Applications of Topological Fixed Point Theory to Nonlocal Differential Equations with Convolution Coefficients

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Preliminaries

We are interested in second-order nonlocal differential equations. A few general examples are the following.

$$-A\left(\int_0^1 |u(s)|^q \, ds\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1)$$

$$-A\left(\int_0^1 |u'(s)|^q \, ds\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1)$$

$$-A\left(\int_\Omega |u(\boldsymbol{s})|^q \, d\boldsymbol{s}\right) (\Delta u)(\boldsymbol{x}) = \lambda g\big(\boldsymbol{x}, u(\boldsymbol{x})\big),\ \boldsymbol{x} \in \Omega \subset \mathbb{R}^n$$

$$-A\left(\int_\Omega |Du(\boldsymbol{s})|^q \, d\boldsymbol{s}\right) (\Delta u)(\boldsymbol{x}) = \lambda g\big(\boldsymbol{x}, u(\boldsymbol{x})\big),\ \boldsymbol{x} \in \Omega \subset \mathbb{R}^n$$

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Our Goal: To develop existence theorems for the ODEs case when the problem is equipped with some boundary data.

To understand the broader context for these types of problems let us recall the classical wave PDE with a source term; it reads:

$$u_{tt} - (\Delta u)(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big).$$

This is a ***local*** PDE. A *nonlocal* version of this was proposed by Kirchhoff in the late 1800s; it takes the following form.

$$u_{tt} - A\left(\int_{\Omega} |Du|^2 \, d\boldsymbol{s}\right)(\Delta u)(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big)$$

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Steady-state solutions solve:

$$0 - A\left(\int_{\Omega} |Du|^2 \, d\boldsymbol{s}\right)(\Delta u)(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big),$$

which is one of the model cases.

$$0 - A\left(\int_{\Omega} |Du|^2 \, d\boldsymbol{s}\right) (\Delta u)(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big)$$

Specialized to the case $n = 1$ (i.e., the one-dimensional setting) we recover the model **O**DE:

$$-A\left(\int_{\Omega} \left|u'(s)\right|^2 \, ds\right) u''(x) = f\big(x, u(x)\big), \; x \in I \subset \mathbb{R}.$$

$$u_{tt} - A\left(\int_{\Omega} |Du|^2 \, d\boldsymbol{s}\right)(\Delta u)(\boldsymbol{x}) = f\big(\boldsymbol{x}, u(\boldsymbol{x})\big)$$

All in all, there is a long history (literally 150 years!) of analyzing nonlocal DEs both in the one-dimensional and higher-dimensional cases.

Let's look at some specific references for a couple reasons.

1. To get a sense of some of the common assumptions in the literature regarding these problems.
2. To provide a brief accounting of how I became interested in these problems.

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Non-local boundary value problems of arbitrary order

J. R. L. Webb and Gennaro Infante

The authors considered problems of the following type, where α and β are linear functionals.

$$u''(t) = f\big(t, u(t)\big),\ t \in (0,1)$$
$$u(0) = \alpha[u]$$
$$u(1) = \beta[u]$$

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$$u(1) = \beta[u]$$

This first piqued my interest in nonlocal problems – specifically, nonlocal boundary conditions.

PERGAMON

Nonlinear Analysis 47 (2001) 3579–3584

Nonlinear Analysis

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Nonlocal elliptic equations

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The following problem was considered:

$$-\varphi''(t) = M\big(f \circ \varphi\big)^{\alpha}(t)\left(\int_0^1 (f \circ \varphi)(s)\, ds\right)^{-\beta}, \; t \in (0,1),$$

subject to $\varphi'(0) = 0 = \varphi(1)$. Existence of a positive solution was shown under the assumption that f is continuous, nondecreasing, and satisfied a growth condition, which I won't state here.

