Duality for Stieltjes integral equations

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Motivation

The scalar first-order linear differential equation

$$x'(t) = p(t)x(t)$$

can be written in the integral form

$$x(t) = x(t_0) + \int_{t_0}^{t} p(s)x(s)\,ds.$$

We replace it by

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\,dP(s), \qquad (1)$$

where $P(t) = \int_{t_0}^{t} p(s)\,ds$.
But Eq. (1) makes sense even if P is discontinuous; then the solution x will be discontinuous as well. Eq. (1) is called a generalized differential equation.

Notation (jumps)

We work with regulated functions defined on an interval $[a,b]$ and use the following notation:

$$\Delta^+ g(t) = \begin{cases} g(t+) - g(t) & \text{if } t \in [a,b), \\ 0 & \text{if } t = b, \end{cases}$$

$$\Delta^- g(t) = \begin{cases} g(t) - g(t-) & \text{if } t \in (a,b], \\ 0 & \text{if } t = a. \end{cases}$$

Also, we let $\Delta g(t) = \Delta^+ g(t) + \Delta^- g(t)$.

Basic results for the Kurzweil-Stieltjes integral

Integration by parts: If $f, g : [a,b] \to \mathbb{R}$ have bounded variation, then

$$\int_a^b f(x-)\,dg(x) + \int_a^b g(x+)\,df(x) = f(b)g(b) - f(a)g(a).$$

Substitution theorem: If $h : [a,b] \to \mathbb{R}$ is bounded, then

$$\int_a^b h(x)\,d\left[\int_a^x f(z)\,dg(z)\right] = \int_a^b h(x)f(x)\,dg(x),$$

whenever either side of the equation exists.

Linear equations (existence and uniqueness)

Let $t_0 \in [a, b]$. Consider a function $P : [a, b] \to \mathbb{R}$, which has bounded variation on $[a, b]$ and satisfies $1 + \Delta^+ P(t) \neq 0$ for all $t \in [a, t_0)$, $1 - \Delta^- P(t) \neq 0$, for all $t \in (t_0, b]$.
Then for every $x_0 \in \mathbb{R}$, there exists a unique function $x : [a, b] \to \mathbb{R}$ such that

$$x(t) = x_0 + \int_{t_0}^{t} x(s)\, dP(s), \quad t \in [a, b].$$

The solution corresponding to $x_0 = 1$ is called the generalized exponential function, and is denoted by $t \mapsto e_{dP}(t, t_0)$, $t \in [a, b]$.

The solution corresponding to a general x_0 is $x(t) = x_0 e_{dP}(t, t_0)$.

Generalized exponential function – explicit formula

If P is continuous, then $e_{dP}(t,t_0) = e^{P(t)-P(t_0)}$.
In the general case, we have the following explicit formula:

$$e_{dP}(t,t_0) = \begin{cases} 1, & t = t_0, \\ \dfrac{e^{P(t-)-P(t_0+)}}{e^{\sum_{s\in(t_0,t)} \Delta P(s)}} \dfrac{\prod_{s\in[t_0,t)}(1+\Delta^+ P(s))}{\prod_{s\in(t_0,t]}(1-\Delta^- P(s))}, & t > t_0, \\ \dfrac{e^{\sum_{s\in(t,t_0)} \Delta P(s)}}{e^{P(t_0-)-P(t+)}} \dfrac{\prod_{s\in(t,t_0]}(1-\Delta^- P(s))}{\prod_{s\in[t,t_0)}(1+\Delta^+ P(s))}, & t < t_0. \end{cases}$$

Relation to other types of equations

Special cases of

$$x(t) = x_0 + \int_{t_0}^{t} x(s)\, dP(s)$$

include:

