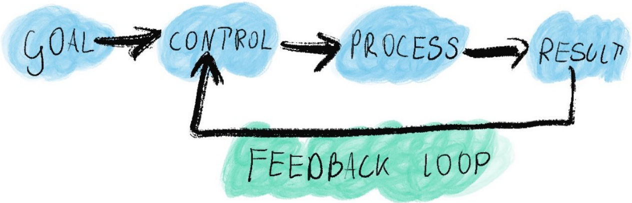
Mackey–Glass equation

In this case, $g(x) = \ln(a/\mu) - \ln(1 + e^{bx})$ and

$$3(g'')^2(x) - g'(x)g'''(x) = \frac{b^4 e^{2bx}(2 + e^{bx})}{(1 + e^{bx})^4} > 0 \, .$$

Control Theory

D. Franco, C. Guiver & H. Logemann, *Acta Applicandae Mathematicae*, 2021.


GOAL
CONTROL
PROCESS
RESULT
FEEDBACK LOOP

Lur'e systems: Linear process with nonlinear feedback:

$$\begin{cases} \dot{x} = Ax + u \\ y = c^T x \\ u = b\,f(y) \end{cases}$$

$x \in \mathbb{R}^n$ is the **state**
y is the **observation** or **output**
u is the **input**

Adding all together

$$\dot{x}(t) = Ax(t) + bf(c^T x(t-h)).$$

Lur'e systems: Linear process with nonlinear feedback:

$$\begin{cases} \dot{x} = Ax + u \\ y = c^T x \\ u = b\, f(y) \end{cases}$$

$x \in \mathbb{R}^n$ is the **state**
y is the **observation** or **output**
u is the **input**

Adding all together

$$\dot{x}(t) = Ax(t) + bf(c^T x(t-h)).$$

Goals

- **Persistence**
- **Tend to a positive steady state (global stability)**
  Allows to plan ahead.

Main processes:

1. **Mortality.**
2. **Interchanges among classes:**
   Migration

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{n} a_{ij} x_j, \quad a_{ij} \geq 0,$$

$$A = \begin{pmatrix} -d_1 & a_{12} & \dots & a_{1n} \\ a_{21} & -d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1n} \\ a_{n1} & \dots & a_{nn-1} & -d_n \end{pmatrix}$$ Metzler and Hurwitz.

Main processes:

1. **Mortality.**
2. **Interchanges among classes:**
   Migration

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{n} a_{ij} x_j, \quad a_{ij} \geq 0,$$

$$A = \begin{pmatrix} -d_1 & a_{12} & \dots & a_{1n} \\ a_{21} & -d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1n} \\ a_{n1} & \dots & a_{nn-1} & -d_n \end{pmatrix}$$

Metzler and Hurwitz.

Birth: a natural feedback

$$\left.\begin{aligned} \dot{x}_1(t) &= -d_1 x_1(t) + \sum_{j=1}^{n} a_{1j} x_j(t) + f\Big(u(t), \sum_{j=1}^{n} c_j x_j(t-h)\Big) + v_1(t), \\ \dot{x}_i(t) &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} x_j(t) + v_i(t), \quad i \in \{2, \ldots, n\}, \end{aligned}\right\}$$

$h > 0$ maturation time; $f\colon [u^-, u^+] \times \mathbb{R}_+ \to \mathbb{R}_+$ birth function;
$c_i \geq 0$ contribution of patch i to births in patch 1;
$u(t) \in [u^-, u^+] \subset (0, \infty)$ and $v_i(t) \geq 0$ controls or disturbances.

$$\dot{x}(t) = Ax(t) + bf(u(t), c^T x(t-h)) + v(t), \quad b := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Observavility

From an observation we know the state.

Theorem

$\dot{x} = Ax, \; y = c^T x$ *is observable iff* $\ker(O(c^T, A)) = \{0\}$.

$$O(c^T, A) := \begin{pmatrix} c^T \\ c^T A \\ c^T A^2 \\ \vdots \\ c^T A^{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

ISS stability

A system is input-to-state stable (ISS) if

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|_\infty), \quad t \geq 0$$

for all admissible initial values and inputs, with

- $\beta(s, t)$ increasing in s, decreasing in t, $\beta(0, t) = 0$ and $\lim_{t \to +\infty} \beta(s, t) = 0$.
- γ is increasing and $\gamma(0) = 0$.

ISS stability

A system is input-to-state stable (ISS) if

$$\|x(t)\| \leq e^{-t}\|x(0)\| + \|u\|_\infty, \quad t \geq 0$$

for all admissible initial values and inputs, with

- $\beta(s,t)$ increasing in s, decreasing in t, $\beta(0,t) = 0$ and $\lim_{t\to+\infty} \beta(s,t) = 0$.
- γ is increasing and $\gamma(0) = 0$.

Assumptions

(P) $\ker(O(c^T, A)) \cap \mathbb{R}^n_+ = \{0\}$.