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Nonlinear Analysis 59 (2004) 1147 – 1155
Nonlinear Analysis
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On positive solutions of nonlocal and nonvariational elliptic problems

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The following problem was considered.

$$-a\left(\|u\|_{L^q}^q\right)u''(t) = h(t)f\big(u(t)\big),\ t \in (0,1) \text{ subject to } u'(0) = 0 = u(1)$$

Importantly, it was assumed that $t \mapsto a(t)$ was nondecreasing and satisfied $a(\mathbb{R}) \subset (0, +\infty)$.

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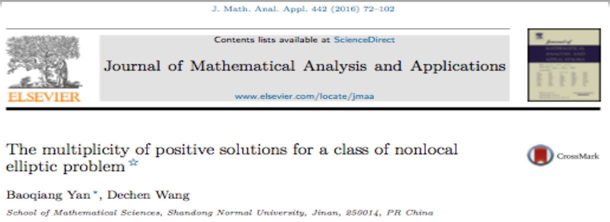
Positive solutions for some nonlocal and nonvariational elliptic systems

João Marcos do Ó[a*], Sebastián Lorca[b], Justino Sánchez[c] and Pedro Ubilla[d]

The following radially symmetric system (and, thus, a system of ODEs) was considered.

$$-A_i\left(\|u_i\|_{L^{q_i}}^{q_i}\right)\Delta u_i = f_i(|\boldsymbol{x}|, \boldsymbol{u}),\ \boldsymbol{x}\in\Omega \text{ subject to } u_i = 0 \text{ on } \partial\Omega$$

Here it was assumed that the A_i functions were nondecreasing and satisfied $A_i(\mathbb{R})\subset[0,+\infty)$. Some additional conditions were imposed on the A_i functions – e.g., a limit condition involving a ratio of the f_i to the A_i.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The multiplicity of positive solutions for a class of nonlocal elliptic problem ☆

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The following problem was considered.

$$-A\left(\int_{\Omega} |u(\boldsymbol{s})|^{\gamma}\, d\boldsymbol{s}\right)\Delta u = \lambda f(\boldsymbol{x}, u(\boldsymbol{x})),\ \boldsymbol{x} \in \Omega \subset \mathbb{R}^n,$$

subject to $u(\boldsymbol{x}) = 0$ on $\partial\Omega$ and $u(\boldsymbol{x}) > 0$ in Ω. It was assumed that $A(t) > 0$ for all $t \geq 0$.

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Positive solutions for a Kirchhoff problem with vanishing nonlocal term ☆

João R. Santos Júnior [a], Gaetano Siciliano [b]

They considered the problem

$$-A\left(\|u\|_{L^2}^2\right)\Delta u = f\left(u(\boldsymbol{x})\right),\ \boldsymbol{x} \in \Omega \subset \mathbb{R}^n,$$

subject to $u(\boldsymbol{x}) = 0$ on $\partial\Omega$. Although A can change sign, it must be positive on an open set having 0 as an accumulation point. And, furthermore, a condition is imposed on a definite integral of A. Also, only the L^2-norm was considered as the argument of A.

Going back to the one-dimensional case, where we will henceforth remain, we consider for a moment the specific problem

$$-A\left(\int_0^1 \left|u(s)\right|^q\, ds\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1).$$

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We summarize the conditions that are very common in the literature.

1. $A(z) > 0$ for all $z \geq 0$
2. A monotonicity condition on A – e.g., that A is monotone increasing.
3. A growth-type condition on A – e.g., some on condition on $A(z)$ as $z \to +\infty$.
4. Conditions involving a ratio of f to A.
5. Considering only an integral of $\left|u(s)\right|^q$ as the argument for the function A.

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Question: Are these *necessary*?

$$-A\left(\int_0^1 \left|u(s)\right|^q\, ds\right)u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1).$$

The methodology that I will discuss does not require any of these assumptions. In fact, $A(z)$ can equal zero infinitely often and can even be negative on sets of infinite measure.