1. Classical ODE $x'(t) = p(t)x(t)$ corresponds to $P(t) = \int_{t_0}^{t} p(s)\, ds$ (P is continuously differentiable)
2. Stieltjes differential equation $x'_g(t) = p(t)x(t)$ corresponds to $P(t) = \int_{t_0}^{t} p(s)\, dg(s)$ (g and P are left-continuous)
3. Δ-dynamic equation $x^{\Delta}(t) = p(t)x(t)$, $t \in \mathbb{T}$, corresponds to $P(t) = \int_{t_0}^{t} p(g(s))\, dg(s)$, where $g(t) = \inf\{s \in \mathbb{T} : s \geq t\}$ (g and P are left-continuous)
4. ∇-dynamic equation $x^{\nabla}(t) = p(t)x(t)$, $t \in \mathbb{T}$, corresponds to $P(t) = \int_{t_0}^{t} p(g(s))\, dg(s)$, where $g(t) = \sup\{s \in \mathbb{T} : s \leq t\}$ (g and P are right-continuous)

Towards duality theory for generalized ODEs

Duality theory for classical 1st order linear ODEs and dynamic equations on time scales:

- The product of solutions to dual (adjoint) equations is a constant function.
- The linear operators corresponding to dual equations satisfy Lagrange's identity.

What is the corresponding theory for generalized ODEs?

An idea: If x is a solution to a generalized ODE, is there a generalized ODE whose solution is $1/x$?

Inverse of the exponential function

Assume that $P:[a,b]\to\mathbb{R}$ has bounded variation,

$$\begin{aligned} 1+\Delta^+P(t) &\neq 0 \text{ for every } t\in[a,b),\\ 1-\Delta^-P(t) &\neq 0 \text{ for every } t\in(a,b]. \end{aligned}$$

Then

$$e_{dP}(t,t_0)^{-1}=e_{d(\ominus P)}(t,t_0),\quad t\in[a,b],$$

where

$$(\ominus P)(t)=-P(t)+\sum_{s\in[t_0,t)}\frac{(\Delta^+P(s))^2}{1+\Delta^+P(s)}-\sum_{s\in(t_0,t]}\frac{(\Delta^-P(s))^2}{1-\Delta^-P(s)}.$$

Possible candidates for dual generalized ODEs

Under suitable assumptions on P, the unique solution of

$$x(t) = x_0 + \int_{t_0}^{t} x(s)\, dP(s), \quad t \in [a,b], \tag{2}$$

is $x(t) = x_0 e_{dP}(t, t_0)$, the unique solution of

$$y(t) = y_0 + \int_{t_0}^{t} y(s)\, d(\ominus P)(s), \quad t \in [a,b], \tag{3}$$

is $y(t) = y_0 e_{d\ominus P}(t, t_0)$, and the product

$$x(t)y(t) = x_0 e_{dP}(t, t_0) y_0 e_{d\ominus P}(t, t_0) = x_0 y_0$$

remains constant. Although (2) and (3) seem be to good candidates for dual equations, there is no satisfactory analogue of Lagrange's identity, and the theory is not compatible with duality theory for dynamic equations on time scales.

Duality for Stieltjes integral equations

It turns out that more reasonable candidates for dual equations are:

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s-)\,dP(s), \quad t \in [a,b], \tag{4}$$

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\,dP(s), \quad t \in [a,b]. \tag{5}$$

Important notes:

- We always assume that P has bounded variation.
- Convention: In the first integral, the value $x(s-)$ should be understood as $x(s)$ when s coincides with $\min(t_0, t)$. In the second integral, the value $y(s+)$ should be understood as $y(s)$ when s coincides with $\max(t_0, t)$.
- (4) and (5) are no longer generalized ODEs in the classical sense, and we speak about Stieltjes integral equations.