0 cannot be L.A.S.

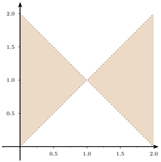
The smallest positive parameter value q for which the additive linear perturbation $qbc^t x(t-h)$ destabilizes $\dot{x} = Ax$ is $\frac{-1}{c^T A^{-1} b}$

Sector bound condition

Let $u^* \in [u^-, u^+]$ and $p = \frac{-1}{c^T A^{-1} b}$.

(N) There exists a unique $y^* > 0$ such that $f(y^*) = py^*$ and

$$|f(u^*, y) - py^*| < p|y - y^*|, \quad y \in \mathbb{R}_+ \setminus \{0, y^*\}$$



Stability

Let $u^* \in U$, then $x^* = -A^{-1}bpy^*$ is the non-zero steady state of the system with $u \equiv u^*$ and $v \equiv 0$.

Theorem

Assume ***(P)*** *and* ***(N)*** *and let $\beta > \alpha > 0$. Then there exist $R \geq 1$ and $\mu > 0$ such that*

$$\|x(t) - x^*\| \leq R\big(e^{-\mu t}\|\xi - x^*\|_{M^1} + \|u - u^*\|_{L^\infty(0,t)} + \|v\|_{L^\infty(0,t)}\big),$$

for all $t \geq 0$, $u \in L(\mathbb{R}_+, U)$, $\xi \in M_+^\infty := \mathbb{R}_+^n \times L^\infty([-h, 0], \mathbb{R}_+^n)$, and $v \in L_+^\infty$ with

$$\|\xi\|_{M^\infty} + \|v\|_{L^\infty} \leq \beta, \quad \|\xi^0\| \geq \alpha.$$

Application

G. Kiss and G. Röst. Controlling Mackey-Glass chaos. Chaos 27, 114321 (2017).

- Process $\rightarrow$ Mackey-Glass equation.
- Feedback $\rightarrow$ Constant, Proportional and Pyragas.

$$\dot{x} = Ax + b(f(c^t x(t-h)) + bk) \quad \rightarrow \quad (A, b, c^T, f + k)$$

$$\dot{x} = Ax - dx + bf(c^t x(t-h)) \quad \rightarrow \quad (A - I\,d, b, c^T, f)$$

$$\dot{x} = Ax - kbx + b(f(c^t x(t-h) + kc^t x(t-h)) \rightarrow (A - k\,I\,b, b, c^T, f + k\mathrm{id})$$

Application

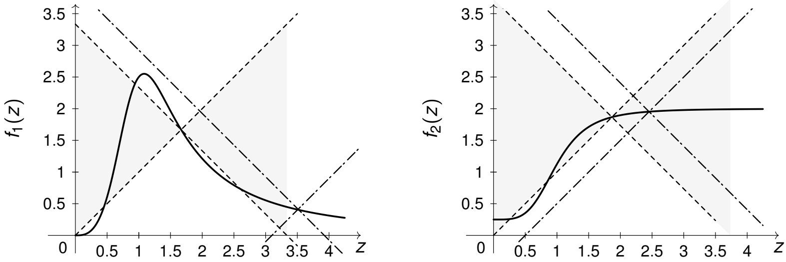
G. Kiss and G. Röst. Controlling Mackey-Glass chaos. Chaos 27, 114321 (2017).

- Process $\rightarrow$ Mackey-Glass equation.
- Feedback $\rightarrow$ Constant, Proportional and Pyragas.

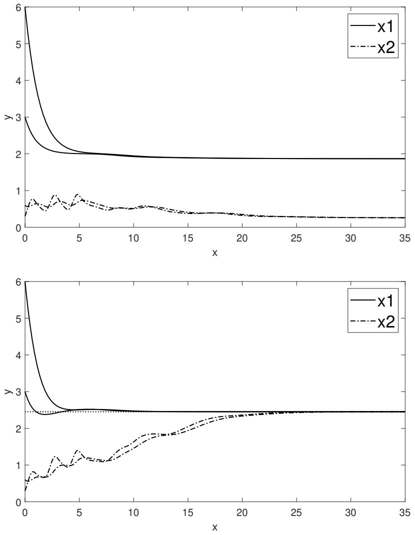
$$\dot{x} = Ax + b(f(c^t x(t-h)) + bk) \quad \rightarrow \quad (A, b, c^T, f + k)$$

$$\dot{x} = Ax - dx + bf(c^t x(t-h)) \quad \rightarrow \quad (A - I\,d, b, c^T, f)$$

$$\dot{x} = Ax - kbx + b(f(c^t x(t-h) + kc^t x(t-h)) \rightarrow (A - k\,I\,b, b, c^T, f + k\mathrm{id})$$



(a) (b)


x1
x2
y
x
0
5
10
15
20
25
30
35
0
1
2
3
4
5
6

Thank you!