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We also will be able to consider the above problem in the following, more general formulation:

$$-A\Big(\big(a * (g \circ u)\big)(1)\Big) u''(t) = \lambda f\big(t, u(t)\big), \, t \in (0,1),$$

where

$$(a * b)(t) := \int_0^t a(t-s)b(s) \, ds, \, t \geq 0,$$

for sufficiently regular functions a and b.

$$-A\Big(\big(a*(g\circ u)\big)(1)\Big)u''(t)=\lambda f\big(t,u(t)\big),\ t\in(0,1),$$

Let's consider for a moment why we would want to consider the above convolution-type nonlocal term.

$$-A\Big(\big(a*(g\circ u)\big)(1)\Big)u''(t)=\lambda f\big(t,u(t)\big),\ t\in(0,1),$$

Let's consider for a moment why we would want to consider the above convolution-type nonlocal term. A primary motivation is from the theory of fractional differential operators. Recall that the Riemann-Liouville fractional integral of order $\alpha>0$ is defined by

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds.$$

$$-A\Big(\big(a*(g\circ u)\big)(1)\Big)u''(t)=\lambda f\big(t,u(t)\big),\ t\in(0,1),$$

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In other words,

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\ ds=(a*u)(t),\text{ where } a(t):=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}.$$

Main Results

With the preceding context in mind we now set forth the specific problem we'll consider.

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$$-A\Big(\big(a * (g \circ u)\big)(1)\Big)u''(t) = \lambda f\big(t, u(t)\big),\ 0 < t < 1,$$

subject to the Dirichlet boundary conditions

$$u(0) = 0 = u(1).$$

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$$u(0) = 0 = u(1).$$

Note that the allowable boundary conditions are quite flexible – we choose Dirichlet conditions just for definiteness and to keep things simpler.

Main Results

With the preceding context in mind we now set forth the specific problem we'll consider. We consider, for $A : \mathbb{R} \to \mathbb{R}$ a continuous function,

$$-A\Big(\big(\mathbf{1} * (g \circ u)\big)(1)\Big)u''(t) = \lambda f\big(t, u(t)\big),\ 0 < t < 1,$$

subject to the Dirichlet boundary conditions

$$u(0) = 0 = u(1).$$

Also to keep things simpler, in the statement of the existence theorem to follow we will assume that $a \equiv \mathbf{1}$, where $\mathbf{1}$ denotes (with abuse of notation) the function that is constantly one – i.e., $\mathbf{1} := \mathbf{1}(x) \equiv 1,\ x \in \mathbb{R}$.

So the following is the problem for which I will state an existence theorem (with conditions on g to be stated momentarily):

$$-A\Big(\big(\mathbf{1} * (g \circ u)\big)(1)\Big)u''(t) = \lambda f\big(t, u(t)\big),\ 0 < t < 1$$

$$u(0) = 0 = u(1).$$

Note that

$$A\Big(\big(\mathbf{1} * (g \circ u)\big)(1)\Big) = A\left(\int_0^1 (g \circ u)(s)\, ds\right).$$

We consider the operator $T : \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ defined by

$$(Tu)(t) := \lambda \int_0^1 \left(A\left(\int_0^1 (g \circ u)(r)\, dr \right) \right)^{-1} G(t,s) f(s, u(s))\, ds,$$

where the kernel $G : [0,1] \times [0,1] \to (0, +\infty)$ is defined by

$$G(t,s) := \begin{cases} s(1-t), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t \le s \le 1 \end{cases}.$$

Note (for $0 < c < d < 1$) that

$$\min_{t \in [c,d]} G(t,s) \ge \eta_0 \mathscr{G}(s),\ s \in [0,1],$$

where $\eta_0 := \min\{c, 1-d\}$ and $\mathscr{G} := \sup_{t \in [0,1]} G(t,s)$.

$$(Tu)(t) := \lambda \int_0^1 \left(A\left(\int_0^1 (g \circ u)(r)\, dr \right) \right)^{-1} G(t,s) f\big(s, u(s)\big)\, ds.$$

Our approach is to find a fixed point of T (i.e., $Tu = u$) by means of topological fixed point theory.