Operator forms

The operator forms of the dual equations

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s-)\, dP(s), \quad t \in [a, b],$$

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\, dP(s), \quad t \in [a, b],$$

are $Lx = 0$ and $L^* y = 0$, where the operators L and L^* are given by

$$Lx(t) = x(t) - x(t_0) - \int_{t_0}^{t} x(s-)\, dP(s),$$

$$L^* y(t) = y(t) - y(t_0) + \int_{t_0}^{t} y(s+)\, dP(s).$$

Again, the limits $x(s-)$ and $y(s+)$ have to be replaced by corresponding function values if s coincides with the left/right endpoint of the integral.

Lagrange's identity

If $x, y : [a,b] \to \mathbb{R}$ have bounded variation and $t_0 \in [a,b]$, then for each $T \in [a,b]$ we have

$$x(T)y(T) - x(t_0)y(t_0) = \int_{t_0}^{T} y(t+)\, dLx(t) + \int_{t_0}^{T} x(t-)\, dL^* y(t),$$

with the convention that $y(t+) = y(t)$ if $t = \max(t_0, T)$, and $x(t-) = x(t)$ if $t = \min(t_0, T)$.

Proof: Use substitution and integration by parts.

Note: The result no longer holds if we abandon the convention concerning the endpoints.

Corollary: If $Lx = 0$ and $L^* y = 0$ on $[a,b]$, then $x \cdot y$ is a constant function on $[a,b]$.

The left-continuous case

Assume that $P : [a,b] \to \mathbb{R}$ has bounded variation, $t_0 \in [a,b]$, P is left-continuous and $1 + \Delta^+ P(t) \neq 0$ for each $t \in [a, t_0)$. Then $x : [a,b] \to \mathbb{R}$ satisfies $Lx = 0$ if and only if

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\, dP(s), \quad t \in [a,b].$$

All solutions have the form $x(t) = x(t_0) e_{dP}(t, t_0)$.

If we in addition assume that $1 + \Delta^+ P(t) \neq 0$ for each $t \in [t_0, b)$, then $y : [a,b] \to \mathbb{R}$ satisfies $L^* y = 0$ if and only if

$$y(t) = y(t_0) + \int_{t_0}^{t} y(s)\, d(\ominus P)(s), \quad t \in [a,b].$$

All solutions have the form
$y(t) = y(t_0) e_{d(\ominus P)}(t, t_0) = y(t_0) e_{dP}(t, t_0)^{-1}$.

The right-continuous case

Assume that $P : [a, b] \to \mathbb{R}$ has bounded variation, $t_0 \in [a, b]$, P is right-continuous and $1 + \Delta^- P(t) \neq 0$ for each $t \in (t_0, b]$. Then $y : [a, b] \to \mathbb{R}$ satisfies $L^* y = 0$ if and only if

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s)\, dP(s), \quad t \in [a, b].$$

All solutions have the form $y(t) = y(t_0) e_{d(-P)}(t, t_0)$.

If we in addition assume that $1 + \Delta^- P(t) \neq 0$ for each $t \in (a, t_0]$, then $x : [a, b] \to \mathbb{R}$ satisfies $Lx = 0$ if and only if

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\, d(\ominus(-P))(s), \quad t \in [a, b].$$

All solutions have the form
$x(t) = x(t_0) e_{d(\ominus(-P))}(t, t_0) = x(t_0) e_{d(-P)}(t, t_0)^{-1}$.

The general case (1)

Is it possible to get rid of the $x(t-)$ and $y(t+)$ terms?

We can calculate them as follows.

1. If $Lx = 0$ on $[a, b]$, then

$$x(t-) = \begin{cases} x(t)\dfrac{1+\Delta^+ P(t)}{1+\Delta P(t)} & \text{if } t \in (a, t_0) \text{ and } 1+\Delta P(t) \neq 0, \\ x(t)\dfrac{1}{1+\Delta^- P(t)} & \text{if } t \in [t_0, b] \text{ and } 1+\Delta^- P(t) \neq 0. \end{cases}$$

2. If $L^* y = 0$ on $[a, b]$, then

$$y(t+) = \begin{cases} y(t)\dfrac{1}{1+\Delta^+ P(t)} & \text{if } t \in [a, t_0] \text{ and } 1+\Delta^+ P(t) \neq 0, \\ y(t)\dfrac{1+\Delta^- P(t)}{1+\Delta P(t)} & \text{if } t \in (t_0, b) \text{ and } 1+\Delta P(t) \neq 0. \end{cases}$$

The general case (2)

After getting rid of the $x(t-)$ and $y(t+)$ terms, it is easy to rewrite the resulting equations as Volterra-Stieltjes integral equations of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s)\,dK(t,s), \quad t \in [a,b],$$

with suitable kernels K.