To this end we consider the following cone and attendant open set:

$$\mathscr{K} := \Big\{ u \in \mathscr{C}([0,1]) \ : \ u \geq 0, \ \min_{t\in[c,d]} u(t) \geq \eta_0 \|u\|, \int_0^1 u(s)\, ds \geq C_0 \|u\| \Big\},$$

$$\widehat{V}_\rho := \Big\{ u \in \mathscr{K} \ : \ \int_0^1 (g \circ u)(s)\, ds < \rho \Big\}.$$

Here

$$C_0 := \inf_{s\in(0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, dt \in (0,1].$$

If $a \neq 1$, then:

$$\mathscr{K} := \Big\{ u \in \mathscr{C}\big([0,1]\big) \ : \ u \geq 0, \ \min_{t \in [c,d]} u(t) \geq \eta_0 \|u\|, \quad (a * u)(1) \geq C_0 \|u\| \Big\},$$

$$\widehat{V}_\rho := \{ u \in \mathscr{K} \ : \ \big(a * (g \circ u)\big)(1) < \rho \}.$$

Here

$$C_0 := \inf_{s \in (0,1)} \frac{1}{\mathscr{G}(s)} \big(a * G(\cdot, s)\big)(1)$$
$$= \inf_{s \in (0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 a(1-t) G(t,s) \, dt.$$

If $a \neq \mathbf{1}$, then:

$$\mathscr{K} := \Big\{ u \in \mathscr{C}\big([0,1]\big) \ : \ u \geq 0, \ \min_{t \in [c,d]} u(t) \geq \eta_0 \|u\|, \\ (a * u)(1) \geq C_0 \|u\| \Big\},$$

$$\widehat{V}_\rho := \left\{ u \in \mathscr{K} \ : \ \big(a * (g \circ u)\big)(1) < \rho \right\}.$$

Note that

$$(\mathbf{1} * u)(1) = \int_0^1 u(s)\, ds$$

and

$$\big(\mathbf{1} * (g \circ u)\big)(1) = \int_0^1 (g \circ u)(s)\, ds.$$

$$\widehat{V}_\rho := \left\{ u \in \mathscr{K} \ : \ \int_0^1 (g \circ u)(s)\, ds < \rho \right\}$$

The key fact about $\widehat{V}_\rho$ is that

$$u \in \partial \widehat{V}_\rho \Longrightarrow \big(a * (g \circ u)\big)(1) = \rho.$$

And this is important because we then have direct control over the nonlocal element.

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And this is important because we then have direct control over the nonlocal element – that is,

$$-A\left(\underbrace{\big(a * (g \circ u)\big)(1)}_{=\rho}\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0, 1).$$

Finally what must we assume about $g : [0, +\infty) \to [0, +\infty)$? A possible (but by no means the only such) collection of conditions is as follows.

1. g is continuous
2. g is strictly increasing
3. Either
    1. g is concave; or
    2. g is convex.

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   1. g is concave; or
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A model case in the concave case is $g(t) := t^q$ for $0 < q < 1$ and a model case in the convex case is the same but with $q > 1$. (Note that $q = 1$ can also be accommodated, in fact.) The choice $g(t) = t^q$ leads to the following model problem:

$$-A\left(\int_0^1 (u(s))^q \, ds\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0, 1),$$

which we mentioned at the beginning.

So, what does a typical existence theorem look like?

We'll need the following notation

- $f^m_{[a,b]\times[c,d]}$ to denote

$$f^m_{[a,b]\times[c,d]} := \min_{(t,y)\in[a,b]\times[c,d]} f(t,y); \textit{ and}$$

- $f^M_{[a,b]\times[c,d]}$ to denote

$$f^m_{[a,b]\times[c,d]} := \max_{(t,y)\in[a,b]\times[c,d]} f(t,y).$$

So, what does a typical existence theorem look like? Here's the case when g is concave.

Assume that there are numbers $0 < \rho_1 < \rho_2$ such that each of the following is true.

1. $A(t) > 0$ for $t \in \left[\rho_1, \rho_2\right]$
2. $$\int_0^1 g\left(\frac{\lambda}{A\left(\rho_1\right)}\left(f^m_{[c,d]\times\left[\eta_0 g^{-1}(\rho_1),\frac{1}{\eta_0}g^{-1}\left(\frac{\rho_1}{d-c}\right)\right]}\right)\int_c^d G(t,s)\,ds\right)\,dt > \rho_1$$
3. $$\frac{\lambda}{A\left(\rho_2\right)}\left(f^M_{[0,1]\times\left[0,\frac{1}{\eta_0}g^{-1}\left(\frac{\rho_2}{d-c}\right)\right]}\right)\int_0^1\int_0^1 G(t,s)\,ds\,dt < g^{-1}\left(\rho_2\right)$$

If $g(0) < \rho_2$, then the operator T has at least one positive fixed point u_0.

In fact, we can say that u_0 must satisfy the localization

$$g^{-1}(\rho_1) \leq \|u_0\| \leq \frac{1}{\eta_0} g^{-1}\left(\frac{\rho_2}{d-c}\right).$$

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$$g^{-1}\left(\rho_1\right) \leq \|u_0\| \leq \frac{1}{\eta_0} g^{-1}\left(\frac{\rho_2}{d-c}\right).$$

Note that

$$g^{-1}\left(\rho_1\right) > 0$$

since

1. $g\big([0,\infty)\big) \subset [0,\infty)$;
2. g is strictly increasing; and
3. $\rho_1 > 0$.

Notice the pointwise-type conditions on A, which, recall, houses the nonlocal element.

Assume that there are numbers $0 < \rho_1 < \rho_2$ such that each of the following is true.

1. $A(t) > 0$ for $t \in \left[\rho_1, \rho_2\right]$
2. $$\int_0^1 g\left(\frac{\lambda}{A\left(\rho_1\right)}\left(f^m_{[c,d]\times\left[\eta_0 g^{-1}(\rho_1), \frac{1}{\eta_0} g^{-1}\left(\frac{\rho_1}{d-c}\right)\right]}\right)\int_c^d G(t,s)\, ds\right)\, dt > \rho_1$$
3. $$\frac{\lambda}{A\left(\rho_2\right)}\left(f^M_{[0,1]\times\left[0, \frac{1}{\eta_0} g^{-1}\left(\frac{\rho_2}{d-c}\right)\right]}\right)\int_0^1\int_0^1 G(t,s)\, ds\, dt < g^{-1}\left(\rho_2\right)$$

If $g(0) < \rho_2$, then the operator T has at least one positive fixed point u_0.

Note that in the model case $g(u) = u^q$, where $0 < q < 1$ and $u \geq 0$, the conditions previously stated become

$$\int_0^1 \left(\frac{\lambda}{A(\rho_1)} \left(f^m_{[c,d]\times\left[\eta_0 \rho_1^{\frac{1}{q}}, \frac{1}{\eta_0}\left(\frac{\rho_1}{d-c}\right)^{\frac{1}{q}}\right]} \right) \int_c^d G(t,s)\, ds \right)^q dt > \rho_1, \quad (1)$$

and

$$\frac{\lambda}{A(\rho_2)} \left(f^M_{[0,1]\times\left[0, \frac{1}{\eta_0}\left(\frac{\rho_2}{d-c}\right)^{\frac{1}{q}}\right]} \right) \int_0^1 \int_0^1 G(t,s)\, ds\, dt < \rho_2^{\frac{1}{q}}. \quad (2)$$