One can now use the existing theory of Volterra-Stieltjes integral equations to derive an existence and uniqueness result for the pair of adjoint equations.

The general case – explicit solution formulas

$$x(t)=\begin{cases} x_0, & t=t_0,\\ x_0\dfrac{e^{P(t-)-P(t_0+)}}{e^{\sum_{s\in(t_0,t)}\Delta P(s)}}(1+\Delta^+P(t_0))\prod_{s\in(t_0,t)}(1+\Delta P(s))(1+\Delta^-P(t)), & t>t_0,\\ x_0\dfrac{e^{\sum_{s\in(t,t_0)}\Delta P(s)}}{e^{P(t_0-)-P(t+)}}\dfrac{1}{1+\Delta^+P(t)}\dfrac{1}{\prod_{s\in(t,t_0)}(1+\Delta P(s))}\dfrac{1}{1+\Delta^-P(t_0)}, & t<t_0. \end{cases}$$

$$y(t)=\begin{cases} y_0, & t=t_0,\\ y_0\dfrac{e^{\sum_{s\in(t_0,t)}\Delta P(s)}}{e^{P(t-)-P(t_0+)}}\dfrac{1}{1+\Delta^+P(t_0)}\dfrac{1}{\prod_{s\in(t_0,t)}(1+\Delta P(s))}\dfrac{1}{1+\Delta^-P(t)}, & t>t_0,\\ y_0\dfrac{e^{P(t_0-)-P(t+)}}{e^{\sum_{s\in(t,t_0)}\Delta P(s)}}(1+\Delta^+P(t))\prod_{s\in(t,t_0)}(1+\Delta P(s))(1+\Delta^-P(t_0)), & t<t_0. \end{cases}$$

Relation to Stieltjes differential equations

If g is a left-continuous nondecreasing function and $P(t) = \int_{t_0}^{t} p(s)\,dg(s)$, then the equations

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s-)\,dP(s), \quad t \in [a,b],$$

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\,dP(s), \quad t \in [a,b].$$

are equivalent to

$$x_g'(t) = p(t)x(t),$$

$$y_g'(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t)$$

(or equivalently $y_g'(t) = -p(t)y(t+)$), which were discussed in the previous talk by Ignacio Márquez Albés.

Relation to dynamic equations on time scales

If $\mathbb{T}$ is a time scale and $p : [a,b]_{\mathbb{T}} \to \mathbb{R}$, let $P(t) = \int_{t_0}^{t} p(g(s))\, dg(s)$, where either $g(t) = \inf\{s \in \mathbb{T} : s \geq t\}$, or $g(t) = \sup\{s \in \mathbb{T} : s \leq t\}$. Then the equations

$$x(t) = x(t_0) + \int_{t_0}^{t} x(s-)\, dP(s), \quad t \in [a,b],$$

$$y(t) = y(t_0) - \int_{t_0}^{t} y(s+)\, dP(s), \quad t \in [a,b],$$

are equivalent either to

$$x^{\Delta}(t) = p(t)x(t), \qquad y^{\Delta}(t) = -p(t)y(\sigma(t))$$

(where σ is the forward jump operator), or to

$$x^{\nabla}(t) = p(t)x(\rho(t)), \qquad y^{\nabla}(t) = -p(t)y(t)$$

(where ρ is the backward jump operator).

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