For example, in case $q = \dfrac{1}{2}$ so that $g(u) = \sqrt{u}$ and $g^{-1}(u) = u^2$, for $u \geq 0$, we see that (1) becomes

$$\int_0^1 \left(\frac{\lambda}{A(\rho_1)} \left(f^m_{[c,d]\times\left[\eta_0\rho_1^2, \left(\frac{\rho_1}{(d-c)\sqrt{\eta_0}}\right)^2\right]} \right) \int_c^d G(t,s)\, ds \right)^{\frac{1}{2}} dt > \rho_1,$$

whereas (2) becomes

$$\frac{\lambda}{A(\rho_2)} \left(f^M_{[0,1]\times\left[0, \left(\frac{\rho_2}{(d-c)\sqrt{\eta_0}}\right)^2\right]} \right) \int_0^1 \int_0^1 G(t,s)\, ds\, dt < \rho_2^2.$$

So, what does a typical application look like?

Consider the problem

$$-A\left(\int_0^1 \left(u(s)\right)^{\frac{1}{2}}\,ds\right)u''(t)=\lambda f\big(t,u(t)\big),\ t\in(0,1)$$

subject to the Dirichlet boundary conditions

$$u(0)=0=u(1).$$

Here we have chosen $q=\dfrac{1}{2}$ and defined $g\ :\ [0,+\infty)\to[0,+\infty)$ by $g(t)=t^{\frac{1}{2}}$ so that g^{-1} is defined by $g^{-1}(t)=t^2$. Furthermore, define the function $A\ :\ [0,+\infty)\to\mathbb{R}$ by

$$A(t):=\begin{cases}-t^3, & 0\leq t\leq 1\\ t\sin\left(\frac{3\pi}{2}t\right), & t>1\end{cases}.$$

Consider the problem

$$-A\left(\int_0^1 (u(s))^{\frac{1}{2}}\, ds\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1)$$

subject to the Dirichlet boundary conditions

$$u(0) = 0 = u(1).$$

Furthermore, define the function $A : [0, +\infty) \to \mathbb{R}$ by

$$A(t) := \begin{cases} -t^3, & 0 \le t \le 1 \\ t \sin\left(\frac{3\pi}{2} t\right), & t > 1 \end{cases}.$$

Note that A is nonpositive on infinitely many intervals of positive measure. Moreover, $A(t) = 0$ for countably infinitely many values of $t \geq 0$. And, in addition, A is not bounded on $[0, +\infty)$. In fact, $\liminf_{t \to +\infty} A(t) = -\infty$.

One can then show that if

$$\lambda f^{m}_{\left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4}\left(\frac{4003}{3000}\right)^2, 4\left(\frac{4003}{1500}\right)^2\right]} > 0.14474,$$

and

$$\lambda f^{M}_{[0,1]\times\left[0,\frac{400}{9}\right]} < \frac{500}{9},$$

then

$$-A\left(\int_0^1 (u(s))^{\frac{1}{2}}\, ds\right) u''(t) = \lambda f\big(t, u(t)\big),\ t \in (0,1)$$

has at least one positive solution u_0 satisfying the localization

$$\left(\frac{4003}{3000}\right)^2 \leq \|u_0\| \leq \frac{400}{9}.$$

Some concluding thoughts.

1. The more general case, in which $a \neq \mathbf{1}$, proceeds similarly. The main difference is that the conditions for existence now contain integrals involving $a(1-s)$ since a is no longer a constant function.
2. The case in which g is convex (rather than concave) also proceeds similarly.
3. One can also consider multiple nonlocal convolution elements – e.g.,

   $$-A\big((a*u^q)(1)\big)u''(t) = \lambda B\big((b*u^p)(1)\big)f\big(t,u(t)\big),\ t \in (0,1).$$

   However, this case is more technical because using a set like $\widehat{V}_\rho := \{u \in \mathscr{K}\ :\ \big(a*(g \circ u)\big)(1) < \rho\}$ only allows us to control *directly* **one** nonlocal element at a time.

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Thank you for your attention!